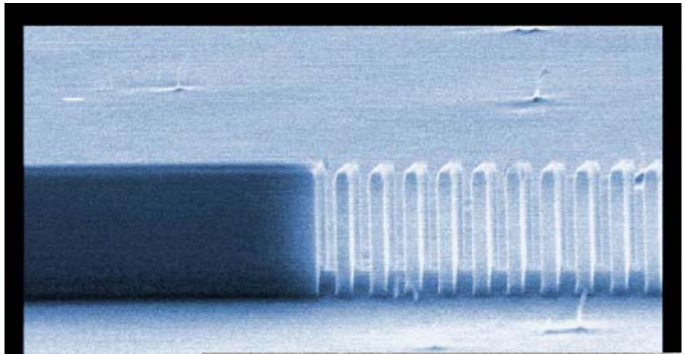
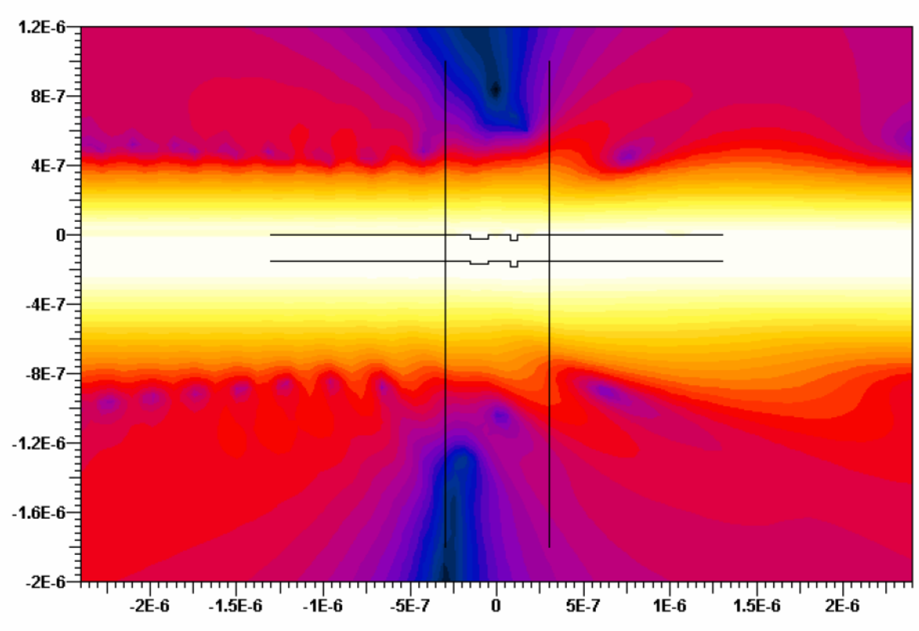
4 Optical Signal Processing and Mode-coupling

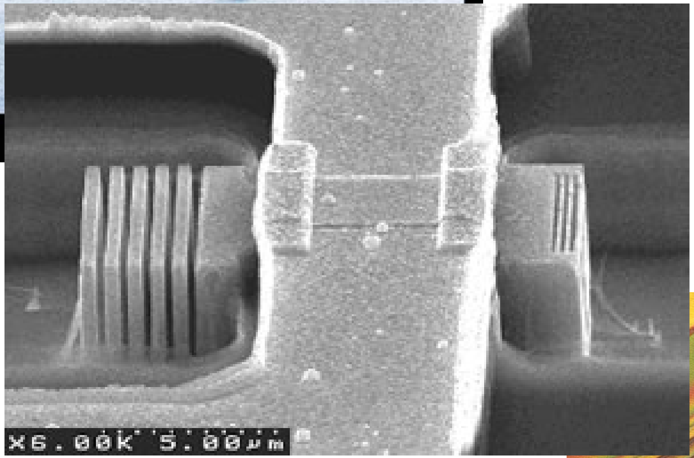
19/02/2010



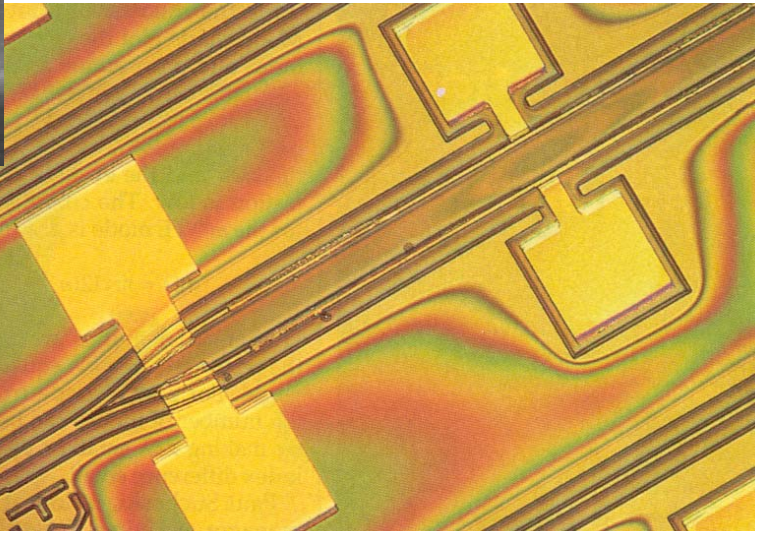
Ridge waveguide
with air-semiconductor
DFB-structure



Simulation of scattered
EM-field of a waveguide
with a small geometrical
disturbance



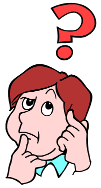
Micro-laser diode with an
air-semiconductor DFB-mirror



Mach-Zehnder
Interferometer modulator
with Y-splitter

4 Optical Signal Processing and Mode-coupling

Goals of the chapter:



- **Theory of waveguide devices for signal processing** (passive manipulation) **of optical waves**

  - filtering, wave splitting, mode-conversion, beam deflection and coupling, mirrors, etc. …
- **Processing requires conversion or coupling of optical modes by controlled passive or active dielectric functional "disturbances" of the WG**

  (Modes without perturbation are orthogonal and can not interact)
- **Mode processing** requires the solution of Maxwell's-equations in complex coupled dielectric structures beyond simple, homogeneous waveguides
- Development of a **perturbation or coupled mode formalism** to describe the interactions between different optical modes and functional dielectric disturbances

Methods for the Solution:

- **Rigorous Solution of Maxwell's equation for coupled dielectric longitudinal, transverse inhomogeneous WGs is difficult ➡ approximate problem as scattering problem in the unperturbed system**
- **Restriction to weak dielectric or geometrical "disturbances " ($\Delta n/n \ll 1$, $\Delta x/\lambda \ll 1$), allows the use of the solutions of the <u>unperturbed</u> system as an approximation of the solution of the perturbed system**
- **Mode Coupling Theory (MCT) describes energy exchange between modes in periodically perturbed structures**
- **Demonstrate important applications of coupled wave devices: WG-couplers and Bragg-Filters**

4. The concept of mode coupling for optical processing of waves

Unperturbed lossless waveguides propagate modes without changing the number, the character and energy of propagating modes because modes are **orthogonal** $\int \vec{E}_i(\vec{r}_T)\bullet\vec{E}_j^*(\vec{r}_T)\,df_T = \Delta_{ij}$ and do not interact (exchange energy).

Functional transverse or longitudinal dielectric disturbances excite new modes in a controlled way by scattering and modify the exciting mode (by reflection, transmission, change of propagation direction, etc.) by a:

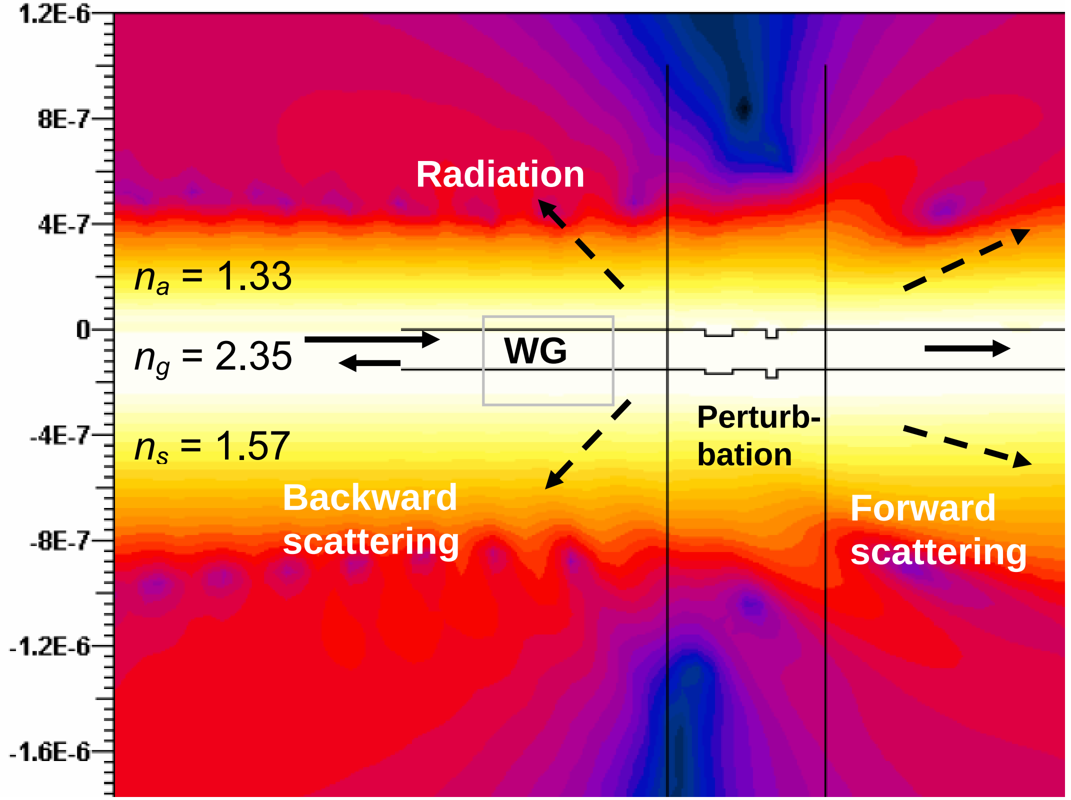
- **change of dielectric properties $\Delta\varepsilon$, Δn**
- **change of geometrical / spatial properties Δd, Δw**

➡ eg. **spatial mode conversion**
(eg. in Y-power splitters)

➡ eg. **frequency selective mode conversion**
(eg. resonances for filters, resonators, etc.)

Concept of controlled coupling:

External RF EM-fields control perturbations $\Delta\varepsilon$, $\Delta\alpha$ by different physical effects leading to modulation of the propagating wave (eg. optical modulator, chap.8):



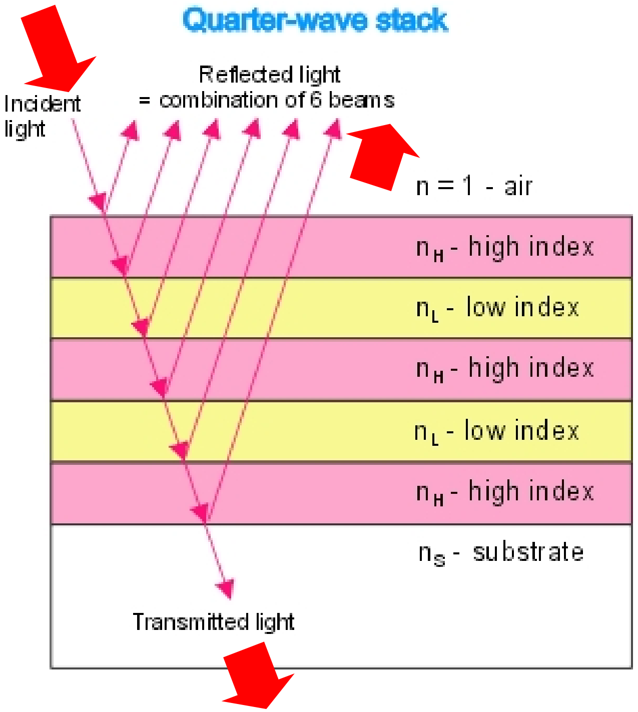
Scattering of an incoming wave by a perturbation of the WG

- electrical field E → Electro-Optic effect $\Delta n(E)$
- electrical field H → Magneto-Optic effect $\Delta n(H)$
- acoustic stress field → Acoustic-Optic effect

- thermal field → Thermo-Optic effect $\Delta n(T)$
- current injection → Plasma effect $\Delta n(n_{carrier})$

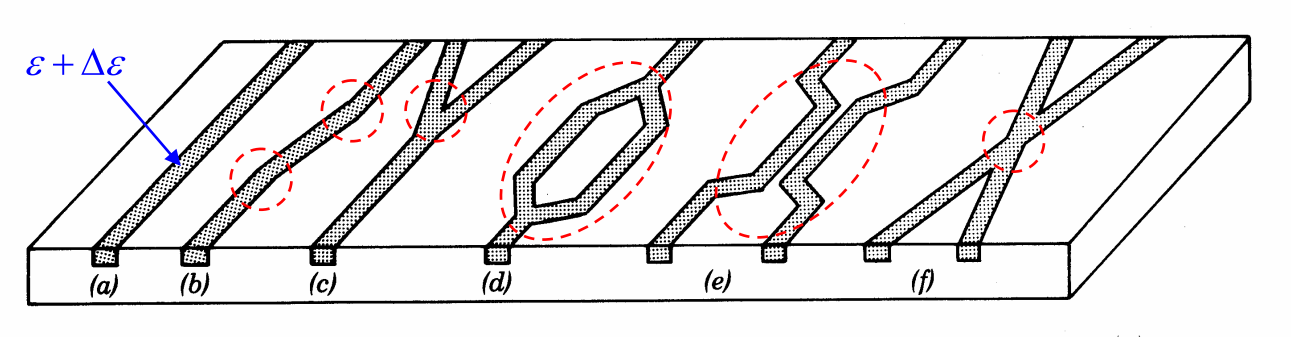
Multilayer grating: example for a fixed scattering / mode coupling process:

- **longitudinal perturbation (coupling)**, no transverse perturbation
- dielectric interfaces n_H-n_L act as disturbance
  (scattering: reflection and transmission)
- forward- and backward scattered partial waves are phase-coherent and modify the exciting wave by interference ➨ **coupling** to the forward or backward propagating wave
- applications: antireflection coatings of surfaces
  filters coatings waveguide filters
  coupling free-space-to-waveguide ….



