Digital control system design

Sampled-data closed-loop

$\frac{G(s)}{T_{ZOH}} K_d(z) + r(k) u(k) u(t) y(t) y(k) − G_{ZOH}(z)$

$G_{ZOH}(z)$ equivalence

$\frac{G(s)}{T_{ZOH}} K_d(z) + r(k) y(k) − G_{ZOH}(z)$
Zero-order hold equivalence — transfer function

\[
\begin{align*}
y(k) & \quad y(t) \\
& \quad G(s) \\
& \quad \text{ZOH} \\
& \quad u(t) \\
& \quad u(k)
\end{align*}
\]

\[G_{ZOH}(z)\]

Input: \[u(k) = \begin{cases} 1 & k = 0, \\ 0 & k \neq 0 \end{cases}, \quad u(t) = \text{step}(t) - \text{step}(t - T).\]

Output: \[y(s) = \left(1 - e^{-Ts}\right) \frac{G(s)}{s}.
\]

We now sample this, and take the \(Z\)-transform,

\[G_{ZOH}(z) = \mathcal{Z}\left\{ \left(1 - e^{-Ts}\right) \frac{G(s)}{s} \right\}\]

\[= (1 - z^{-1}) \mathcal{Z}\left\{ \frac{G(s)}{s} \right\}.
\]

Zero-order hold equivalence — state space

\[
\begin{align*}
y(k) & \quad y(t) \\
& \quad G(s) \\
& \quad \text{ZOH} \\
& \quad u(t) \\
& \quad u(k)
\end{align*}
\]

Integrating \(\Phi(t)\) over a single sample period \((kT \text{ to } kT + T)\):

\[x(kT + T) = e^{AT} x(kT) + \int_{kT}^{kT+T} e^{A(kT+T-\tau)} B u(\tau) d\tau,
\]

\[
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \quad \text{ZOH-equivalence} \implies \begin{bmatrix} e^{AT} \int_{0}^{T} e^{A\eta} B d\eta \\ C \int_{0}^{T} e^{A\eta} B d\eta \int_{0}^{T} e^{A\eta} B d\eta \end{bmatrix}
\]
Zero-order hold equivalence — frequency domain

Example:  \[ G(s) = \frac{(2 - s)}{(2s + 1)(s + 2)}, \quad T = 0.6 \]

Digital control system design

Sampled-data closed-loop

\[
\begin{align*}
y(k) & \quad \downarrow \quad T \quad \downarrow \quad y(t) \\
 & \quad \downarrow \quad G(s) \quad \downarrow \quad u(t) \quad \downarrow \quad u(k) \quad \downarrow \quad r(k)
\end{align*}
\]

\[ K(s) \] approximation

\[
\begin{align*}
y(k) & \quad \downarrow \quad T \quad \downarrow \quad y(t) \\
 & \quad \downarrow \quad G(s) \quad \downarrow \quad ZOH \quad \downarrow \quad K_d(z) \quad \downarrow \quad r(k)
\end{align*}
\]
Design approaches

$G(s)$ $\rightarrow$ $K(s)$

$G_{ZOH}(z)$ $\rightarrow$ $K_d(z)$

Continuous-time design
Discrete-time design

$ZOH$-equivalence of $G(s)$ and sample/hold
Approximation of $K(s)$ with $K_d(z)$

Sampled-data design

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Design by approximation

1. Design a continuous-time controller, $K(s)$
   ▶ Verify stability, performance and bandwidth
   ▶ Verify margins and robustness

2. Select a sample-rate, $T$

3. Find $K_d(z)$ approximating $K(s)$

4. Calculate the ZOH-equivalent $G_{ZOH}(z)$

5. Check the stability of the $G_{ZOH}(z), K_d(z)$ loop

6. Simulate $K_d(z)$ with $G(s)$ (including sample/hold).
   ▶ Verify simulated performance
   ▶ Examine intersample behaviour

Controller approximation

Approach: approximating the integrators

If $F(z) \approx 1/s$, then, $s \approx F^{-1}(z)$, \[ \implies K_d(z) = K(s) \big|_{s=F^{-1}(z)} \]

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Integration

\[ y(t) = y(0) + \int_0^t x(\tau) \, d\tau, \]

The signal, \( y(t) \), over a single \( T \) second sample period is,

\[ y(kT + T) = y(kT) + \int_{kT}^{kT+T} x(\tau) \, d\tau. \]

Trapezoidal approximation

\[ \hat{y}(kT + T) = \hat{y}(kT) + T x(kT) + (x(kT + T) - x(kT))T/2. \]

Taking \( z \)-transforms,

\[ z \hat{y}(z) = \hat{y}(z) + T x(z) + \frac{T}{2} (z - 1) x(z), \]

Approximation:

\[ \frac{\hat{y}(z)}{x(z)} = F(z) = \frac{T}{2} \frac{z + 1}{z - 1}. \]

So the substitution is,

\[ s \leftarrow \frac{2 \, z - 1}{T \, z + 1}. \]

This is known as a bilinear (or Tustin) transform.
Frequency mapping

Pole locations under bilinear transform:

\[
\{ s \mid \text{real}(s) < 0 \} \xrightarrow{\text{bilinear}} \{ z \mid |z| < 1 \}
\]

\[K(s) \text{ stable } \iff K_d(z) \text{ stable}.\]

Bilinear frequency distortion

\[\Omega : \text{discrete-frequencies: } e^{j\Omega T}, \; \Omega \in (-\pi, \pi].\]

Frequency mapping:

Continuous frequencies, \(\omega\) to discrete frequencies, \(\Omega\).

