# A MIMO Right-Half Plane Zero Example

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The performance and robustness limitations of MIMO right-half plane (RHP) transmission zeros are illustrated by example. A two-input, two-output system with a RHP zero is studied. The zero is not obvious from Bode plots, or from plots of the SVD of the frequency response matrix.

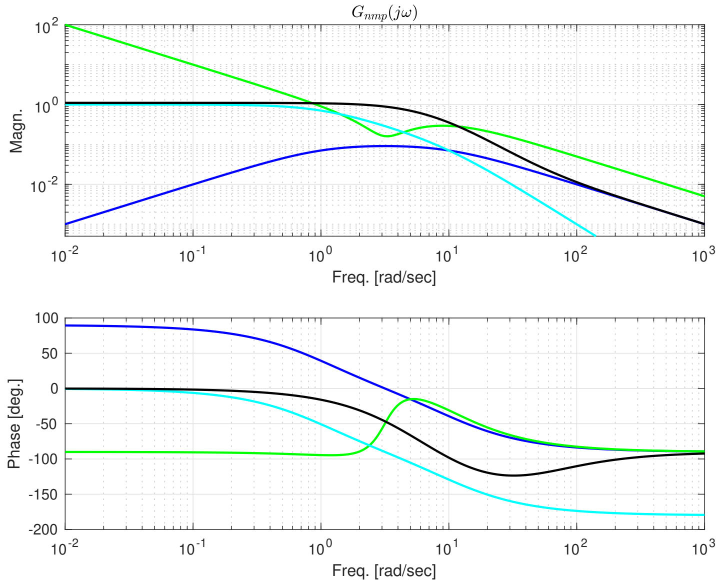
A feedback control design is performed with the goal of achieving a certain closed-loop bandwidth. As the desired bandwidth approaches the frequency of the RHP zero the performance (sensitivity function) and robustness (complementary sensitivity function) deteriorate. The example also illustrates the use of IMC for MIMO system feedback design.

## 1 MIMO system

The plant to be considered is,

$$G_{\mathrm{NMP}}(s) = \begin{bmatrix} \dfrac{s}{s^2+11s+10} & \dfrac{5s^2+10s+50}{s^3+15s^2+50s} \\ \dfrac{10}{s^2+11s+10} & \dfrac{s+55}{s^2+15s+50} \end{bmatrix}.$$

The plant has poles at $s = \{0, -1, -5, -10\}$. A Bode plot is shown below.



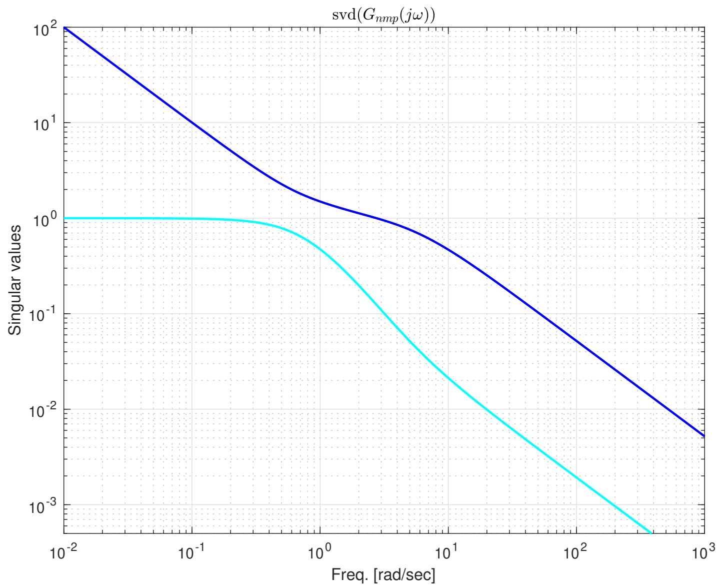

The integrator is obvious from the Bode plot; all of the other poles are stable. What is not so obvious is that the plant has a RHP zero at $s_z = 10$. The output direction vector is

$$y_z = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

and so,

$$y_z^T G_{\mathrm{NMP}}(s) \mid_{s=10} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

Although $G_{\mathrm{NMP}}(s)|_{s=10}$ loses rank, this is not detected in the Bode magnitude plot of the singular values of $G(j\omega)$ shown below.