Passive planar waveguides with local and distributed „perturbations“:

- **tranversal (and longitudinal) perturbation (coupling)**

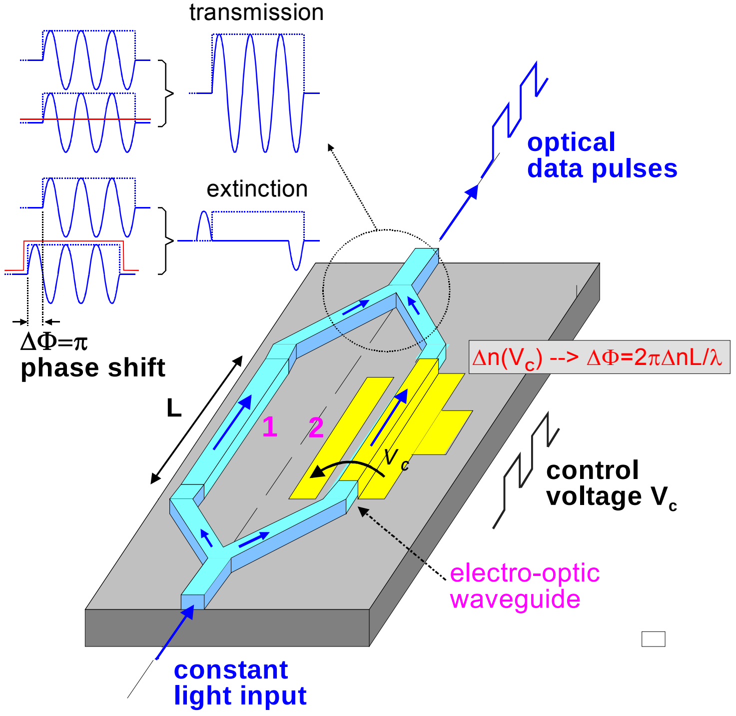


(a) straight waveguide,
(b) waveguide S-bend,
(c) Y-branch, power splitter
(d) Mach-Zehnder-Interferometer
(e) directional coupler
(f) waveguide crossing

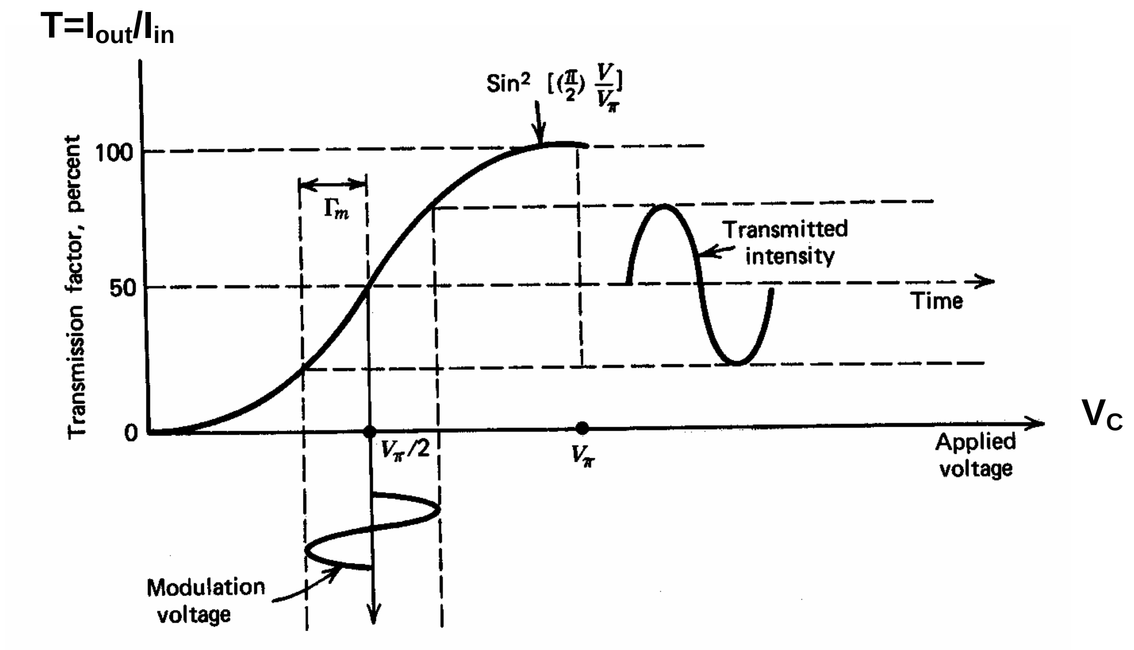
Active Electro-Optic Mach-Zehnder Interferometer (MZI)waveguide Modulator:

The coupling is modulated by an applied external electrical field V_C (see chap.8)

Device structure:



Voltage Controlled Transmission Characteristic:



Operation Principle: $T(V_C)$

- the RF Voltage VC at the electrodes changes the refractive index of the right interferometer branch $\Delta n(E_C)$
- Δn introduces a controlled phase difference $\Delta\Phi$ between the 2 optical waves in the MZI arms
- the combined waves at the output might change from constructive interference (transmission) to destructive interference (no transmission)

Concept of Mode Coupling and Perturbation Calculation:

- Disturbance couples exciting wave to the scattered waves
- **Scattered waves of the perturbed structure** are expanded mathematically by **sums of orthogonal wave solutions of the unperturbed structure** (approximation valid only for weak perturbations)

➡ The solution of the perturbed problem can be expanded by the modes of the unperturbed problem, because these modes form a complete set of basis functions and are orthogonal.

Functions form a complete set if any other function can be expanded by a sum of the functions of the complete set.

- To solve the problem we have to determine the complex amplitudes of the modes of the complete set.

➡ Coupled differential equations for these mode amplitudes can be obtained by the repeated applications of the orthonormality on the MX-equations.

- For mathematical simplicity we consider the field as a scalar, neglecting the vector field continuity requirements at the disturbances

Alternatives: Transmission Matrix-Formalism

Longitudinal perturbations (eg. Bragg-Gratings) can also be described by transmission matrices A of each elementary perturbation and the **total transmission- or reflection-function** is obtained by the **matrix-product of all elementary matrices**.

The method is flexible and applicable for relative strong perturbations, but leads less directly to analytic expressions, potential for numerical methods (see eg. Lit. L. Coldren).

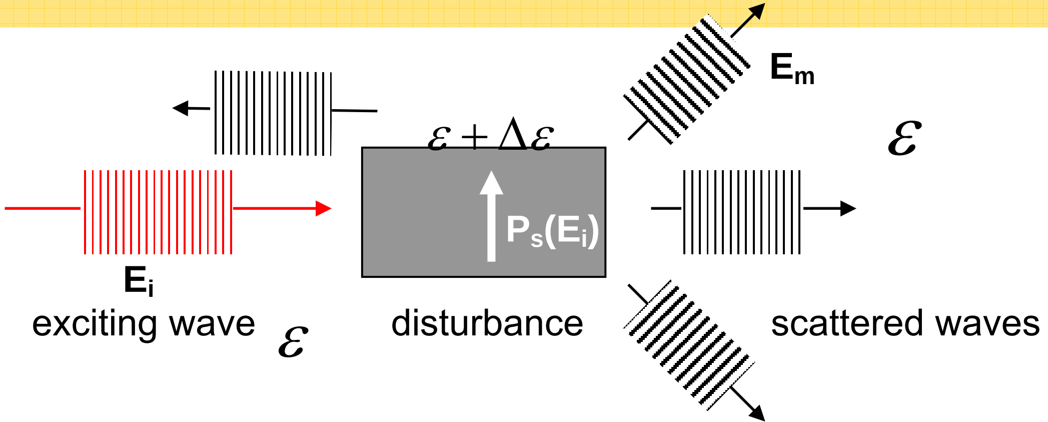
4.1 Theory of Perturbation and Mode-coupling (MC):

1) Assuming a **weak** **(Δε<<ε)** disturbance, we represent the dielectric or geometrical **disturbance** by the addition of a **„disturbance-polarization“ P_s excited by the unperturbed mode E_i**: ➡ **$P_s(E_i, \Delta\varepsilon)$**

2) The excited polarization of the disturbance **P_s creates a complex scattered field** $\sum \vec{E}_m$ which superposes with the exciting field $\vec{E}_i$ to the total field $\vec{E} = \vec{E}_i + \sum_m \vec{E}_m$. **The perturbation Δε couples the modes.**

3) The possible modes E_m are the unperturbed mode of the problem, forming a **complete, orthonormal set** $\langle f_i | f_j \rangle = \delta_{ij}$, used to express the total field of the perturbed structure as $\vec{E} = \vec{E}_i + \sum_m \vec{E}_m$ ($\vec{E}_m$ base- or expansion functions)

4) The total field E fulfills Maxwell’s eq. approximately - the **perturbation polarization P_s**(E_i) **acts as a source**



Limitations of the approximation: Weak Perturbation

The rigorous alternative is solving Maxwell’s-equation exactly for the perturbed problem ($\Delta\varepsilon \neq 0$) – this exact solution might not be well expandable by base-functions of the unperturbed problem ($\Delta\varepsilon = 0$) precisely – therefore we require only **weak perturbations ($\Delta\varepsilon << \varepsilon$)**

Mathematical formulation MC for a transversal 1D-perturbation (scalar field only):

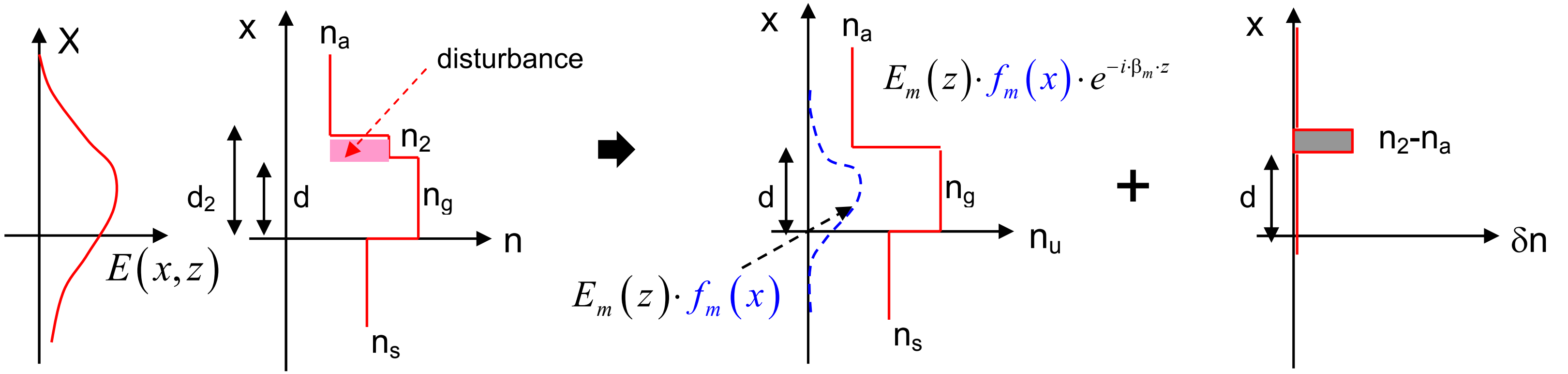
As generic perturbation situation we consider the weak transversal perturbation of a 3-layer WG(n_a, n_g, n_s, d) by an additional 4th layer (n_2, $(d_2\text{-}d)<<d$ ➞ "weak") forming a 4-layer WG.

Solution Idea: the 4th layer WG is a perturbation of a 3-layer WG !

➨ the field in the weakly "perturbed" 4-layer WG can be approximated by unperturbed modes of the 3-layer WG.

Simplification: 1) modes are propagating and scattering only in the z-direction of a planar 3 layer waveguide. Off-axis scattering (transverse directions x, y) is neglected.

2) only time harmonic fields with $f(t)=e^{j\omega t}$



4-layer WG (vertical disturbed 3-layer WG)

3-layer WG (unperturbed)

perturbation (3 layer WG)

Weak perturbation:

n_2-n_a <<n

d_2-d <<d_2

Expansion of the total field E (perturbed):

$$E(x,z,t)=E(x,z)e^{j\omega t}$$

$$E(x,z)=\sum_{\substack{m=\text{all possible}\\ \text{modes of}\\ \text{the problem}}} E_m(z)\cdot f_m(x)\cdot e^{-i\cdot\beta_m\cdot z}$$

🡄 **Unperturbed mode m E_m:**

$f_m(x)$ = transverse mode profile (of eg. the $E_Z(r_T)$-component)

$E_m(z)$ = slowly varying z-dependent field amplitude (envelope) of mode m

$\beta_m(\omega)$= propagation constant of unperturbed mode m

Concept of analysis procedure: what do we want to achieve ?

The 4th layer is the perturbation (addition of the layer d_2-d, n_2-n_a) to the 3-layer structure, which we assume to be known at a frequency ω by it's mode set ($f_m(x)$, β_m).
The modes ($f_m(x)$, b_m) fulfill Maxwell's -, resp. Helmholz equation.

The 4th layer adds of course dielectric constant, resp. additional polarization P_S~(n_2-n_a) driven by the field E.

1) We assume that the unknown field solution E(x,z) of the 4-layer structure is expandable by the complete set of 3-layer modes $E_m(x,z)=E_m(z)fm(x)e^{-j\beta mz}$. $E_m(z)$ takes into account that the amplitude (envelope) of the modes might depend on the the propagation direction z:

$$E(x,z)\simeq\sum_m E_m(z)\cdot f_m(x)\cdot e^{-i\cdot\beta_m\cdot z}$$

2) we use Maxwell's equation in the polarization form, - the perturbing polarization difference (n_2-n_a) of the 4th layer is kept on the right side of the Maxwell's eq. but not the unperturbed 3-layer dielectric structure itself (is lkept on the left side) !!!

