Digital implementation of control systems

Why digital?
▶ Easily reprogrammed or modified.
▶ Complex algorithms (or optimisations) can be implemented.
▶ Integration with remote systems (via internet).

Why analogue?
▶ Simple and cheap in mass production.
▶ Highly reliable.
▶ Very high frequency operation.
▶ On-chip integrated systems.
Components:

- Plant: $G(s)$, continuous-time;
- Controller: $K_d(z)$, discrete-time;
- Sampler (A/D converter): $y(k) = y(t) |_{t=kT}$ for $k = 0, 1, 2, \ldots$
- Zero-order hold (D/A converter): $u(t) = u(kT)$, $kT \leq t < kT + T$.

$T$ is the sampling period.
Components: zero-order hold

\[ u(t) = u(k), \quad \text{for } kT \leq t < kT + T. \]

Discrete sequence:

Sample index

\[ \begin{array}{ccccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
0 & 1 & 0 & 1 & 0 & 1 \\
\end{array} \]

Continuous signal:

Time (seconds)

\[ \begin{array}{ccccccc}
1T & 2T & 3T & 4T & 5T & 6T \\
0 & 1 & 0 & 1 & 0 & 1 \\
\end{array} \]

Quantisation

Potential error: \( \pm \frac{1}{2} \text{ LSB} \) in the best case.

Example: 12 bit A/D and D/A on a \( \pm 10 \) volt scale

1 LSB = 0.00488 volts.
Sampled-data reconstruction

\[ \tilde{x}(t) \xrightarrow{ZOH} x(k) \xrightarrow{T} x(t) \]

Input signal: \( x(t) \)

Output signal: \( \tilde{x}(t) \)

Sampling

\[ y(k) \xrightarrow{T} y(t) \]

Example: single pole signal

Consider \( y(t) = \begin{cases} e^{-at}, & t \geq 0 \\ 0, & t < 0 \end{cases} \) with \( a > 0 \).

Laplace transform: \( y(s) = \frac{1}{s + a} \).

Sampled signal: \( y(k) = y(t) \bigg|_{t=kT} = e^{-akT} = \left(e^{-aT}\right)^k \).

Z-transform: \( y(z) = \frac{z}{z - e^{-aT}} \).

The \( s \)-plane pole is at \( s_1 = -a \), and the corresponding \( z \)-plane pole is at \( z_1 = e^{-aT} \).
**Sampling**

**General case:**

Sampling maps the $s$-domain poles to the $z$-domain via: $z_i = e^{s_i T}$. Stability preserving: $\{\text{real}(s_i) < 0\}$ maps to $\{|z_i| < 1\}$.

**Pole locations under sampling:**

![Diagram showing pole locations in the s-plane and their mapping to the z-plane via $z_i = e^{s_i T}$]

**Sampling (in detail)**

**Z-plane**

<table>
<thead>
<tr>
<th>Real</th>
<th>Imaginary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_n = 0.6\pi/T$</td>
<td>$N = 4$</td>
</tr>
<tr>
<td>$\omega_n = 0.7\pi/T$</td>
<td>$N = 3$</td>
</tr>
<tr>
<td>$\omega_n = 0.8\pi/T$</td>
<td>$N = 2$</td>
</tr>
<tr>
<td>$\omega_n = 0.9\pi/T$</td>
<td></td>
</tr>
</tbody>
</table>

| $\zeta = 0.1$ | $\omega_n = 0.5\pi/T$ |
| $\omega_n = 0.4\pi/T$ | $N = 6$ |
| $\omega_n = 0.3\pi/T$ | $N = 8$ |
| $\omega_n = 0.2\pi/T$ | $N = 10$ |
| $\omega_n = 0.1\pi/T$ | $N = 20$ |

**Lines of constant frequency**

**Lines of constant damping**

**Samples per oscillation**

**Changing the sampling frequency.**

Decreasing $T$: decrease decay rate ($r \to 1$) decrease oscillation frequency ($\theta \to 0$) poles track constant damping curves towards 1
Sampled signals: aliasing

Example:

55 Hz signal sampled at 275 Hz

Consider \( y(t) = \cos \omega_1 t \)

Laplace: \( y(s) = \frac{s}{s^2 + \omega_1^2} \).

Continuous poles: \( s_{1,2} = \pm j\omega_1 \)

Sampled poles: \( z_{1,2} = e^{\pm j\omega_1 T} \)
Sampled signals: aliasing

Example:

55 Hz signal sampled at: \( \frac{1}{T_1} = 275 \) Hz
\( \frac{1}{T_2} = 130 \) Hz
\( \frac{1}{T_3} = 65 \) Hz

Real
Imaginary
Z-plane

Amplitude

Time (seconds)

2019-5-21 12.13
Sampled signals: aliasing

Example:

55 Hz signal sampled at: 
\[ \frac{1}{T_1} = 275 \text{ Hz} \]
\[ \frac{1}{T_2} = 130 \text{ Hz} \]
\[ \frac{1}{T_3} = 65 \text{ Hz} \]
Sampled signals: aliasing

The unit disk can only represent signals of frequency up to 1/2 the sampling frequency. (Nyquist frequency).