As in the SISO case there is a factorization into a non-minimum phase all-pass factor and a minimum phase factor. For this particular plant this factorization[1] is,

$$G_{\mathrm{NMP}}(s) = \underbrace{\begin{bmatrix} \dfrac{s}{s+10} & \dfrac{10}{s+10} \\ \dfrac{10}{s+10} & \dfrac{s}{s+10} \end{bmatrix}}_{B_z(s)^{-1}} \underbrace{\begin{bmatrix} \dfrac{1}{s+1} & \dfrac{5}{s+5} \\ 0 & \dfrac{1}{s} \end{bmatrix}}_{G_{\mathrm{MP}}(s)}.$$

The factor $B_z(s)$ is called a Blaschke product. It has the form,

$$B_z(s) = I + \frac{2\,\mathrm{real}(s_z)}{(s - s_z)} y_z y_z^T.$$

As written, $B_z(s)$ is an all-pass system with a pole at $s = s_z$ and output pole direction equal to $y_z$. In our case we invert $B_z(s)$ to get a zero in the RHP.

## 2 Feedback design

We will use an Internal Model Control (IMC) approach[2] to design the controller. This is based on parametrizing all stabilizing controllers in terms of a variable $Q(s)$, where the stability of $Q(s)$ is equivalent to closed-loop stability of the feedback system.

To do this we define $Q(s)$ as

$$Q(s) = K(s)(I + G_{\mathrm{NMP}}(s)K(s))^{-1}.$$

Note that this transformation is invertible;

$$K(s) = (I - Q(s)G_{\mathrm{NMP}}(s))^{-1}Q(s).$$

We can express the complementary sensitivity function linearly in $Q(s)$ via,

$$T(s) \;=\; G_{\mathrm{NMP}}(s)K(s)(I + G_{\mathrm{NMP}}(s)K(s))^{-1} \;=\; G_{\mathrm{NMP}}(s)Q(s).$$

Suppose we specify an "ideal" complementary sensitivity function, $T_{\mathrm{ideal}}(s)$. Conceptually we could solve for $Q(s)$ via,

$$Q(s) = G_{\mathrm{NMP}}(s)^{-1}T_{\mathrm{ideal}}(s),$$

which would give $T(s) = T_{\mathrm{ideal}}(s)$. However the problem is that $G_{\mathrm{NMP}}(s)$ can't be inverted. The fact that $G_{\mathrm{NMP}}(s)$ is improper isn't a big problem—it's only $G_{\mathrm{NMP}}(s)^{-1}T_{\mathrm{ideal}}(s)$ that has to be proper. But the RHP zero in $G_{\mathrm{NMP}}(s)$ will give a RHP pole in $Q(s)$ and our closed-loop system will not be stable.

[1] This factorization is not obvious. See Section 6.2.4 in Skogestad & Posthlethwaite for details.

[2] See Section 4.8 in Skogestad & Posthlethwaite for a description of IMC. Much more detail can be found in Morari & Zafiriou, *Robust Process Control,* Prentice Hall, 1988.

Recall that for the closed-loop system to be stable any RHP zeros in $G_{\text{NMP}}(s)$ must also be in $T(s)$. So we will use the same idea as in the SISO case with inverse-based feedback: invert only the non-minimum phase part of $G_{\text{NMP}}(s)$. The part that can't be inverted is the Blaschke product, $B_z(s)^{-1}$, which is the all-pass function containing the RHP zero.

So let's define an "ideal" complementary sensitivity function of the form,

$$T_{\text{ideal}}(s) = \frac{1}{\left(\frac{s}{\omega_c}\right)^2 + \sqrt{2}\left(\frac{s}{\omega_c}\right) + 1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This is actually a 2nd order low-pass Butterworth filter with a cut-off frequency of $\omega_c$. This implies that we want integral reference tracking on each channel with a bandwidth of approximately $\omega_c$. Later we will look at the consequences of different choices of $\omega_c$.