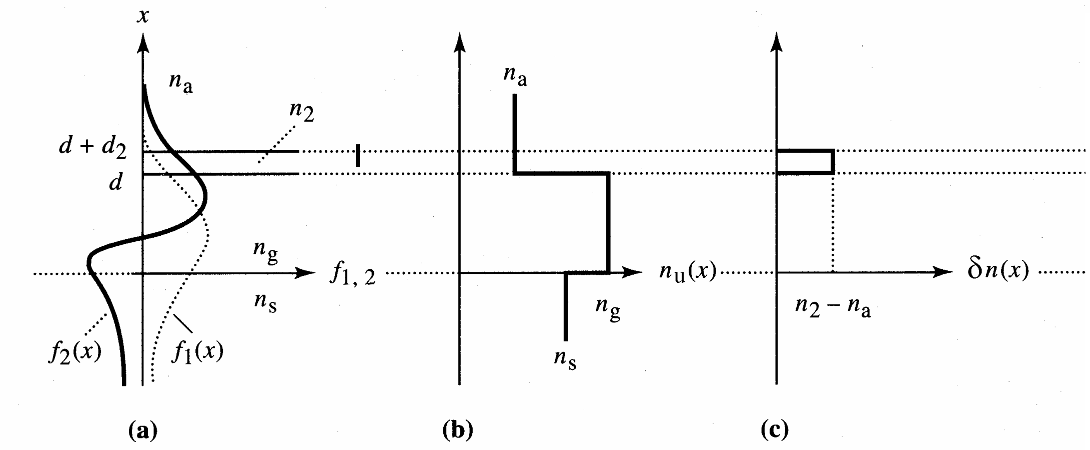
$$\underbrace{\left(\Delta-\mu_0\varepsilon_u\cdot\frac{\partial^2}{\partial t^2}\right)}_{\substack{unperturbed\\ 3\text{–}layer\ \varepsilon_{u,i}\ \ i=a,g,s}}\vec{E} = \underbrace{\mu_0\cdot\frac{\partial^2}{\partial t^2}\vec{P}_s\left(\vec{E}\right)}_{\substack{\text{perturbation term,}\\ \text{4th layer , } \mathrm{n}_2\text{-}\mathrm{n}_\mathrm{a}\to\delta n^2}} \quad\to\quad \left(\Delta+k_0^2 n_u^2\right)\vec{E}(z,x) = -k_0^2\delta n^2\cdot\vec{E}(z,x)$$

3) we insert E(x,z) on both sides of Maxwell's eq. and obtain by using the orthonormality of 3-layer modes and the fact that all 3-layer modes fulfill their Maxwell's eq. the coupled mode differential equation for the field amplitudes of all modes $E_m(z)$:

$$\frac{\partial E_\ell}{\partial z} + i\kappa_{\ell\ell}\cdot E_\ell = -i\cdot\sum_{m\neq\ell}E_m\cdot\kappa_{\ell m}\cdot e^{-i\cdot\delta_{\ell m}\cdot z} \qquad \kappa_{\ell m}=\frac{k_0^2(\omega)}{2\beta_\ell(\omega)}\cdot\langle f_\ell|\delta n^2|f_m\rangle$$

We assume that dispersion $\beta_m(\omega)$ and mode profile $f_m(x)$ for **all** possible modes m for the unperturbed 3-layer WG is known.

Convention: m>0 are **right-propagating** $\beta_m>0$, m<0 are assumed to be **left-propagating** $\beta_{-m}<0$!



modes 1, 2 unperturbed perturbation

Mode coupling $f_2 \rightarrow f_1$ by a perturbed (n_2-n_a)-layer planar film WG:

(a) unperturbed film (thickness d) and core index n_g.

Two unperturbed mode profiles $f_1(x)$ and $f_2(x)$ are considered as an example.

The perturbation is an additional film of thickness d_2 and index n_2-n_a.

(b) unperturbed index profile $n_u(x)$.

(c) perturbation $\delta n(x)=n_2-n_a$.

Orthonormal base of unperturbed 3-layer modes: (without proof)

$$\int_{-\infty}^{+\infty} f_m^*(x) f_n(x) dx = \langle f_m | f_n \rangle = \delta_{nm} \quad \text{with} \quad \delta_{nm} = 1 \text{ for } n = m \quad \text{and } \delta_{nm} = 0 \text{ for } n \neq m$$

Base m: only guided, normalized modes (must be proven!)

Separation of dielectric disturbance: $n=\sqrt{\varepsilon_r}$

refractive index: $n(x) = n_u(x) + \delta n(x)$

dielectric constant: $\varepsilon(x) = \varepsilon_u(x) + \delta\varepsilon(x) = n_u^2(x) + \delta n^2(x)$

unperturbed $n_u(x)$ and perturbation $\delta n(x)$

Observe: $\delta\varepsilon = \delta n^2 = 2 n_u \delta n$!

Key step: represent the perturbation by its polarization

Diel. **perturbation** δn^2 ➡ **creates driven by the exciting field $\vec{E}$ an additional "perturbation polarization" $\vec{P}_s$**

$$\varepsilon\vec{E} = \varepsilon_0\vec{E} + \vec{P}(\vec{E}) \quad (\text{definition})$$

$$\boxed{\vec{P} = \underbrace{\vec{P}_u}_{\substack{\text{unperturbed}\\ n_u}} + \underbrace{\vec{P}_s}_{\substack{\text{perturbation}\\ \delta n^2 \equiv (n_2, d_2)}}}$$

(decomposition)

Express disturbance by perturbation δn^2 ➞ Disturbance Polarization P_s:

$$\vec{D} = \varepsilon\vec{E} = \varepsilon_0\vec{E} + \vec{P}(\vec{E}) = \varepsilon_0\vec{E} + \vec{P}_u + \vec{P}_s = \varepsilon_0\left(n_u^2 + \delta n^2\right)\vec{E}$$

$$\rightarrow \vec{P}_u = \varepsilon_0\left(n_u^2 - 1\right)\vec{E} \quad \text{unperturbed}$$

$$\rightarrow \vec{P}_s = \varepsilon_0\left(\delta n^2\right)\vec{E} \quad \text{perturbation}$$

Inserting the assumed total polarization $\vec{P} = \vec{P}_u + \vec{P}_s$ into Maxwell's equations we get the for the total field E:

Inhomogenous Helmholtz equation

$$\underbrace{\left(\Delta - \mu_0\varepsilon_u \cdot \frac{\partial^2}{\partial t^2}\right)}_{unperturbed}\vec{E} = \underbrace{\mu_0 \cdot \frac{\partial^2}{\partial t^2}\vec{P}_s\left(\vec{E}\right)}_{\substack{\text{excitation term,}\\ \text{(pertrurbation term)}}}$$

For harmonic fields: $\frac{\partial}{\partial t} = j\omega$ and with the linear perturbation – polarization $\vec{P}_s = \varepsilon_0\delta n^2\vec{E}$ (separation of space z and time t)

$$\left(\Delta + \omega^2\mu_0\varepsilon_u\right)\vec{E}(z,x) = \left(\Delta + k_0^2 n_u^2\right)\vec{E}(z,x) = -\omega^2\mu_0 \cdot \vec{P}_s(z,x) \qquad \text{(2D: x,z)}$$

Using $\omega^2\mu_0\varepsilon_0 n^2 = k_0^2 n^2$ *for x-dependent* $n_u(x)$ *and* $\delta n^2(x\pi)$ →

$$\boxed{\left(\underbrace{\Delta}_{a)} + \underbrace{k_0^2 n_u^2}_{b)}\right)\vec{E}(z,x) = -\underbrace{k_0^2\delta n^2}_{c)}\cdot\vec{E}(z,x)}$$

inhomogenous Helmholtz equation (with disturbance δn^2)

Assuming that the disturbance (c) is small and that we have <u>analyzed the unperturbed</u> (without corrugation $\delta n^2=0$) <u>system for all</u> $f_m(x,\omega)$ <u>and</u> $\beta_m(\omega)$, we express the perturbed field by <u>**a sum**</u> of unperturbed mode fields

Insertion of the „Ansatz“ of the total “right propagating m>0” perturbed field (x-z-separation)

$$E(x,z) = \sum_m E_m(z)\cdot f_m(x)\cdot e^{-i\cdot\beta_m\cdot z}$$

(expansion by orthonormal unperturbed modes, $E_m(z)$ is the field amplitude at z of mode m)

1) Determination of the 2D-Laplace-operator $\Delta E(x,z)$; $\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}\right)$

a) by insertion of E(x,z)

$$\Delta E = \Delta\left\{\sum_m E_m(z)\cdot f_m(x)\cdot e^{-i\cdot\beta_m\cdot z}\right\} = \sum_m \Delta\left\{E_m(z)\cdot f_m(x)\cdot e^{-i\cdot\beta_m\cdot z}\right\} =$$

$$= \sum_m\left\{E_m\frac{\partial^2}{\partial x^2}f_m + f_m\cdot\left[\left(\cancel{\frac{\partial^2}{\partial z^2}} - \beta_m^2\right)E_m - 2i\beta_m\frac{\partial}{\partial z}E_m\right]\right\}\cdot e^{-i\cdot\beta_m\cdot z}$$

the 1.order differential term $\frac{\partial}{\partial z}$ remains !

with the <u>weak disturbance assumption</u>: $\delta n^2 << n_u^2$ the amplitude E(z) varies very slowly $\partial^2/\partial z^2 << |\beta_m^2| \to 0$:

$$\Delta E \simeq \sum_m\left\{E_m\frac{\partial^2}{\partial x^2}f_m + f_m\cdot\left[-\beta_m^2 E_m - 2i\beta_m\frac{\partial}{\partial z}E_m\right]\right\}\cdot e^{-i\cdot\beta_m\cdot z} \qquad (4.10).$$

b) $$k_0^2 n_u^2 \cdot E \;=\; k_0^2 n_u^2 \cdot \sum_m E_m \cdot f_m \cdot e^{-i\cdot\beta_m\cdot z}$$

c) $$-\,k_0^2 \delta n^2 \cdot E \;=\; -\,k_0^2 \delta n^2 \cdot \sum_m E_m \cdot f_m \cdot e^{-i\cdot\beta_m\cdot z}$$ and

by using the **homogeneous Helmholtz-equation** for the unperturbed ($\delta n^2=0$) mode m:

$$\left[\Delta-\mu_0\varepsilon_u\frac{\partial^2}{\partial t^2}\right]E_m(x,z,t)=\left[\Delta-\mu_0\varepsilon_u\frac{\partial^2}{\partial t^2}\right]E_m(z)f_m(x)e^{-\beta_m z}e^{-j\omega t} \;\Rightarrow\; \text{for all unperturbed modes m}: \underbrace{\left[\frac{\partial^2}{\partial x^2}+\left(k_0^2 n_u^2-\beta_m^2\right)\right]}_{=0} f_m(x)=0$$

we eliminate several terms from the **inhomogeneous** Helmholtz-eq. and get:

$$\left(\Delta+k_0^2 n_u^2\right)\vec{E} \;=\; \sum_m\left\{E_m\cdot\cancel{\left[\frac{\partial^2}{\partial x^2}+\left(k_0^2 n_u^2-\beta_m^2\right)\right]}f_m-2i\beta_m\frac{\partial}{\partial z}E_m\cdot f_m\right\}\cdot e^{-i\cdot\beta_m\cdot z} \;=\; \underbrace{-\,k_0^2\delta n^2\cdot\sum_m E_m\cdot f_m\cdot e^{-i\cdot\beta_m\cdot z}}_{perturbation} \quad \Rightarrow$$

$$2i\cdot\sum_m\left\{\beta_m\cdot\frac{\partial}{\partial z}E_m(z)\cdot f_m(x)\right\}\cdot e^{-i\cdot\beta_m\cdot z} \;=\; k_0^2\delta n^2(x)\cdot\sum_m E_m(z)\cdot f_m(x)\cdot e^{-i\cdot\beta_m\cdot z}$$

this equation depends on x by $f_m(x)$

2) remove the x-dependence and **isolate a** $\partial E_l/\partial z$**-term** by making use of the orthonormality of the modes by:

a) right-multiplying the equation by $f_\ell(x)^*$ and

b) subsequent integration in the transverse x-direction $\int dx$ using the ortho-normality of the solution-base $\left\langle f_m(x)\middle|f_l(x)\right\rangle=\delta_{ml}$.

$\delta n^2(x)$ may be a function of x (transverse coupling) and/or z (longitudinal coupling):

$$\boxed{\frac{\partial}{\partial z}E_\ell \;=\; \underbrace{E_\ell\cdot\frac{k_0^2}{2i\beta_\ell}\cdot\int_S f_\ell^*(x)\delta n^2(x)f_\ell(x)\,dx}_{self\ coupling\ \kappa_{ll}\quad l\to l} \;+\; \underbrace{\sum_{m\neq\ell}E_m\cdot\frac{k_0^2}{2i\beta_\ell}\cdot\int_S f_\ell^*(x)\delta n^2(x)f_m(x)\,dx\cdot e^{-i\cdot(\beta_m-\beta_\ell)\cdot z}}_{mutual\ coupling\ \kappa_{lm}\quad l\to m}}$$

with: $\kappa_{\ell m} = \frac{k_0^2}{2\beta_\ell}\cdot\int_S f_\ell^*(x)\cdot\delta n^2(x)\cdot f_m(x)\,dx = \frac{k_0^2}{2\beta_\ell}\cdot\langle f_\ell|\delta n^2|f_m\rangle$ as weighted transverse overlap integral

$$\boxed{\frac{\partial E_\ell}{\partial z} + i\kappa_{\ell\ell}\cdot E_\ell = -i\cdot\sum_{m\neq\ell} E_m\cdot\kappa_{\ell m}\cdot e^{-i\cdot\delta_{\ell m}\cdot z}}$$

Mode coupling equation for E(z) (system of linear coupled diff.eq.)

l= 1, 2, 3 .....

with the **mutual coupling constant l→m**: $\kappa_{\ell m} = \frac{k_0^2}{2\beta_\ell}\cdot\int_S f_\ell^*(x)\cdot\delta n^2(x)\cdot f_m(x)\,dx = \frac{k_0^2}{2\beta_\ell}\cdot\langle f_\ell|\delta n^2|f_m\rangle = \kappa_{\underbrace{\ell}_{to}\ \underbrace{m}_{from}}$ and

with the detuning: $\delta_{\ell m} = \beta_m - \beta_\ell =$ (difference of propagations constants)

Interpretation:

The coupling constants κ_{lm} describes the z-dependent variation of mode l caused by mode m (energy transfer l → m)

The coupling constants κ_{lm} is the overlap-integral of mode l and m weighted by the x-dependent perturbation $\delta n^2(x)$

$$\boxed{\kappa_{\ell m} = \frac{k_0^2}{2\beta_\ell}\cdot\int_S f_\ell^*(x)\cdot\delta n_{(m)}^2(x)\cdot f_m(x)\,dx = \frac{k_0^2}{2\beta_\ell}\cdot\langle f_\ell|\delta n^2|f_m\rangle}$$

no function of z (only transverse coupling)

κ_{lm} measures the excitation of mode l by the evanescent field of mode m (=overlap integral weighted by disturbance)

κ_{ll} **self-coupling** measures the influence of the disturbance on the exciting mode l (slight modulation of $E_l(z)$)

The **phase-function** $e^{-i\cdot\delta_{\ell m}\cdot z}$ in the mode coupling eq. describes the z-dependent **spatial phase difference between the modes**, resp. difference of the propagation constants of mode l and m. (eg. inphase – anti-phase coupling)

$$\boxed{\delta_{\ell m}(\omega) = \beta_m(\omega) - \beta_\ell(\omega)}$$

phase-difference (of the unperturbed modes)

Interpretation:

δ_{lm} measures the difference of the phase-velocities (phase changes) of the interacting modes l and m.