Maps the horizontal strip from \(-j\pi/T\) to \(j\pi/T\) onto the \(z\)-plane.

And \(\text{real}(s) < 0\) in this strip maps to the inside of the unit disk.

Sampling also maps the next strip (from \(j\pi/T\) to \(j3\pi/T\)) onto the whole \(z\)-plane and adds it into the result.

Also true for all (infinite) \(2\pi/T\) wide strips above and below the lowest frequency strip.
### Sampled signals: aliasing

**Pole locations under sampling:**

![Pole locations diagram](image)

\[ z = e^{sT} \]

Aliased high frequency disturbances are indistinguishable from low frequency disturbances.

The controller responds at the wrong frequency.

### Sampled signals: aliasing

**Consequences of aliasing:**

- **Ambiguity.** Our computer/controller cannot distinguish between frequencies inside the \(-\pi/T\) to \(\pi/T\) range and those outside of it.
  - Controller will respond incorrectly to an aliased signal.
  - An aliased signal cannot be reconstructed (signal processing).

**Amelioration of the problem:**

- **Anti-aliasing filter.** Low pass, rejecting \(|\omega| > \pi/T\).
  - High frequency signals no longer enter loop erroneously.
  - High frequency disturbances/errors are "invisible."
The anti-aliasing filter, $F_a(s)$, will add phase to the loop (Potentially destabilizing!)

ZOH response

Pulse input: $u(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0 \end{cases}$, gives the output,

Equivalently, the pulse response is:

$$u(t) = \text{step}(t) - \text{step}(t - T), \quad (\text{step}(t) = \text{unit step function})$$
ZOH response

The discrete-time transfer function is the $z$-transform of the sampled pulse response.

For a pulse, $u(k)$, the plant input is,

$$u(t) = \text{step}(t) - \text{step}(t - T).$$

The ZOH output (in the Laplace domain) is

$$u(s) = \left( 1 - e^{-Ts} \right) \frac{1}{s}.$$

ZOH properties

"Frequency response"

The ZOH frequency response (for the fundamental frequency only) is given by:

$$ZOH(j\omega) = \frac{1 - e^{-j\omega T}}{j\omega}$$

$$= e^{-j\omega T/2} \left( \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j} \right) \frac{2j}{j\omega}$$

$$= Te^{-j\omega T/2} \frac{\sin(\omega T/2)}{\omega T/2}$$

$$= Te^{-j\omega T/2} \text{sinc}(\omega T/2)$$
Zero-order hold equivalence

\[ y(k) \rightarrow G(s) \rightarrow ZOH \rightarrow u(k) \]

\[ y(t) \rightarrow u(t) \]

Input: \[ u(k) = \begin{cases} 1 & k = 0, \\ 0 & k \neq 0 \end{cases}, \quad u(t) = \text{step}(t) - \text{step}(t - T). \]

Output: \[ y(s) = \left(1 - e^{-Ts}\right) \frac{G(s)}{s}. \]

We now sample this, and take the Z-transform,

\[ G_{ZOH}(z) = Z\left\{ \left(1 - e^{-Ts}\right) \frac{G(s)}{s} \right\} \]

\[ = (1 - z^{-1}) Z\left\{ \frac{G(s)}{s} \right\}. \]

Easily calculated (c2d or zohequiv in MATLAB).

Sampling

Sample rate effects:

\[ z_i = e^{s_i T}, \text{ changing } T \text{ changes the pole positions.} \]

Continuous closed-loop step response:

\[ G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \]

\[ \zeta = 0.6, \quad \omega_n = 5 \text{ rad./sec.}, \quad s_{1,2} = -3 \pm 4i. \]
Sampling

Sample rate effects:
Continuous pole positions: \( s_{1,2} = -3 \pm 4i \).

Sample period \( T_1 \): \( z_{1,2} = 0.255 \pm 0.398i \)
Sample period \( T_2 \): \( z_{1,2} = 0.650 \pm 0.306i \)

Continuous time: root-locus analysis of closed-loop stability

\[
G(s) = \frac{a}{s + a}, \quad a > 0.
\]

Controller (proportional):
\[
K(s) = K_p.
\]

\( G(s)K(s) \) has one pole and no zeros.

Theoretically stable for \(-1 < K_p \leq \infty\).
Discrete time: root-locus analysis of closed-loop stability

\[ G(s) \times \text{ZOH} \times K + r(k)u(k)u(t)y(t)y(k) = \frac{1 - e^{-aT}}{z - e^{-aT}} \]

\[ G_{\text{ZOH}}(z)K \] has one pole and no zeros.

Unstable for \( K > \frac{1 + e^{-aT}}{1 - e^{-aT}} \)

The additional phase from the ZOH is also potentially destabilizing.