Now if we choose

$$Q(s) \;=\; G_{\text{MP}}(s)^{-1} T_{\text{ideal}}(s),$$

then we would get a complementary sensitivity function,

$$T(s) \;=\; G_{\text{NMP}}(s)Q(s) \;=\; B_z(s)^{-1}G_{\text{MP}}(s)Q(s) \;=\; B_z(s)^{-1}T_{\text{ideal}}(s).$$

It's clear that $Q(s)$ is stable and that $T(s)$ satisfies the interpolation conditions for closed-loop stability.

But there's a problem with this design choice. Although $B_z(s)^{-1}$ is all-pass (all of its singular values are equal to one), it's nothing like the identity matrix at low frequencies. So our reference tracking objective will not be satisfied. There's one more thing we can do to at least partially fix this problem—choose,

$$Q(s) \;=\; G_{\text{MP}}(s)^{-1} B_z(0) T_{\text{ideal}}(s),$$

which gives a complementary sensitivity function,

$$T(s) \;=\; B_z(s)^{-1} B_z(0) T_{\text{ideal}}(s).$$

This is a much better choice. At low frequencies $T(s)$ will be close to an identity matrix—at zero frequency it will exactly be the identity—and as we approach the frequency of the RHP zero the product $B_z(s)^{-1}B_z(0)$ will rotate away from the identity. As $B_z(s)^{-1}B_z(0)$ is still all-pass the complementary sensitivity, $T(s)$, will roll-off at $s = \omega_c$.

To illustrate this method we will choose a cut-off frequency and look at the resulting design. Take

$$\omega_c \;=\; 3 \text{ radians/second}.$$

Now $Q(s)$ is given by,

$$Q(s) = \frac{1}{\frac{1}{(s+1)}\frac{1}{s} - 0\frac{5}{(s+5)}} \begin{bmatrix} \frac{1}{s} & \frac{-5}{(s+5)} \\ 0 & \frac{1}{(s+1)} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \frac{1}{\left(\frac{s}{3}\right)^2 + \sqrt{2}\left(\frac{s}{3}\right) + 1}$$

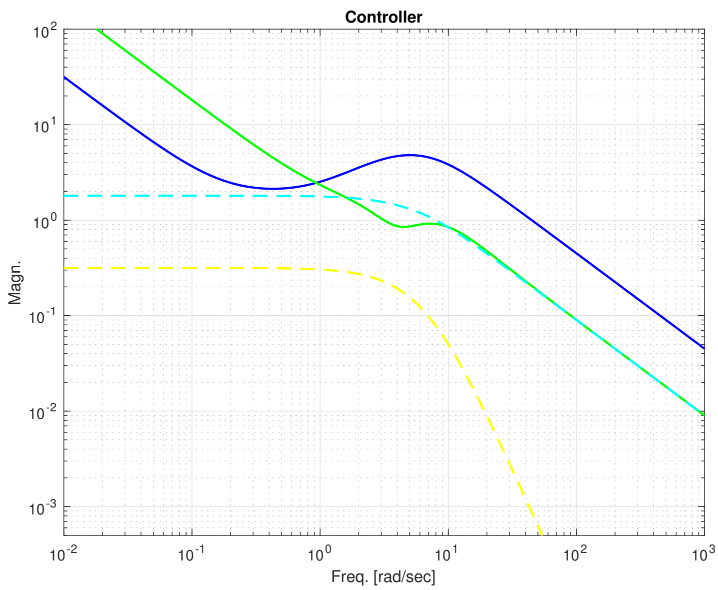
$$= \frac{s(s+1)}{\left(\frac{s}{3}\right)^2 + \sqrt{2}\left(\frac{s}{3}\right) + 1} \begin{bmatrix} \frac{1}{s} & \frac{-5}{(s+5)} \\ 0 & \frac{1}{(s+1)} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-45s^2 - 45s}{s^3 + (5+3\sqrt{2})s^2 + (9+15\sqrt{2})s + 45} & \frac{9s}{s^2 + 3\sqrt{2}s + 9} \\ \frac{9s+9}{s^2+3\sqrt{2}s+9} & 0 \end{bmatrix}.$$