The mode coupling equations describes the change per unit length of the z-dependent field E(z) of mode l due to the interaction (scattering) to/from all modes. E(z) has the character of an amplitude-modulated envelop.

Mode l and m couple only efficiently if the phase function does not oscillate fast over the interaction length – otherwise the distributed coupling contribution cancel each other and are integrated out. For strong coupling $\delta_{lm} \to 0$.

The role of Self-Coupling: κ_{ll}

κ_{ll} describes the “self”-modification of the exciting mode l due to the dielectric perturbation.

So it is useful to consider self-coupling and its solution alone to simplify the mode-coupling equation afterwards.

For analysis purpose consider the hypothetical situation $\kappa_{lm}=0$ for $l \neq m$ (only self-coupling $\kappa_{ll} \neq 0$):

$$\frac{\partial E_\ell}{\partial z} + i\kappa_{\ell\ell} \cdot E_\ell = 0$$

(eq. contains no phase factor)

This homogeneous MC-equation has a simple **exponential solution** for the E(z)-envelope by an exponential:

$$E_\ell(z) = A_\ell \cdot e^{-i\cdot\kappa_{\ell\ell}\cdot z},$$

(A_l=const.) resp. for the total propagating field $e^{-j\beta z}$:

$$E(x,z) = A_\ell \cdot f_\ell(x) \cdot e^{-i\cdot(\beta_\ell+\kappa_{\ell\ell})\cdot z}$$

A_l= Amplitude value of mode l

Modification of mode propagation constant by **self-perturbation** of mode l:

$\beta_\ell' = \beta_\ell + \kappa_{\ell\ell}$ (perturbed propagation constant)

➨ **The dielectric disturbance modifies the effective propagation constant of the original mode l $\beta_\ell \to \beta_\ell'$ but leaves the mode energy constant.**

Generalization for a generic solution:

For the general case with mutual- and self-mode coupling we may assume a solution of the previous form for all the self-coupled coupled modes:

Definition of $A_l(z)$

$$\boxed{E_\ell(z) = A_\ell(z)\cdot e^{-i\cdot\kappa_{\ell\ell}\cdot z}}$$

resp. $E_l(x,z,t) = A_\ell(z)\cdot f_\ell(x)\cdot e^{-i\cdot\{(\beta_\ell+\kappa_{\ell\ell})\cdot z-\omega t\}}$ (with a slowly varying amplitude *$A_l(z)$*)

Inserting the assumed solution into the general MC-coupling equation we obtain for the spatial field amplitude $A_l(z)$ the simplified MC-differential equation:

(system of coupled linear diff.eq. for field amplitudes)

$$\boxed{\frac{\partial A_\ell}{\partial z} = -i\cdot\sum_{m\neq\ell} A_m\cdot\kappa_{\ell m}\cdot e^{-i\cdot\delta'_{\ell m}\cdot z}}$$

modified mode coupling equation for $A_l(t)$

l= 1, 2, 3 .....

coupling detuning at position z

and for the ***modified phase difference $\delta'_{\ell m}$*** we have (all modes are characterized by their perturbed propagation constant β_ℓ')

$$\boxed{\delta'_{\ell m} = \beta'_m - \beta'_\ell = \beta_m + \kappa_{mm} - (\beta_\ell + \kappa_{\ell\ell}) = \delta_{\ell m} + (\kappa_{mm} - \kappa_{\ell\ell})}$$

modified phase deviation

Summarizing the formal procedure for the solution of mode coupling:

- Unperturbed structure: determine the Eigenfunctions $f_m(x)$ and the Eigenvalues β_m, characterized by the unperturbed profile of the refractive index $n_u(x)$.
- Considering now the perturbation $\delta n^2(x)$: Calculation of the coupling constants $\kappa_{\ell m}$, the self-coupling constant $\kappa_{\ell \ell}$ (4.16), the modified propagation constant β'_ℓ , the phase deviations $\delta_{\ell m}$ (4.17) resp. the modified phase deviation $\delta'_{\ell m}$ (4.23).
- Solve the system of differential equations of the coupled modes using the direct or the modified mode coupling equations with the corresponding boundary and/or initial conditions.

Concept of analysis procedure: what do we want to achieve ?

For the following directional coupler we consider the coupling between to adjacent WG where the modes overlap and therefore couple.

The system contains only 2 identical fundamental modes by design.

In the coupling integral the adjacent waveguide acts as the perturbation and the modes are spatially separated in the 2 WGs.

The MC-equation becomes a simple system of two coupled differential equations, which can be solved analytically.

4.2 Codirectional mode coupling — the directional coupler

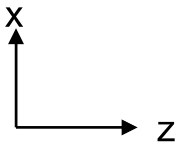
Codirectional couplers consist of two closely spaced, homogenous in z-direction, single mode waveguides, which are so close that the **transverse evanescent mode fields couple (overlap)**.

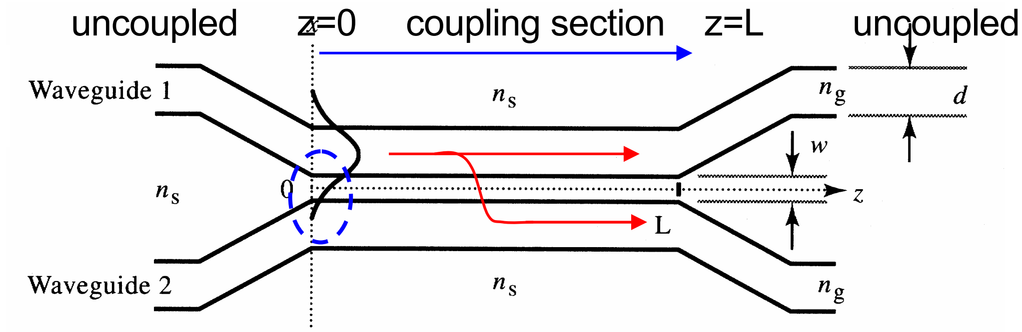
WG width and separation distance are d, resp. w. Both WGs are assumed identical $\beta_1=\beta_2$ und fundamental mode.

➡ **WG2 (1) is a perturbation to WG1 (2) and vice versa**.

input: uncoupled z=0 coupling section z=L uncoupled **output:**

Excitation port 1 ➡



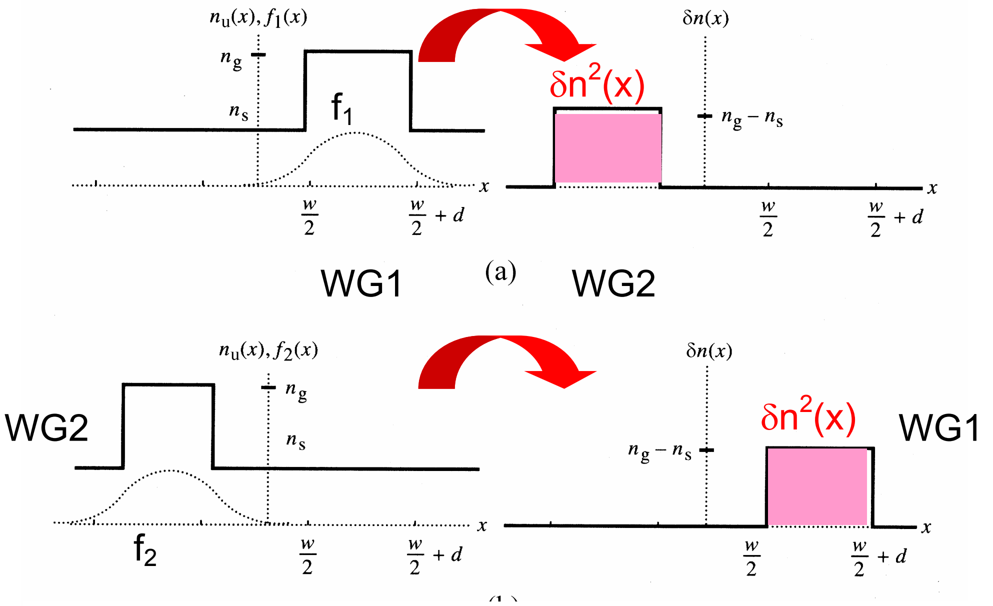


n_g= “core” index

n_s= “cladding” index

Considering WG2 as the disturbance for WG1 and vice versa WG1 is the disturbance of WG2:

The system contains only 2 identical right propagating modes 1, 2.



Coupling 1 ➡ 2

Coupling 2 ➡ 1

unperturbed perturbation

a) Analysis of the symmetric **($\beta_1=\beta_2$, $\kappa_{12}=\kappa_{21}^*$ → $\delta'_{12}=0$ no detuning)** directional coupler (DC)

We assume that in the directional coupler only two codirectionally propagating modes exist (single mode waveguide) (WG1 and WG2 are assumed to be identical for simplicity).

$$\boxed{\begin{aligned}\frac{\partial A_1}{\partial z} &= -i\kappa_{12}\cdot A_2\\ \frac{\partial A_2}{\partial z} &= -i\kappa_{21}\cdot A_1\end{aligned}}$$

modified coupled mode (MCM) equation for only 2 modes amplitudes $A_1(z)$ and $A_2(z)$

From the symmetry ($f_1(x)=f_2(x)$) of the 2 waveguides follows $\underline{\kappa_{12}=\kappa_{21}^*=\kappa=\text{real}}$

Differentiating one of the above equation and inserting into the other one leads to:

$$\left(\frac{\partial^2}{\partial z^2}+\kappa^2\right)A_l = 0 \quad ; \quad l=1,2$$

with $\kappa_{12} = \frac{k_0^2}{2\beta}\cdot\int_S f_1^*(x)\cdot\delta n_{(2)}^2(x)\cdot f_2(x)\,dx = \frac{k_0^2}{2\beta}\cdot\langle f_1|\delta n^2|f_2\rangle=\kappa_{21}^*$

With the condition that the total power must be preserved in both WGs:

$$\sum_{\ell=1}^{2}|A_\ell|^2=\sum_{\ell=1}^{2}A_\ell A_\ell^* = \text{constant} \quad \rightarrow \quad \frac{\partial}{\partial z}\sum_{\ell=1}^{2}|A_\ell|^2=\sum_{\ell=1}^{2}A_\ell\frac{\partial}{\partial z}A_\ell^*+A_\ell^*\frac{\partial}{\partial z}A_\ell=-i\cdot\sum_{m\neq\ell}^{2}\left(\kappa_{\ell m}-\kappa_{m\ell}^*\right)\cdot A_m A_\ell^* = 0$$

We obtain as solution of the MCM-eq. for $A_1(z)$ and $A_2(z)$ harmonic functions (sin, cos(κz)): ➡

$$\begin{aligned}A_1(z) &= a\cdot sin(\kappa\cdot z)+b\cdot cos(\kappa\cdot z)\\ A_2(z) &= c\cdot sin(\kappa\cdot z)+d\cdot cos(\kappa\cdot z)\end{aligned}$$

The boundary conditions (eg. input 1 with intensity I_1, input 2 I_2=0) of the DC define the unknown constants a, b, c, d:

$$A_1(0) = \sqrt{I_1} = b \quad \text{input}$$
$$A_2(0) = 0=d \quad \text{unidirectionality}$$

➡ d=0

and from the mode-coupling eq. $A_1(0) = -\frac{1}{i\kappa}\cdot\frac{\partial A_2}{\partial z} = -\frac{c}{i} = b \quad ; \quad A_2(0) = -\frac{1}{i\kappa}\cdot\frac{\partial A_1}{\partial z} = -\frac{a}{i} = 0$ follows

Solution of the mode coupling equation:

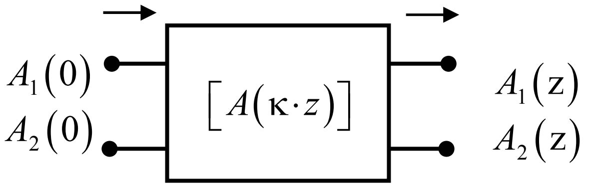
$c = -i\cdot b$ and $a = 0$ leads to

$$A_1(z) = \cancel{-i\,A_2(0)\cdot sin(\kappa\cdot z)} + A_1(0)\cdot cos(\kappa\cdot z) = \sqrt{I_1}\cdot cos(\kappa\cdot z)$$

$$A_2(z) = -i\,A_1(0)\cdot sin(\kappa\cdot z) + \cancel{A_2(0)\cdot cos(\kappa\cdot z)} = -i\sqrt{I_1}\cdot sin(\kappa\cdot z)$$