Substitute \(s = j\omega\) and \(z = e^{j\Omega T}\) into \(s = \frac{2}{T} \frac{z - 1}{z + 1}\):

\[j\omega = \frac{2}{T} \frac{(1 - e^{-j\Omega T})}{(1 + e^{-j\Omega T})} = \frac{2}{T} \frac{j \sin(\Omega T/2)}{\cos(\Omega T/2)} = \frac{2}{T} j \tan(\Omega T/2).\]

Frequency distortion:

\[\Omega = \frac{2}{T} \tan^{-1}(\omega T/2)\]
Bilinear frequency distortion

\[ \Omega = \frac{2}{T} \tan^{-1}(\omega T/2) \]

The \( \Omega = \omega T \) line is the sampling mapping.

Prewarping

\[ s = \frac{\alpha(z - 1)}{(z + 1)}, \quad \alpha \in (0, \pi/T), \quad \text{maps} \{ \text{real}\{s\} < 0 \} \to \{ |z| < 1 \}. \]

Modifying the frequency distortion

Select a frequency “prewarping frequency”, \( \omega_{pw} \).

Solve for \( \alpha \) such that \( K(j\omega_{pw}) = K_d(e^{j\omega_{pw}T}) \).

The “prewarped” transform makes \( K(j\omega) = K_d(e^{j\omega T}) \) at \( \omega = 0 \) and \( \omega = \omega_{pw} \).

\[ j\omega_{pw} = \frac{\alpha(e^{j\omega_{pw}T} - 1)}{(e^{j\omega_{pw}T} + 1)} = j\alpha \tan(\omega_{pw}T/2), \]

which implies that: \( \alpha = \frac{\omega_{pw}}{\tan(\omega_{pw}T/2)}. \)
Prewarping

Frequency distortion (bilinear): \( \Omega = \frac{2}{T} \tan^{-1}(\omega T/2). \)

Frequency distortion (with prewarping): \( \Omega = \frac{2}{T} \tan^{-1}(\omega/\alpha) \)

Controller approximations

Bilinear approximation: \( K_{bl}(e^{j\omega}) \), Prewarped Tustin approximation: \( K_{pw}(e^{j\omega}) \)

\[ K(j\omega) \]
\[ K_{bl}(e^{j\omega}) \]
\[ K_{pw}(e^{j\omega}) \]
\[ \log \omega \text{ (rad/sec)} \]
Choosing a prewarping frequency

The prewarping frequency must be in the range: \( 0 < \omega_{pw} < \pi/T \).

- \( \alpha = 2/T \) (standard bilinear) corresponds to \( \omega_{pw} = 0 \).

Possible options for \( \omega_{pw} \) depend on the problem:

- The cross-over frequency (helps preserve the phase margin);
- The frequency of a critical notch;
- The frequency of a critical oscillatory mode.

The best choice depends on the most important features in your control design.

**Remember:** \( K(s) \) stable implies \( K_d(z) \) stable.

But you **must** check that \( (1 + G_{ZOH}(z)K_d(z))^{-1} \) is stable!

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Example

Plant model:

\[
G(s) = 5(1 - s/\zeta_{rhp}) \frac{(s^2 + 2\zeta\eta\omega_m s + \eta^2\omega_m^2)}{(1 + \tau s)(s^2 + 2\zeta\omega_m s + \omega_m^2)} \frac{1}{\eta^2}
\]

where \( \tau = 0.5 \), \( \zeta_{rhp} = 70 \), \( \omega_m = 20 \), \( \zeta = 0.05 \), and \( \eta = 1.2 \).