The poles of $Q(s)$ are $s = \{-5, -2.12 \pm j2.12, -2.12 \pm j2.12, -2.12 \pm j2.12\}$, and as $Q(s)$ is stable, the closed-loop will also be stable. We now solve for $K(s)$ and calculate

$$S(s) = (I + G_{\mathrm{NMP}}(s)K(s))^{-1}.$$

The sensitivity function has poles at $s = \{-10, -2.12 \pm j2.12, -2.12 \pm j2.12\}$ which is stable as expected. A Bode magnitude plot of the four components of $S(j\omega)$ is shown below.



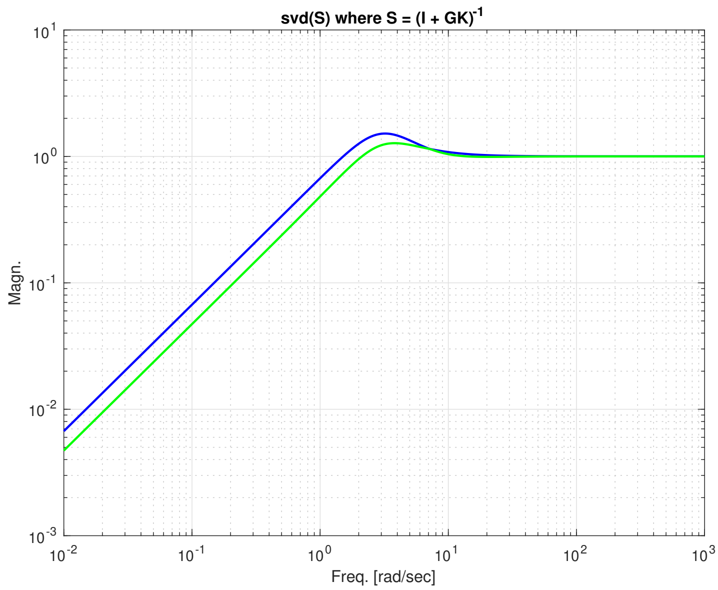

The diagonal elements of the output sensitivity function show that integral tracking control is achieved, and the closed-loop bandwidth is between 1.0 and 2.0

radians/second. The only concern with the reference tracking aspect of the performance is the cross-channel error coupling shown in the green plot. This is approximately a 20% error at a frequency of 3 radians/second.

A Bode magnitude plot of the output complementary sensitivity function,

$$T(s) = G_{\rm NMP}(s)K(s)(I + G_{\rm NMP}(s)K(s))^{-1},$$

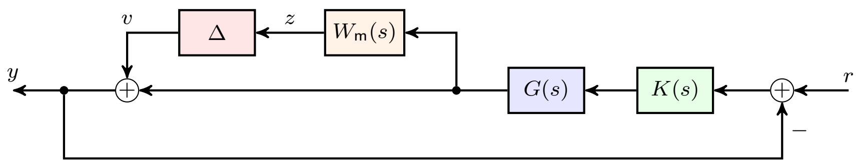
is shown below, and illustrates the cross-channel coupling error. This is to be expected at a frequency of around 3 radians/second the Blaschke product is rotating $T_{\rm ideal}(s)$ away from a diagonal matrix.




We will subsequently see that the output complementary sensitivity function allows us to assess robustness with respect to perturbations at the plant output. These might be used to model uncertainty in sensor dynamics, or uncertainty in the plant dynamics with respect to the output. At the frequencies where

$$T(j\omega) = G_{\rm NMP}(j\omega)K(j\omega)(I + G_{\rm NMP}(j\omega)K(j\omega))^{-1},$$

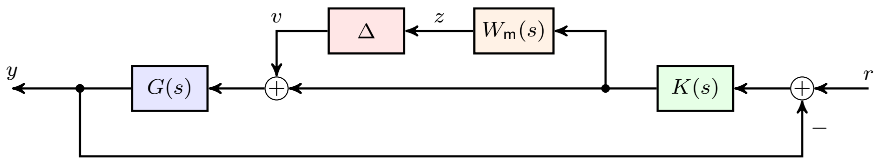
is large (in singular value) then we can tolerate only small perturbations. To be more precise, we can tolerate perturbations, $W_{\rm m}(j\omega)\Delta(j\omega)$ up to size $1/\overline{\sigma}\,(T(j\omega))$. The output perturbation configuration is illustrated below.