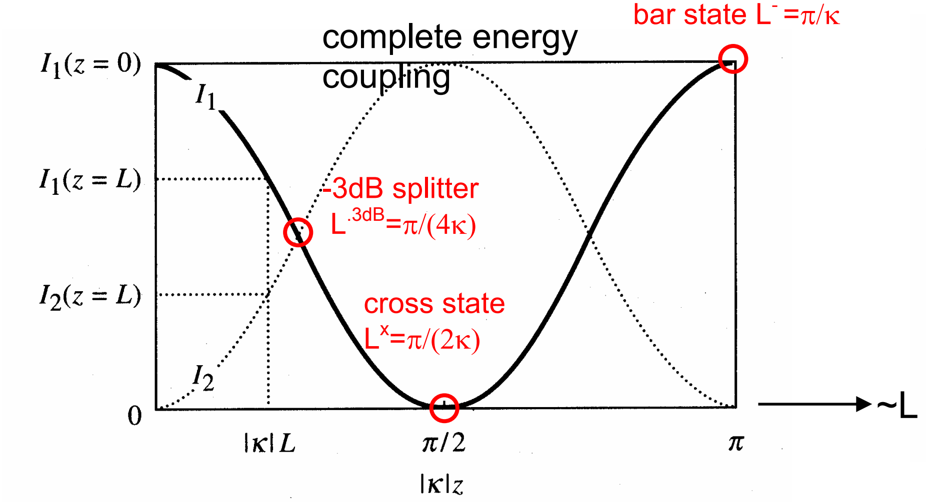
In A-matrix-form for the general situation I_1, $I_2 \neq 0$:

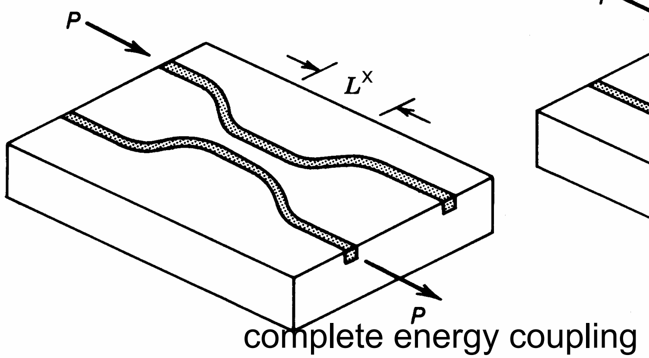
$$\boxed{\begin{bmatrix} A_1(z) \\ A_2(z) \end{bmatrix} = \begin{bmatrix} cos(\kappa\cdot z) & -i\cdot sin(\kappa\cdot z) \\ -i\cdot sin(\kappa\cdot z) & cos(\kappa\cdot z) \end{bmatrix}\cdot\begin{bmatrix} A_1(0) \\ A_2(0) \end{bmatrix} \qquad \begin{bmatrix} A_1(z) \\ A_2(z) \end{bmatrix} = \left[A(\kappa\cdot z)\right]\cdot\begin{bmatrix} A_1(0) \\ A_2(0) \end{bmatrix}}$$



The intensity distribution is calculated from $|A_i(z)|^2 \propto I_i(z)$: (Transfer characteristic)

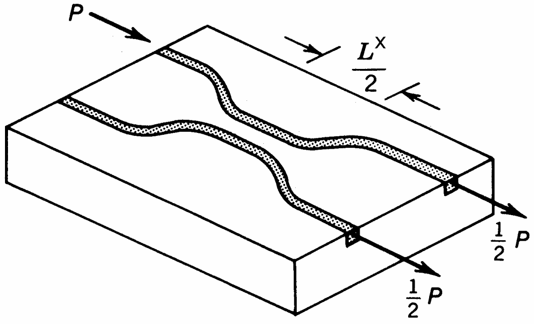
$I_1(z) = I_1\cdot cos^2(\kappa z)$ and $I_2(z) = I_1\cdot sin^2(\kappa z)$ For **complete** power transfer: $\kappa z = \pi/2$ and $L^x = \pi/(2\kappa)$





complete energy coupling

cross state



-3dB Power splitter

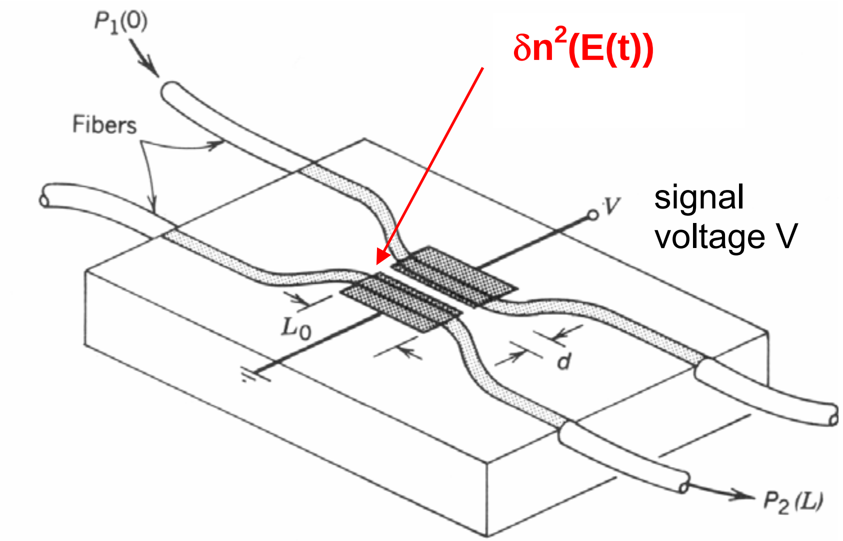
Choosing the device-length z=L allows to functionalize the directional coupler at a particular frequency ω (κ=κ(ω) !).

Summary:

- In symmetric codirection couplers the field energy oscillates back and forth completely (!) between the two waveguides with the coupling length L=π/κ only if $\delta'_{\ell m} = 0$ (no phase difference).
- The coupling length L=π/κ₁₂ and the couple constant $\kappa_{12} = \frac{k_0^2}{2\beta}\cdot\int_S f_1^*(x)\cdot\delta n_{(2)}^2(x)\cdot f_2(x)\,dx = \frac{k_0^2}{2\beta}\cdot\langle f_1|\delta n^2|f_2\rangle$ can be modified by changing the refractive index profile of the coupler n(x,C) by an external control mechanism C. C can be an electric or magnetic field, a thermal field, a stress-field etc.

  This allows to control the power in one WG or switch the light field between the two outputs of the coupler resulting in an optical modulator, see chap.8.
- The MC-theory in this form is only valid for weak perturbations which do not modify the mode pattern strongly. (applicability of the unperturbed mode solutions as a base for expansion)

Schematic of an Electro-Optic (EO) Modulator:



complete energy coupling

The electro-optic effect induced by the electrical field E~V/d modifies **δn²(E)**, resp. the coupling constant **κ(E)** between the 2 WGs.

The modulated coupling modifies the power ratio at the WG output → Electro-optic modulator (switching)

b) The asymmetric ($\beta_1 \neq \beta_2$, $\kappa_{12} \neq \kappa_{21}^* \rightarrow \delta'_{12} \neq 0$ detuned) directional coupler: (self-study)

The previous analysis can be generalized to the **asymmetric directional coupler**, where the two WGs are **different**.

For the lossless asymmetric coupler the waves propagate at different velocities and are detuned.

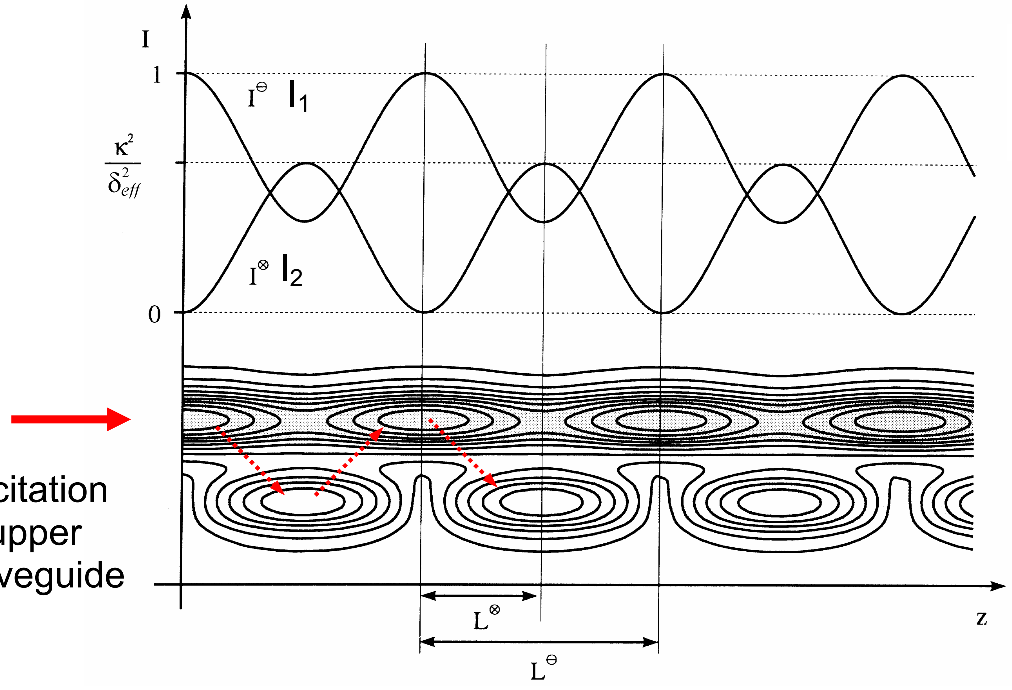
$\beta_1 \neq \beta_2$ ➡ $\delta'_{12} = -\delta'_{21} = \delta' \neq 0$ *detuning*

This leads to the general modified mode coupling equation for two modes:

$$\boxed{\begin{aligned} \frac{\partial A_1}{\partial z} &= -i\kappa_{12} \cdot A_2 \cdot e^{-i \cdot \delta' \cdot z} \\ \frac{\partial A_2}{\partial z} &= -i\kappa_{21} \cdot A_1 \cdot e^{i \cdot \delta' \cdot z} \end{aligned}}$$

by decoupling the eq. $\left(\frac{\partial^2}{\partial z^2} \pm i\,\delta' \cdot \frac{\partial}{\partial z} + \kappa^2 \right) A_{1,2} = 0$ with $\kappa = \sqrt{\kappa_{12}\kappa_{21}}$ (A_1: + sign, A_2: - sign)

Result: Incomplete coupling between the asymmetric waveguides



Excitation in upper waveguide

Intensity distribution in an asymmetric directional coupler

➡ Incomplete coupling in asymmetric directional couplers (eg. fabrication tolerances)

→ Bar state

⋯→ Cross state

Derivation of coupling transfer matrix of the asymmetric codirectional coupler (selfstudy):

We want to find a solution to the equations:

$$\boxed{\begin{aligned}\frac{\partial A_1}{\partial z} &= -i\kappa_{12}\cdot A_2\cdot e^{-i\cdot\delta'\cdot z}\\ \frac{\partial A_2}{\partial z} &= -i\kappa_{21}\cdot A_1\cdot e^{i\cdot\delta'\cdot z}\end{aligned}}$$

We are using a new definition of an effective phase difference: $\delta_{eff} = \sqrt{\delta'^2 + 4\cdot\kappa^2}/2$

and the solution-„Ansatz“ for $A_{1,2}(z) \propto e^{qz}$ or e^{-qz} ; q=propagation constant of the envelop

Inserting $A_{1,2}$ into the MC-equation leads to the 2.order characteristic equation for the propagation constant q:

$q^2 \pm i\cdot\delta' q + \kappa^2 = 0$ ➡ 2 solutions: $q(\delta',\kappa) = q_1, q_2$

$$q_{1,2} = \mp\delta'/2 \pm i\sqrt{\delta'^2 + 4\kappa^2}/2 = \mp\delta'/2 \pm i\delta_{eff}$$

and finally finding for the **solutions for $A_1(z)$ and $A_2(z)$**: (without details)

$$A_1(z) = \left\{a\cdot sin\left(\delta_{eff}z\right) + b\cdot cos\left(\delta_{eff}z\right)\right\}\cdot e^{-i\cdot\frac{\delta'}{2}\cdot z}$$

$$A_2(z) = \left\{c\cdot sin\left(\delta_{eff}z\right) + d\cdot cos\left(\delta_{eff}z\right)\right\}\cdot e^{+i\cdot\frac{\delta'}{2}\cdot z}$$

◯ remark: additional phase terms compared to the symmetric case

Definition of two possible excitation conditions at z=0:

Excitation of WG1:

$$i)\begin{cases}A_1(0) = \sqrt{I_1} = b\\ A_2(0) = 0\end{cases}$$

Excitation of WG2:

$$ii)\begin{cases}A_1(0) = 0\\ A_2(0) = \sqrt{I_2} = d\end{cases}$$

2 equations for $a,\ \underbrace{b}_{A_1=0},\ c,\ \underbrace{d}_{A_2=0} \rightarrow a, c$ by elimination from A(z)

Determination of the constants a, b, c, d and matrix representation for the solution:

From the the eq. for Ai(z) and the boundary conditions at z=0, L we get

$$\boldsymbol{A(z) = T\,A(0)}$$

$$\begin{bmatrix} A_1(z) \\ A_2(z) \end{bmatrix} = \begin{bmatrix} T_{11}(z) & T_{12}(z) \\ T_{21}(z) & T_{22}(z) \end{bmatrix} \cdot \begin{bmatrix} A_1(0) \\ A_2(0) \end{bmatrix}$$

$$T_{11}(z) = \left\{ cos\left(\delta_{eff}\, z\right) + \tfrac{i\,\delta'}{2\,\delta_{eff}} \cdot sin\left(\delta_{eff}\, z\right) \right\} \cdot e^{-i\cdot\frac{\delta'}{2}\cdot z}$$

$$T_{12}(z) = -\tfrac{i\,\kappa_{12}}{\delta_{eff}} \cdot sin\left(\delta_{eff}\, z\right) \cdot e^{-i\cdot\frac{\delta'}{2}\cdot z}$$

$$T_{21}(z) = -\tfrac{i\,\kappa_{21}}{\delta_{eff}} \cdot sin\left(\delta_{eff}\, z\right) \cdot e^{i\cdot\frac{\delta'}{2}\cdot z}$$

$$T_{22}(z) = \left\{ cos\left(\delta_{eff}\, z\right) - \tfrac{i\,\delta'}{2\,\delta_{eff}} \cdot sin\left(\delta_{eff}\, z\right) \right\} \cdot e^{i\cdot\frac{\delta'}{2}\cdot z}$$

The matrix T is a 2-port description of the amplitude and phase transfer properties of a section with length z of coupled WGs. T depends on coupling, effective detuning and length L.