**IMC design**

\[
T_{ideal}(s) = 3rd \text{ order Butterworth filter with bandwidth: } 25 \text{ [rad./sec.]} \\
Q(s) = T_{ideal}(s)G_m^{-1}(s) \\
K(s) = (I - Q(s)G(s))^{-1}Q(s).
\]
Step response

Output

Amplitude $y(t)$

Actuation

Amplitude $u(t)$
Bilinear/Trapezoidal/Tustin transform

Bilinear transform

Nyquist frequency: \( 100 \) radians/second \( \implies T = \frac{\pi}{100} \).

\[ K_{bl}(z) = K(s) \bigg|_{s = \frac{2}{T} \frac{z - 1}{z + 1}} \]

Discrete-time analysis

\[ G_{ZOH}(z) \]

\[ \begin{array}{c}
 y(k) \\
 \downarrow T \\
 G(s) \\
 \uparrow ZOH \\
 K_d(z)
\end{array} \]

\[ r(k) \]

Loopshapes: bilinearly transformed controller

\[ \begin{array}{c}
 K_d(e^{j\omega}) \\
 G(e^{j\omega}) \\
 G_{ZOH}(e^{j\omega}) \\
 L_d(e^{j\omega}) \\
 L(e^{j\omega}) \\
 K(e^{j\omega})
\end{array} \]

\[ \begin{array}{c}
 \text{Magnitude} \\
 \log \omega \text{ (rad/sec)} \\
 \text{Phase (deg.)} \\
 \log \omega \text{ (rad/sec)}
\end{array} \]
Sensitivity and complementary sensitivity: bilinearly transformed controller

Step response: bilinearly transformed controller
Step response: bilinearly transformed controller

Actuation

Prewarped Tustin transform

Nyquist frequency: 100 radians/second \[ \Rightarrow T = \frac{\pi}{100}. \]

Select the prewarping frequency at \( \omega_{pw} = \omega_m \) (20 radians/sec.).

\[ K_{pw}(z) = K(s) \bigg|_{s=\alpha \frac{z-1}{z+1}} \]

where,

\[ \alpha = \frac{\omega_{pw}}{\tan(\omega_{pw}T/2)}. \]
Loopshapes: prewarped Tustin controller

Sensitivity and complementary sensitivity: prewarped Tustin controller
Step response: prewarped Tustin controller

Output

Amplitude

$y(t)$

$y(k)$

$-0.2$

$0$

$0.2$

$0.4$

$0.6$

$0.8$

$1.0$

$1.2$

$1.4$

Time (sec)

$0.5$

$1.0$

$1.5$

$2.0$

Actuation

Amplitude

$u(t)$

$u(k)$

$-0.2$

$0$

$0.2$

$0.4$

$0.6$

$0.8$

$1.0$

$1.2$

$1.4$

Time (sec)

$0.5$

$1.0$

$1.5$

$2.0$
Sample rate selection

Sample rate selection is critical to digital control design.

Main considerations

- Sampling/ZOH will (approximately) introduce a delay of $T/2$ seconds.
- Anti-aliasing filters will need to be designed and these will also introduce phase lag.
- The system runs “open-loop” between samples.
- Very fast sampling can introduce additional noise.
- Very fast sampling makes all of the poles appear close to 1. The controller design can become numerically sensitive.

Designing for digital implementation

Sampled-data implementation

Continuous-time design
We want a similar discrete sensitivity function up to the frequency where $|\hat{S}_d(j\omega)|$ returns to 1.

$\hat{S}_d(s) = (I + F_a(s)G(s)e^{-sT_1/2}K(s))^{-1}$  

Approximate discrete sensitivity

In this example, for $\omega > 20$ rad./sec.,

$|1 - \hat{S}_d(j\omega)| \ll 1 \implies \pi/T = 20$ is about the minimum.
Loop-shaping interpretation

For $\omega$ up to where $|F_a(j\omega)G(j\omega)K(j\omega)| < \epsilon$ and remains very small,
we want $F_a(j\omega)G(j\omega)K(j\omega)e^{-j\omega T_1/2} \approx G_d\left(e^{j\omega T_1}\right)K_d\left(e^{j\omega T_1}\right)$.

($G_d(z)$ is the ZOH-equivalent of $F_a(s)G(s)$)

Discrete-time sensitivity function

What is the actual discrete sensitivity function,
$S_d(s) = \frac{1}{(1 + G_d(e^{j\omega})K_d(e^{j\omega}))}$ for $\pi/T_1 = 20.9$ [rad./sec.]?

$K_d(z)$ is a prewarped Tustin approximation with $\omega_{pw} = \omega_B$. 

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Discrete-time loopshape: \( L_{d_1}(s) = G_d(s)K_d(s), T_1 = 0.15 \) seconds

\( G_d(s) \) is a ZOH-equivalent transform of \( G(s) \).
\( K_d(z) \) is a prewarped Tustin approximation with \( \omega_{pw} = \omega_B \).

Choosing a slower sample rate.

Choosing \( T_2 = 0.5 \) seconds \( \Rightarrow \pi/T_2 = 6.28 \) rad./sec.

\( K_d(z) \) is a prewarped Tustin approximation with \( \omega_{pw} = \omega_B \).
Choosing a slower sample rate.

Choosing $T_2 = 0.5$ seconds $\implies \pi/T_2 = 6.28$ rad./sec.

Fast sampling

Fast sampling period: $T_f$.

Control appropriate (slower) sampling period: $T_s$ (typically $T_s = MT_f$ for integer $M > 1$).