Because $B_z(s)^{-1}B_z(0)$ is all-pass the singular values of $T(s)$ are simply those of $T_{\rm ideal}(s)$. In our case these are the diagonal elements. The robustness requirement is easy to satisfy as we have complete control over $\overline{\sigma}\left(T(j\omega)\right)$ in the design process. Notice that the RHP zeros do not influence this as they appear only in the all-pass function which does not change the singular values of $T(s)$.

This design method has specified $T(s)$ exactly—as long as we take into account the requirement that $T(s)$ must also contain the RHP zeros of $G_{\mathrm{NMP}}(s)$. We can see that this imposes some limits on the achievable sensitivity function, $S(s)$. If one of the outputs was less important than the other we could design $T(s)$ and $S(s)$ in such a way that the errors resulting from the RHP zero appear on the less important output.[3]

At first glance it appears that RHP zeros in MIMO systems do not have as much of a performance limitation as RHP zeros in SISO systems. However, we are not taking everything into account. In MIMO systems we may also have to consider robustness with respect to perturbations at the input. Placing a perturbation at the input of the plant would model the effect of uncertainty in the actuator dynamics or plant dynamic uncertainty with respect to the actuation inputs. The block diagram below illustrates where the perturbations enter the loop.




In SISO systems input and output multiplicative perturbations have the same effect on the robustness analysis as everything commutes. In MIMO systems this is not true and for a robustness analysis of the effect of the input perturbations we must consider the complementary sensitivity function *defined with respect to the plant input,*

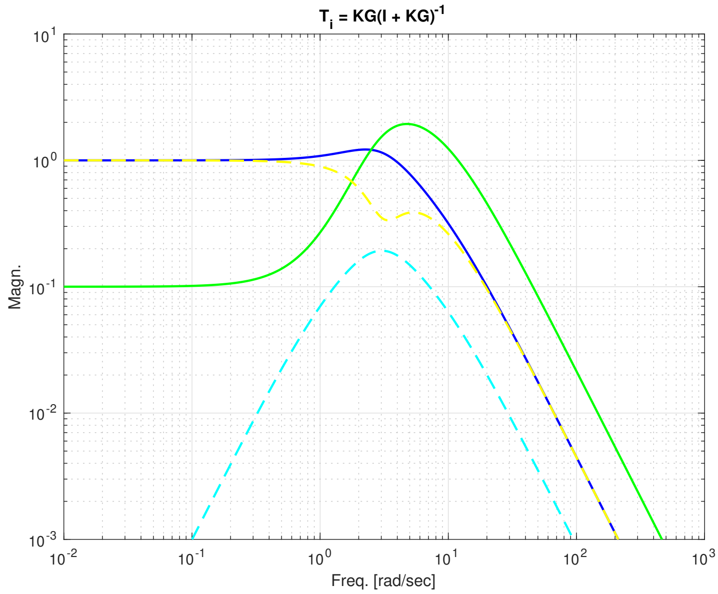
$$T_i(s) = K(s)G_{\mathrm{NMP}}(s)(I + K(s)G_{\mathrm{NMP}}(s))^{-1}.$$

The singular values of $T_i(j\omega)$ are shown in the figure below. As we can see these are significantly larger than the singular values of $T(j\omega)$ at high frequencies. The maximum singular value of $T_i(j\omega)$ peaks at approximately 2 between 4 and 5 radians/second. It remains above 1 until beyond 10 radians/second. This means that we require the input uncertainty to be small up to frequencies above 10 radians/second. In contrast we are only requiring the output uncertainty to be small up to $\omega_c = 3$ radians/second.