Intensity distribution I(x)=IA(x)I^2:

$$\frac{I_2(L)}{I_1(0)} = \left|T_{21}(L)\right|^2 = \left|\frac{\kappa}{\delta_{eff}}\right|^2 \cdot sin^2\left(\delta_{eff}\, L\right)$$

$$\frac{I_1(L)}{I_1(0)} = \left|T_{11}(L)\right|^2 = 1 - \left|\frac{\kappa}{\delta_{eff}}\right|^2 \cdot sin^2\left(\delta_{eff}\, L\right)$$

(can be obtained by just squaring the expressions for A(z)

with the definition from p.18: $\delta_{\text{eff}}(\omega) = \frac{1}{2}\sqrt{\delta'(\omega)^2 + 4\kappa(\omega)^2} = \frac{1}{2}\sqrt{\left\{\beta_1(\omega) - \beta_2(\omega)\right\}^2 + 4\kappa(\omega)^2}$

Interpretation:

δ_{eff} L → coupling period

$\kappa/\ \delta_{\text{eff}}$ → coupling amplitude

Conclusions:

- Intensity transfer in asymmetric directional couplers is **incomplete**
- The maximum transferred intensity is proportional to $(\kappa / \delta_{eff})^2$
- The cross-length $L^{\times} = \pi / (2 \cdot \delta_{eff})$ and the bar-length $L^{-} = \pi / \delta_{eff}$ are shorter than in the symmetric coupler
- The optical frequency ω dependence of the cross-port is a **bandpass-filter** with a $[sin(x) / x]^2$- characteristic

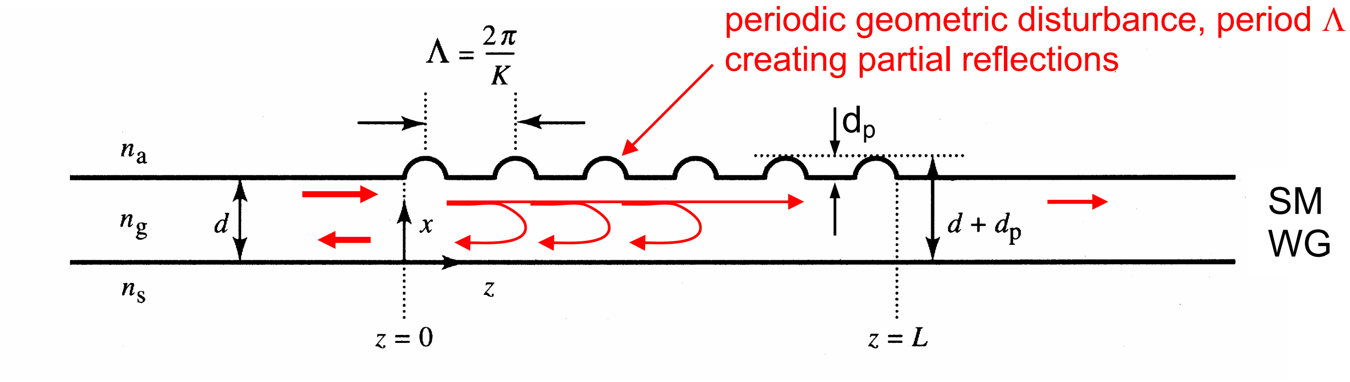
4.3 Contradirectional Mode Coupling – Bragg-Filters and Mirrors

A second very important mode coupling structure is the **longitudinally periodically disturbed waveguide** used for integrated low loss mirrors, narrow band filters, single frequency laser diodes, multi-layer coatings etc..

The coupling is not by a transverse evanescent field in a homogenous (in propagation direction z) disturbance, but a **localized, periodic** (period Λ) **longitudinal disturbance $\delta n^2(z)$** of the WG, creating **multiple, interfering reflections and transmissions**, thus coupling back and forward propagating wave, by

1) periodic variations of the refractive index n of the WG or by
2) periodic variations of the WG geometry (eg. corrugation by variation of thickness d → 2D-problem).

Example of a disturbed (perturbation period Λ, corrugation d_p) 3-layer fundamental mode film-WG consisting of the core n_g with a unperturbed thickness d , the refractive indices of the substrate and cladding are n_s , resp. n_a.



periodic geometric disturbance, period Λ
creating partial reflections

Definitions:

Λ perturbation period

Top cladding: refractive index n_a

Unperturbed core: n_g with thickness d

Bottom cladding (substrate): n_S

Perturbation: d_p or δn_{eff}

Schematic Representation of a waveguide grating (distributed Bragg reflector, DBR)

The z-periodic perturbation can be eg. a transverse geometry or longitudinal index perturbation acting as periodic local reflection centers.

Intuitive picture of the operation principle of the Distributed Bragg Reflector (DBR): Coherent additions of distributed reflections

- for a particular frequency ω_B (Bragg-frequency) the back-reflected wave from each local disturbances add up in phase (coherently) at the input

  ➡ strong reflection , small transmission

- the reflection phaseshift over the distance 2Λ must be a multiple i of 2π for <u>constructive</u> interference at the Bragg-resonance ω_B

  $2\Lambda = i\ \lambda_B/n_{eff} = I\ (2\pi c_0/n_{eff})/\omega_B \quad \rightarrow \underline{\omega_{B,i} = i(\pi c_0/n_{eff}/\Lambda)}$

- for $\omega \neq \omega_B$ the reflections add up out of phase and interfere to zero <u>distructive</u>

  ➡ transmission, small reflections

Possible technical realization of a Bragg-Grating (BG) mirror:

Planar DFB-Laser with built-in waveguide core

Vertical Surface Emitting Laser (VCSEL) with layered BG

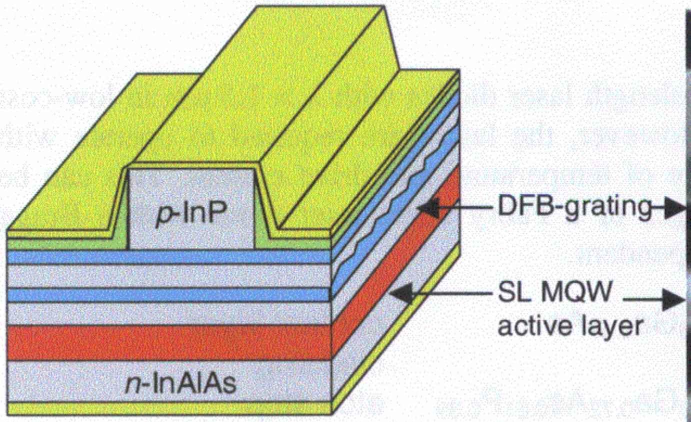


Fig. 2(a): Schematic structure of Distributed feedback laser

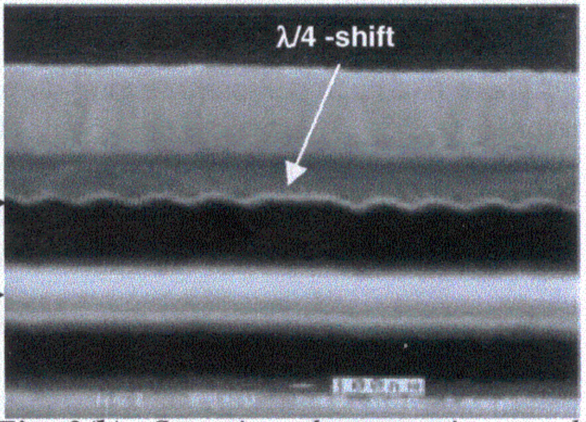
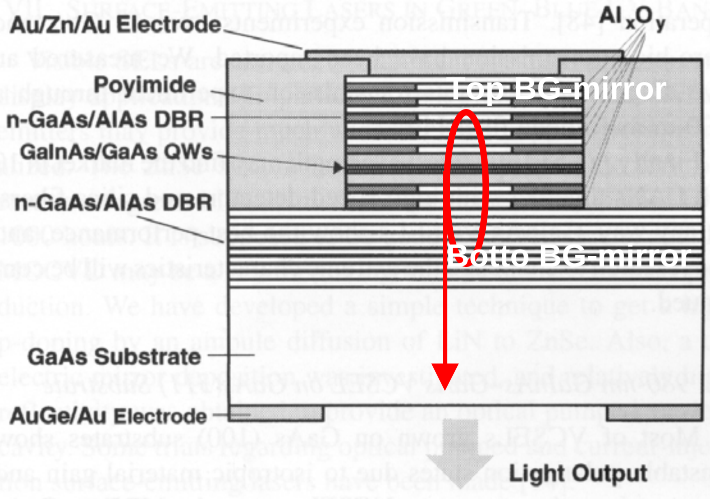


Fig. 2(b): Scanning electron micrograph of a λ/4-shifted corrugation



Concept of analysis procedure: what do we want to achieve ?

1) We modify the perturbation n(x,z) which is now 2-dimensional, transverse and longitudinal for the Bragg-grating contra-directional coupler.

2) We assume again that the WG is fundamental mode, meaning there is only 1 fore-ward and 1 back-ward propagating mode, which couple due to the periodic grating.

3) MC-eq. is similar but contains an additional summation over the spatial harmonics p of the corrugation. The coupling coefficient are also similar but contain the x-dependent Fourier-coefficient of the corrugation $c_p(x)$ instead of the transverse index distribution.

In addition the phase-factor $\delta^p_{\ell m}$ contains the space vector of the grating $K_G = 2\pi/\Lambda$

$$\underbrace{\frac{\partial E_\ell}{\partial z}}_{\text{Coupling effect}} = \underbrace{-i\cdot\sum_p\sum_m E_m\cdot\kappa^p_{\ell m}\cdot e^{-i\cdot\delta^p_{\ell m}\cdot z}}_{\text{coupling of all modes into l including selfcoupling } \ell\to\ell}$$

$$\delta^p_{\ell m} = \beta_m - \beta_\ell - p\cdot K_G = \delta_{\ell m} - p\cdot K_G \quad with \quad \beta_i(\omega) = 2\pi/\lambda_i = \omega n_{eff,i}/c_0$$

The shape of the corrugation determine the spectrum of spatial Fourier-coefficients $c_p(x)$ (eg. higher harmonics of K_G due to sharp features).

Observe the **convention for the direction of mode propagation**:

Right propagating wave: m >0 , β_m>0

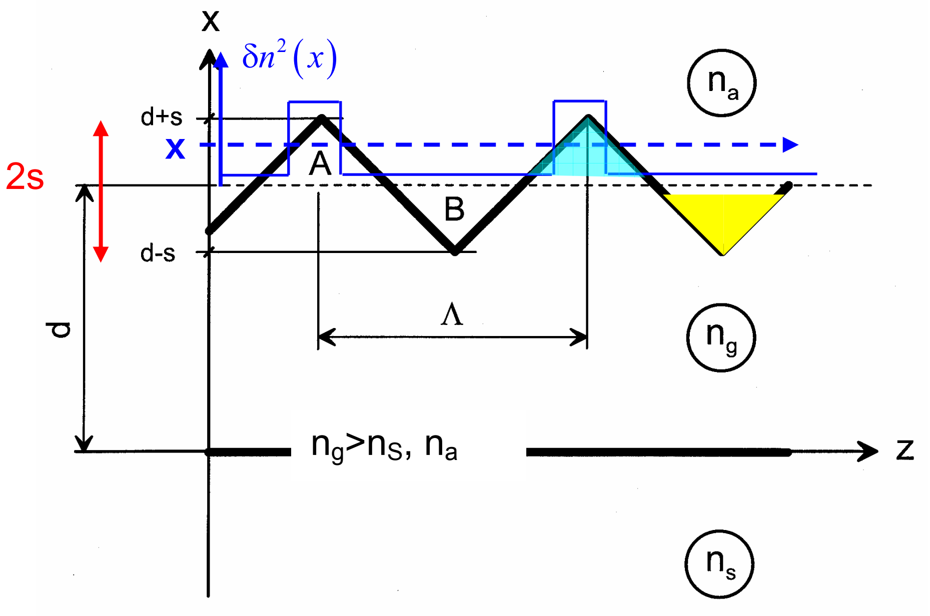
Left propagating wave: –m<0 , β_{-m}<0

p spatial harmonics, p>0, p<0 ???

For efficient coupling the detuning $\delta^p_{\ell m}(\omega)\to 0$ should vanish – observe that $\beta(\omega)$ and $\delta^p_{\ell m}(\omega)$ are frequency dependent and define the frequency dependence of the DBR-transmission/reflection (stop-band characteristic)

2D **(transverse & longitudinal)**-Corrugation Grating Model:

$\delta n^2(x,z)$ is a mix of protrusions A and indentations B depending on x and y:



Disturbance by geometrical dielectric corrugation of the WG interface: (periodic Λ in z- propagation direction**)**

The Index-Profile $\delta n^2(x,z)$ is a rectangular function in z with period Λ, but the pulse width depends on x.

Assumption of weak perturbation: $d >> 2 \cdot s$ and using $\varepsilon_r = n^2$

Rectangular dielectric profile function at x:

$$\delta n^2(x,z) = \begin{cases} n_g^2 - n_a^2 > 0 & \forall (x,z) \in A \\ n_a^2 - n_g^2 < 0 & \forall (x,z) \in B \end{cases}$$

🟦 Index increase

🟨 Index depression

(observe: $\delta n^2(x,z)$ is dependent on x and z)

Method of representation of $\delta n^2(x,z)$: spatial Fourier-transform

- For a given x-coordinate the perturbation $\delta n^2(x,z)$ is a bipolar (increase / decrease) rectangular profile function of z with a period Λ and x-dependent “pulse length”.
- As a simplification we assume that we can decompose $\delta n^2(x,z)$ into a x-dependent spatial Fourier-series along z, meaning that the Fourier-coefficients $c_p(x)$ are x-dependent with respect to a variable duty-cycle.