So if we precisely control the output complementary sensitivity and the reference tracking error or output disturbance rejection (both specified by the output sensitivity function) to the extent possible, then we place stringent requirements on the level of uncertainty at the plant input. The converse is also true—precisely

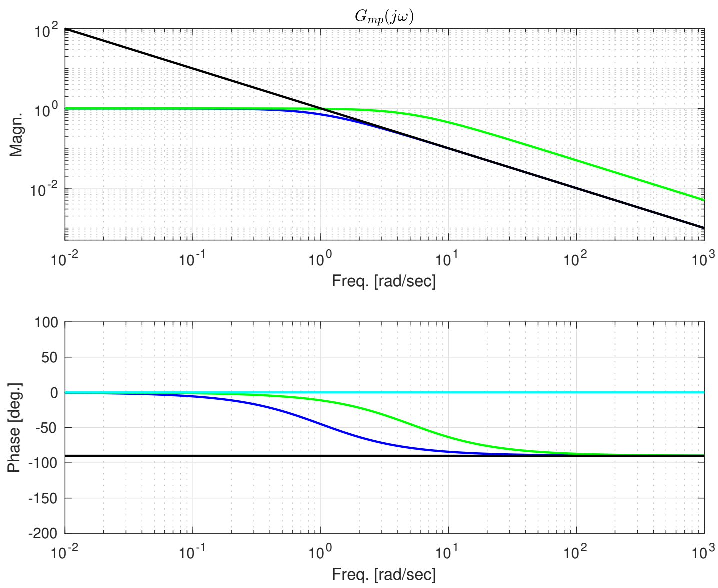
[3] This is discussed further in Section 6.5 of Skogestad & Postlethwaite.

specifying the input sensitivity will place stringent requirements on the level of uncertainty at the plant output.




## 3 Minimum phase plant comparison

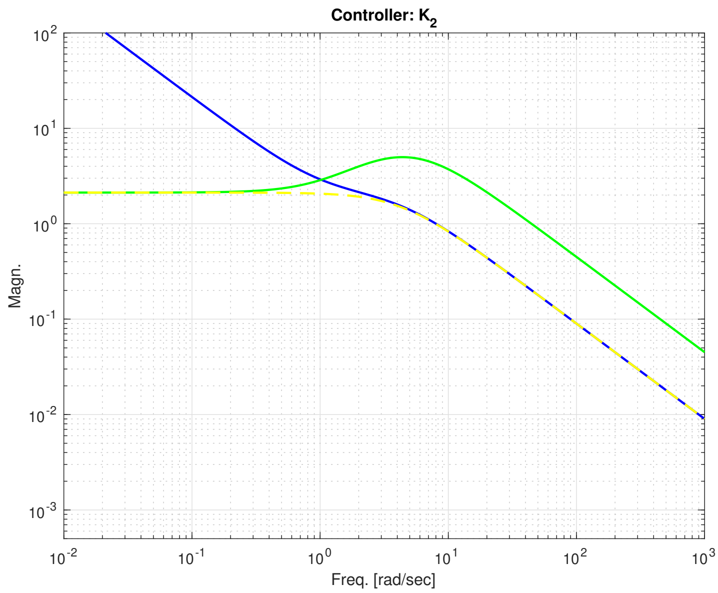
To illustrate that the above limitations arise from the RHP zero we can look at doing a similar design for the minimum phase part of the plant, $G_{\rm MP}(s)$. The Bode plot of $G_{\rm MP}(s)$ is given below.




Repeating the IMC design procedure gives,

$$T(s) = T_{\rm ideal}(s),$$

and the sensitivity function shown below. This is essentially a perfectly shaped sensitivity function.



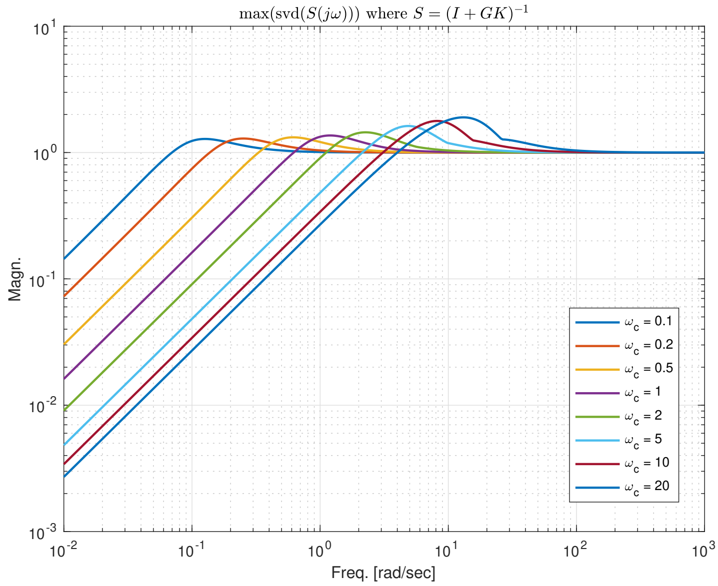

Note that we have no cross-coupling errors in the closed-loop control of the minimum phase plant, $G_{\mathrm{MP}}(s)$.

## 4 Bandwidth limits design

As the ideal complementary sensitivity function,

$$T_{\mathrm{ideal}}(s) = \frac{1}{\left(\dfrac{s}{\omega_c}\right)^2 + \sqrt{2}\left(\dfrac{s}{\omega_c}\right) + 1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

is expressed in terms of a bandwidth, $\omega_c$, we can investigate the effect of the RHP zero as a function of increasing closed-loop bandwidth. The Bode plot below shows the maximum singular value of the output sensitivity function for a range of values of $\omega_c$.

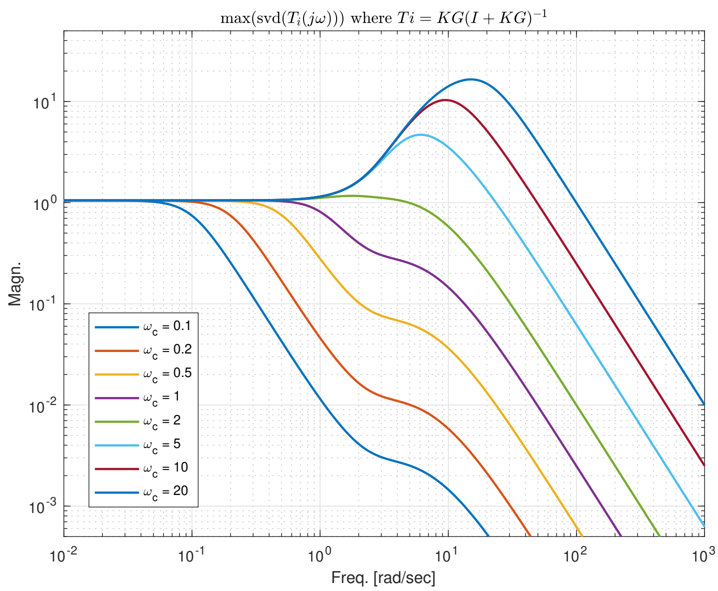



Note that as $\omega_c$ increases above 2 radians/second the peak of the sensitivity function increases. The deterioration is in the cross-channel coupling which at higher frequencies rises to be 100% of the reference as $\omega_c$ is increased. Attempting to increase $\omega_c$ beyond $w_c = 3$ increases the cross-channel errors without providing any additional tracking performance for the channels on the diagonal. Even though $\omega_c$ increases the actual bandwidth achieved comes to a limit.

A much more significant deterioration is evident in the input complementary sensitivity function,

$$T_i(s) = K(s)G_{\mathrm{NMP}}(s)(I + K(s)G_{\mathrm{NMP}}(s))^{-1},$$

illustrated below for a range of choices of $\omega_c$.



We can see that for $\omega_c$ above 2 radians/second the level of precision required at high frequencies rapidly becomes unrealistic for any practical system.