Spatial **Fourier-Series representation** (z-direction) **of the rectangular** $\delta n^2(x,z)$-function:

$$\underline{\delta n^2(x,z) = \sum_{p=-\infty}^{\infty} c_p(x) \cdot e^{i \cdot p \cdot K_G \cdot z}} \quad and \quad c_p(x) = \frac{1}{\Lambda} \cdot \int_0^{\Lambda} \delta n^2(x,z) \cdot e^{-i \cdot p \cdot K_G \cdot z} dz \;\; and \;\; c_{-p}(x) = c_p{}^*(x) \qquad \forall \; K_G = \frac{2\pi}{\Lambda}$$

Observe: the Fourier coefficient $c_p(x)$ are x-dependent.

Definition : $K_G = 2\pi/\Lambda$ is the spatial wave number of the periodic spatial perturbation (Λ).
p is the number of the spatial grating harmonics (p can be positive or negative).

Mode Coupling Equation:

Each x-dependent spatial Fourier-component $c_p(x)$ of the perturbation acts as a continuous sinusoidal perturbation in the z-direction.

We use the original 2D mode coupling equation (p.4-9 before x-integration) with the perturbation polarization of the corrugation and develop the right hand side scattering term:

$$\left(\underbrace{\Delta}_{a)} + \underbrace{k_0^2 n_u^2}_{b)}\right) \vec{E}(z,x) = - \underbrace{k_0^2 \delta n^2(z,x)}_{c)\ \text{depends also on } z} \cdot \vec{E}(z,x)$$

inserting $\delta n^2(x,z)$: (p. 4-11)

$$2i \cdot \sum_m \left\{ \beta_m \cdot \frac{\partial}{\partial z} E_m \cdot f_m(x) \right\} \cdot e^{-i \cdot \beta_m \cdot z} = k_0^2 \delta n^2(x,z) \cdot \sum_m E_m \cdot f_m(x) \cdot e^{-i \cdot \beta_m \cdot z}$$ and replace $\delta n^2(x,z)$ by its Fourier-series

$$2i \cdot \sum_m \left\{ \beta_m \cdot \frac{\partial}{\partial z} E_m \cdot f_m \right\} \cdot e^{-i \cdot \beta_m \cdot z} = k_0^2 \cdot \left(\sum_p c_p(x) \cdot e^{i \cdot p \cdot K_G \cdot z} \right) \cdot \sum_m E_m \cdot f_m \cdot e^{-i \cdot \beta_m \cdot z}$$

leading to:

phase term

$$2i \cdot \sum_m \left\{ \beta_m \cdot \frac{\partial}{\partial z} E_m \cdot f_m \right\} \cdot e^{-i \cdot \beta_m \cdot z} = k_0^2 \cdot \sum_p \sum_m E_m \cdot c_p(x) \cdot f_m \cdot e^{i \cdot (p \cdot K_G - \beta_m) \cdot z}$$

As before we 1) multiply again both sides with $f_\ell(x)^*$ and 2) integrate $\int ...dx$ using a) the orthonormality relation $\langle f_m | f_l \rangle = \delta_{ml}$ of the mode profiles f(x) and 3) making use of the weak perturbation assumption s<<d:

$$\boxed{\frac{\partial E_\ell}{\partial z} = -i\cdot \sum_{\substack{p\\(perturbation)}} \sum_{\substack{m\\(modes)}} E_m \cdot \frac{k_0^2}{2\beta_\ell}\cdot\langle f_\ell | c_p(x) | f_m\rangle \cdot e^{-i\cdot(\beta_m-\beta_\ell-p\cdot K_G)\cdot z}}$$

and introducing the new parameters for:

the **coupling constant $\kappa^p_{\ell m}$ between mode l and m due to the p^th Fourier-component** and the **phase difference $\delta^p_{\ell m}$** we write the above equation:

$$\boxed{\kappa^p_{\ell m} = \frac{k_0^2}{2\beta_\ell}\cdot\int_S f_\ell^*(x)\cdot c_p(x)\cdot f_m(x)\,dx = \frac{k_0^2}{2\beta_\ell}\cdot\langle f_\ell|c_p|f_m\rangle \quad = f(\omega)}$$

Definition: **coupling constant** between mode *l* and *m* due to the p^th component of the perturbation

Using $\kappa^{-p}_{ml} = \kappa^{p}_{lm}{}^{*}$

$$\boxed{\delta^p_{\ell m} = \beta_m - \beta_\ell - p\cdot K_G = \delta_{\ell m} - p\cdot K_G}$$

Definition: **phase factor** between mode *l* and *m*

➡ mode coupling equation (only z-dependent equation)

$$\boxed{\frac{\partial E_\ell}{\underbrace{\partial z}_{\substack{\text{Coupling}\\\text{effect}}}} = \underbrace{-i\cdot\sum_p\sum_m E_m\cdot\kappa^p_{\ell m}\cdot e^{-i\cdot\delta^p_{\ell m}\cdot z}}_{\substack{\text{coupling of all modes into l}\\\text{including selfcoupling } \ell\to\ell\\ \\ \text{Coupling origin}}}}$$

Mode coupling (MC) equation (including self-coupling) (4.45), coupling is only effective if $\delta^p{}_{lm} \rightarrow 0$ (synchronization of the modes)

For a sinusoidal grating: p= 0, ±1 and fundamental mode operation of the WG: l= -m= -1 (left), m=1 (right propagating).

The MC-eq. simplify to:

$$\frac{\partial E_\ell}{\partial z} = -i \cdot \sum_p \sum_m E_m \cdot \kappa^p_{\ell m} \cdot e^{-i \cdot \delta^p_{\ell m} \cdot z}$$

Interpretations:

- m represents all possible modes of the unperturbed WG (guided and potentially unguided, scattered modes)
- p represents the p^{th} spatial harmonic of the corrugation. c_p spectrum depends on 2D corrugation shap.
- for the waveguide Bragg-reflector we assume for simplicity that only guided modes are relevant (off-axis scattering is neglected) and that the WG is fundamental mode.
- for the Bragg-reflector we assume that only one propagating (m) and one contra-propagating (-m) mode coupling exists m=±1
- κ^p_{lm} is a measure of the strength of the coupling between mode m and l due to the p-harmonic of the perturbation

  κ^p_{lm} is a function of ω and is approximately ~ω
- $\delta^p_{lm}(\omega)$ is a measure of the detuning between the forward and backward wave and the grating, resp. in the frequency domain the difference between the signal ω and the Bragg-resonance frequency $\omega_B = \frac{\pi c_0}{n_{eff}\Lambda}$ of the grating
- if higher harmonics of c_p (p>1) are present, then the grating might also resonate at harmonics of $\omega_{B,p} = p\omega_B = p\frac{\pi c_0}{n_{eff}\Lambda}$

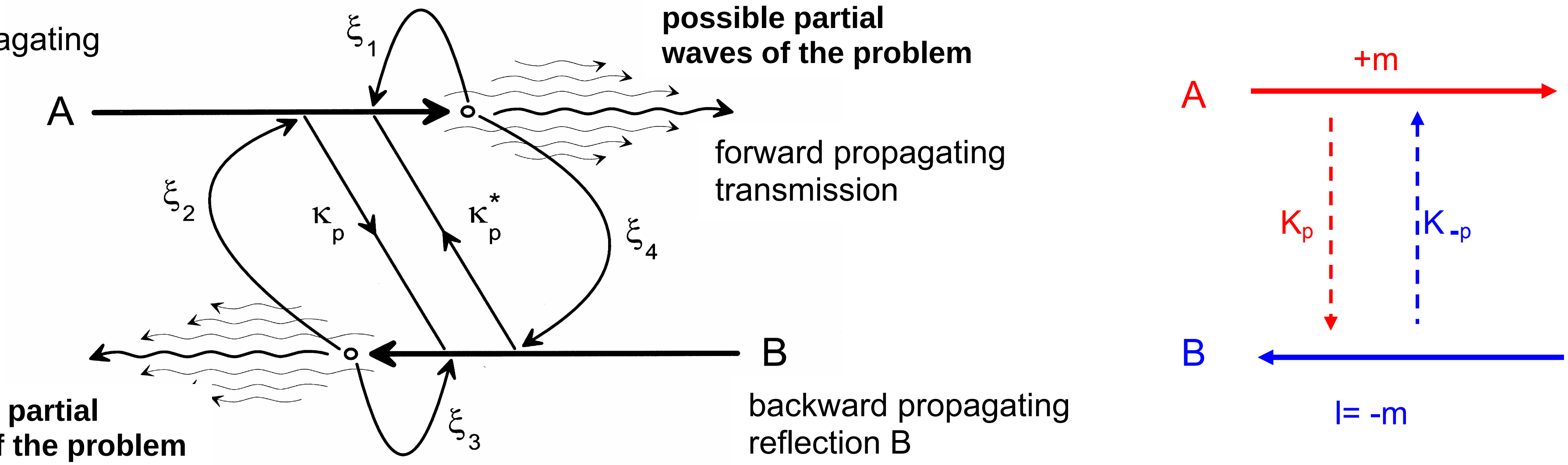
Elimination of explicit self-coupling in the mode coupling equation:

As the core thickness d of the corrugated core is not uniquely defined in the corrugated area, we can always adjust d mathematically in such a way that d → d' in order to eliminate the self-coupling coefficient term $\kappa^0_{ll} \rightarrow 0$ (assumption only)

$$\kappa^0_{\ell m}(d') = \frac{k_0^2}{2\beta_\ell} \cdot \langle f_\ell | c_0(d') | f_m \rangle \equiv 0 \qquad \text{for } l=m \qquad \rightarrow \mathbf{d'}$$

Remark: the above equation delivers an equation for the determination of d'.

Illustration of Coupling in Bragg-Reflectors for the pth spatial harmonic of the corrugation:



forward propagating excitation A

possible partial waves of the problem

forward propagating transmission

backward propagating reflection B

possible partial waves of the problem

Graphical illustration of the coupling between exciting, forward propagating wave A, which undergoes self- and mutual coupling and could excite a multitude of the possible partial waves of the problem.

Assumptions, conventions and definitions:

- for simplicity we assume that in the Bragg-reflector only **one coupled backward propagating wave (B≡l)** is excited.
- forward propagating modes **(A≡m)** are described by positive mode indices m>0 and positive propagation constants β_m
- backward propagating modes (B) are described by negative mode indices l<0 and negative propagation constants β_l
- coupling in the MC-eq. for coupling A→B is only effective by the term $\kappa_{m,l}^{+p}$ (p>0) and for coupling B→A is only effective by the term $\kappa_{m,l}^{-p}$ (p<0).
- we assume a <u>sinusoidal</u> corrugation **p=±1** of the grating with a spatial vector $K_G = 2\pi / \Lambda$

Conditions for energy exchange:

In order to realize an **energy exchange** between <u>forward</u> and <u>backward</u> propagating modes we must request:

1. **<u>Synchronization:</u>**

<u>Directional coupling</u> m → l (backward mode couples into forward mode):

for maximum energy exchange m → l the phase difference $\delta_{\ell m}^{p}$ should be 0 (resp. independent of z) for co-propagating modes

$$\delta_{\ell m}^{p} = \beta_m - \beta_\ell - p \cdot K_G = \delta_{\ell m} - p \cdot K_G \rightarrow 0$$

$$\text{because } l = -m \text{ we have } \beta_\ell = \beta_{-m} = -\beta_m$$

$$\delta_{-mm}^{p} = 2\,\beta_m - p \cdot K_G = 0 \quad \textit{has only solutions for } \underline{p > 0}, \textit{resp.} + p\,! \quad \rightarrow \kappa_{-mm}^{p} \textit{ is effective}$$

m → l (forward propagating)

Interpretation: for maximum positive interference the partial reflections at Λ / p should have a 2π phase difference.

2. **<u>Contra-directional coupling</u>** m ← l (forwardward mode couples into backward mode)

the desired coupling should be between forward and backward propagating mode of the same type *l*

$$\delta^{p}_{m-m} = \beta_{-m} - \beta_{m} - p\cdot K_G = \delta_{m-m} - p\cdot K_G \rightarrow 0$$

because $l = -m$ *we have* $\beta_\ell = \beta_{-m} = -\beta_m$

$\delta^{p}_{-mm} = -2\,\beta_m - p\cdot K_G = 0$ *has only solutions for* $\underline{p<0}$*, resp.* $-p$! $\rightarrow \kappa^{-p}_{m-m} = -\,\kappa^{p}_{-mm}$ *is effective*

$\beta_\ell = \beta_{-m} = -\beta_m \rightarrow \beta_m$ (design goal)

3. **Grating resonance of order p:** both mode are of the same type, except opposite propagation direction

i) $\ell = -m \;\; , \;\; \beta_\ell = \beta_{-m}$

ii) we consider only a particular harmonic p c_p of the grating periodic corrugation

1. – 3. result in the **condition for the phase difference for synchronization**

$$\delta^{p}_{-m,m} = \delta_{-m,m} - p\cdot K_G = 2\cdot\beta_m - p\cdot K_G \rightarrow 0 \quad ;$$

$$\rightarrow \underline{Bragg-condition: \quad \beta_m = \beta_B\left(\omega_{B,p}\right) = p\cdot K_G / 2} \quad \omega_{B,p} = Bragg-resonance\ frequency$$

$$with \quad K_G = 2\pi/\Lambda \quad ; \quad \beta_m = 2\pi/\lambda_m = \omega n_{eff,m}/c_0$$

$$\underline{\omega_{B,p} = p\frac{\pi c_0}{\Lambda n_{eff,m}} = p\,\omega_B}$$

→

We use the definition of the effective refractive index of the mode: $n_{eff} = \beta_m c_0 / \omega$

For the p^{th} Bragg-resonance the grating constant Λ must be p-times the half wavelength $\lambda_m/2$ (in the medium of mode *m*)

➡ $p \cdot \frac{\lambda_{B,p}}{2 \cdot n_{eff,m}} = \Lambda \rightarrow \lambda_{B,p} = \frac{\Lambda}{p} 2 \cdot n_{eff,m}$ Bragg-Resonance wavelength $\lambda_{B,p}$ of order p

High p-order Bragg-grating at a given corrugation Λ length are more difficult to fabricate than 1.order grating because Λ~p.
High p-order Bragg-grating at a given wavelength λ_m are easier to fabricate than 1.order grating.

(In addition higher order Bragg-gratings can couple to radiation modes – this may be a desirable device feature)

Summary of the **p^{th} Bragg-resonance wavelength $\lambda_{B,p}$** and **frequency $\omega_{B,p}$**:

$$\boxed{\lambda_{B,p} = 2 \cdot n_{eff,m} \Lambda / p \quad ; \quad \omega_{B,p} = p\pi c_o /(n_{eff}\Lambda) = pK_G c_o / 2n_{eff}}$$

Remark: sinusoidal gratings have only one Bragg-resonance, where as rectangular or triangular gratings have a large number of corresponding Bragg-resonances due to the high number of spatial frequencies.

Determination of the forward and backward propagating modes in the Bragg-Reflector:

Because Bragg-reflectors are very important in many applications (mirrors and filters) we derive additional design equations explicitly.

From the original mode coupling equations we get for only 2 contra-directionally coupling waves:

$A = E_m$ forward propagating mode m

$B = E_{-m}$ backward propagating mode -m

➡

$$\boxed{\begin{aligned} \frac{\partial B}{\partial z} &= -i \cdot A \cdot \kappa^{p}_{-m,m} \cdot e^{-i\cdot\delta^{p}_{-m,m}\cdot z} && \left(back\ coupling,\ p > 0\right) \\ \frac{\partial A}{\partial z} &= -i \cdot B \cdot \kappa^{-p}_{m,-m} \cdot e^{-i\cdot\delta^{-p}_{m,-m}\cdot z} && \left(forward\ coupling,\ p = -p < 0\right) \end{aligned}}$$

CM equation for modes *m, -m, and p, -p*

Simplifications:

1) for the back-propagating mode $-m$ (B) synchronization and high coupling with the grating occurs for $-p$ leading to:

$p{\cdot}K_G \rightarrow -p{\cdot}K_G$ (only p and –p spatial frequencies couple)

and $\delta_{m,-m} = -\delta_{-m,m}$ (only co- and counter-propagating modes with opposite propagation vectors couple)

➡ $(\delta_{-m,m} - p{\cdot}K_G) \rightarrow (-\delta_{m,-m} + p{\cdot}K_G)$ ➡ $\delta^{-p}_{m,-m} = -\delta^{p}_{-m,\,m}$

2) for lossless materials with real δn^2 we get per definition of c_p:

$c_{-p} = c^*_p$ because $\langle f_{-m} \,|\, c^*_p \,|\, f_m \rangle = \langle f_{-m} \,|\, c_p \,|\, f_m \rangle^*$

3) again per definition of $\kappa^p_{l,m}$ we have the relation for back and forward coupling constants (as a consequence of 2)):

$\kappa^p_{-m,m} = -\kappa^p_{m,-m}.$

With these relations resulting from the symmetry of the problem for forward (m) and backward (-m) propagating modes:

$$\boxed{\begin{aligned}\frac{\partial A}{\partial z} &= -i\cdot\kappa_p^*\cdot B\cdot e^{i\cdot\delta_p\cdot z}\\ \frac{\partial B}{\partial z} &= i\cdot\kappa_p\cdot A\cdot e^{-i\cdot\delta_p\cdot z}\end{aligned}}$$

simplified MC differential equation for right-propagating A(z) , left-propagating B(z)

with – $\kappa_p = \kappa^p{}_{-m,\,m}$, $\kappa_{ml}^{-p} = \kappa_{lm}^{p}{}^*$ and $\delta_p = \delta^p{}_{-m,\,m} = 2\beta_m - pK_G$ and the

boundary conditions: eg.
$$\begin{aligned}&A(0) = \sqrt{I} = A_0 \quad (input\ at\ z=0) \quad ; \quad A(L)\neq 0\\ &B(L) = 0 \qquad\qquad\quad (no\ input\ at\ z=L) \quad ; \quad B(0)\neq 0\end{aligned}$$
for single right propagating exciting wave

Skipping details of the non-trivial solution of the mode coupling differential equations we obtain the following **solution of the differential equation for A(z) and B(z) in T-matrix-form** (see appendix p.4-51):

$$\boxed{\begin{array}{ll} out: & in: \\ \begin{bmatrix}A(z)\\B(z)\end{bmatrix} = \begin{bmatrix}T_{11}(z) & T_{12}(z)\\ T_{21}(z) & T_{22}(z)\end{bmatrix}\cdot & \begin{bmatrix}A(0)\\B(0)\end{bmatrix}\end{array}}$$

$$\boxed{\begin{aligned}T_{11}(z) &= \left\{cosh\left(\kappa_{eff}\,z\right) - \frac{i\delta_p}{2\kappa_{eff}}\cdot sinh\left(\kappa_{eff}\,z\right)\right\}\cdot e^{i\cdot\frac{\delta_p}{2}\cdot z}\\ T_{12}(z) &= -\frac{i\kappa_p^*}{\kappa_{eff}}\cdot sinh\left(\kappa_{eff}\,z\right)\cdot e^{i\cdot\frac{\delta_p}{2}\cdot z}\\ T_{21}(z) &= \frac{i\kappa_p}{\kappa_{eff}}\cdot sinh\left(\kappa_{eff}\,z\right)\cdot e^{-i\cdot\frac{\delta_p}{2}\cdot z}\\ T_{22}(z) &= \left\{cosh\left(\kappa_{eff}\,z\right) + \frac{i\delta_p}{2\kappa_{eff}}\cdot sinh\left(\kappa_{eff}\,z\right)\right\}\cdot e^{-i\cdot\frac{\delta_p}{2}\cdot z}\end{aligned}}$$

Transmission-Matrix T

$$T\left(z\kappa_{eff},\delta_p,\kappa_p\right)$$

using $\delta_p\ ,\kappa_p\ ,\kappa_{eff}\ ,z$

and **defining** the new **effective coupling constant** κ_{eff}

effective coupling constant κ_{eff}:

$$\boxed{\kappa_{eff}(\omega) = \sqrt{\kappa_p \kappa_p^*(\omega) - \left(\frac{\delta_p(\omega)}{2}\right)^2} = \frac{1}{2}\cdot\sqrt{4\cdot\left|\kappa_p\right|^2 - \delta_p^2} = f(\omega)}$$

with $\delta_p = \delta^p_{-m,\,m} = 2\beta_m - pK_G$

➡ κ_{eff} describes <u>envelope</u> functions A(z), B(z) containing hyperbolic functions of (κ_{eff} **z**) !

Observe that κ_{eff} can be <u>real or imaginary</u> depending on detuning δ resp. ω !

<u>Conclusions:</u>

1) the envelop amplitude functions A(z) and B(z) contain the exponentials phase terms of the type

$$e^{\pm i \frac{\delta_p}{2} z} \left(\text{phase detuning}\right) \text{and} \quad e^{\pm \kappa_{\mathrm{eff}} z} \left(\text{coupling strength}\right)$$

from the sinh-, cosh(κ_{eff}z) -envelop functions

2) the field functions $E_m(z)$ are related to the envelop functions $A_m(z)$, $B_m(z)$ by spatial $e^{\pm i\beta_m z}$

(spatial carrier wave)

➡ **the total wave vectors β of the propagating and counter-propagating waves** $\sim \underbrace{e^{\pm i\frac{\delta_2}{2}z} e^{\pm k_{eff} z}}_{envelope} \underbrace{e^{\pm i\beta_m z}}_{carrier}$ **are:**

$$\boxed{\beta(\omega) = -\frac{\delta_p}{2} + i\kappa_{\mathrm{eff}} + \beta_m = \frac{K_G}{2} + \kappa_{\mathrm{eff}} = \frac{K_G}{2} \pm \frac{i}{2}\sqrt{4\left|\kappa_p\right|^2 - \delta_p^2}}$$

(complex growth/attenuation)

➡ $\underline{E^+(z,t) = E_m^+(z)\, e^{-i\beta z} e^{i\omega t}}$ (example of right propagating wave)

Assuming that κ_p is frequency independent around the Bragg-resonance ω_B we get as an approximation for the propagation constant $\beta(\omega)$ in the grating:

$$\delta_p(\omega) = 2\beta(\omega) - p\mathrm{K} = \frac{4\pi}{\lambda} - p\frac{2\pi}{\Lambda} = \frac{2n_{eff}}{c_0}(\omega - \omega_B)$$

$$\beta(\omega) \simeq \underbrace{\frac{\mathrm{K_G}}{2}}_{\text{real}} \pm \underbrace{\frac{i}{2}\sqrt{4|\kappa_p|^2 - \left(\frac{2n_{\text{eff}}}{c_0}\right)^2(\omega - \omega_B)^2}}_{\text{real or complex}}$$

Dispersion relation of the Bragg-Grating close to resonance

Discussion of $\beta(\omega)$: (for detailed discussion see p.4-50)

$\beta(\omega)$ can become real or complex, depending on detuning, resp. frequency ω:

a) a **complex** propagation constant β, $\omega \rightarrow \omega_B$ means a **decaying or not propagating wave** inside a transmission **stop-band** (formation of a bandgap, with a high Bragg-reflection) of spectral width $\Delta\omega\sim\kappa_p$ (coupling constant, independent of length L)

b) a **real** propagation constant β, $\omega >> \omega_B + \Delta\omega/2$ or $\omega << \omega_B - \Delta\omega/2$, gives rise to a **propagating wave** in the **pass bands**.

Remark:

$$sinh(x) = \frac{e^x - e^{-x}}{2} \quad ; \quad cosh(x) = \frac{e^x + e^{-x}}{2}$$

Properties of the Bragg-Reflector: Reflection and Transmission

1) Reflection coefficient

Motivation: Bragg reflector are narrowband, virtually loss-less dielectric mirrors, much better than their broadband metallic counterparts.

For the reflection behaviour of the Bragg-Grating of length L we assume an incoming (x=0) forward propagating wave A and a reflected backward propagating wave B with no input at x=L:

$$A(0) = \sqrt{I} = A_0 \quad (\text{input})$$
$$B(L) = 0 \quad ; \quad \mathrm{A}(L) \geq 0 \quad (\text{transmitted wave})$$
$$B(0) = ? \quad (\text{reflected wave})$$

Using the BR-Transmission-Matrix $\begin{bmatrix} A(z) \\ B(z) \end{bmatrix} = \begin{bmatrix} T_{11}(z) & T_{12}(z) \\ T_{21}(z) & T_{22}(z) \end{bmatrix} \cdot \begin{bmatrix} A(0) \\ B(0) \end{bmatrix}$ we obtain from the second boundary condition:

$$B(L) = T_{21}(L) \cdot A_0 + T_{22}(L) \cdot B(0) = 0 \quad \rightarrow B(0) = -A_0 T_{21}(L) / T_{22}(L) \cdot$$

This equation allows the determination of the **reflected wave amplitude B(0) at the input, resp. the field reflection coefficient r**:

$$\boxed{r(\omega) = \frac{B(0)}{A_0} = -\frac{T_{21}(L)}{T_{22}(L)} = -\frac{i \cdot \kappa_p \cdot sinh(\kappa_{eff} L)}{\kappa_{eff} \cdot cosh(\kappa_{eff} L) + \frac{i\delta_p}{2} \cdot sinh(\kappa_{eff} L)}}$$

Bragg-Reflection Coefficient (stop-band characteristics)

The field reflection coefficient depends only on the **product** $(\kappa_{eff} L)$ and the detuning δ_p:

$$\kappa_{eff}(\omega) = \frac{1}{2} \cdot \sqrt{4 \cdot |\kappa_p|^2 - \delta_p^2} \quad \text{and} \quad \delta_p(\omega) = \frac{2n_{eff}}{c_0}(\omega - \omega_B)$$

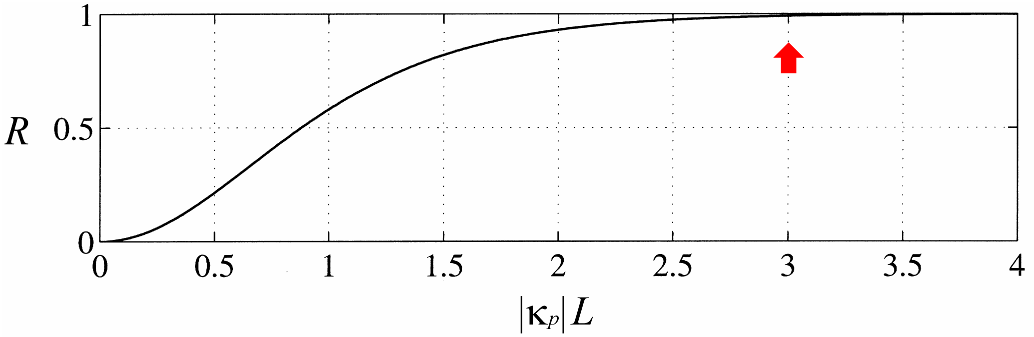
At the Bragg-Resonance $(\delta_p \rightarrow 0)$ we obtain for r:

$$\boxed{r = -i \cdot \frac{\kappa_p \cdot tanh\left(\kappa_{eff} L\right)}{\kappa_{eff} + \frac{i\delta_p}{2} \cdot tanh\left(\kappa_{eff} L\right)} \xrightarrow{\delta_p \rightarrow 0 \;\; ; \;\; \kappa_{eff} \rightarrow |\kappa_p|} r\left(\omega_B\right) = -i \cdot tanh\left(\left|\kappa_p\right| \cdot L\right)}$$

The *Bragg-Resonance* $(\delta_p \rightarrow 0)$ can be expressed as an optical frequency ω_B or wavelength λ_B:

$$\lambda_B = \frac{2}{p} \cdot \Lambda \cdot n_{eff,B} \quad \longleftrightarrow \quad \omega_B = \frac{p \cdot \pi \cdot c_0}{\Lambda \cdot n_{eff,B}}$$

For the **intensity (power) reflection R(ω_B)** at the Bragg-resonance $R = r \cdot r^*$ depends on the grating length L:



A *Bragg-grating* where the condition

$|\kappa_p| \cdot L \approx 3$

is fulfilled, reflects more than 99 % of the incoming radiation and is a very good mirror.

Metallic mirrors @1550nm reflect only ~80-90%.

- The power reflection coefficient R depends only on the **product** $\left(\kappa_p L\right)$
- For $\left(\kappa_{eff} L\right)$<1 $\quad R\left(\omega_B\right) \sim \left(\left|\kappa_p\right| \cdot L\right)^2$

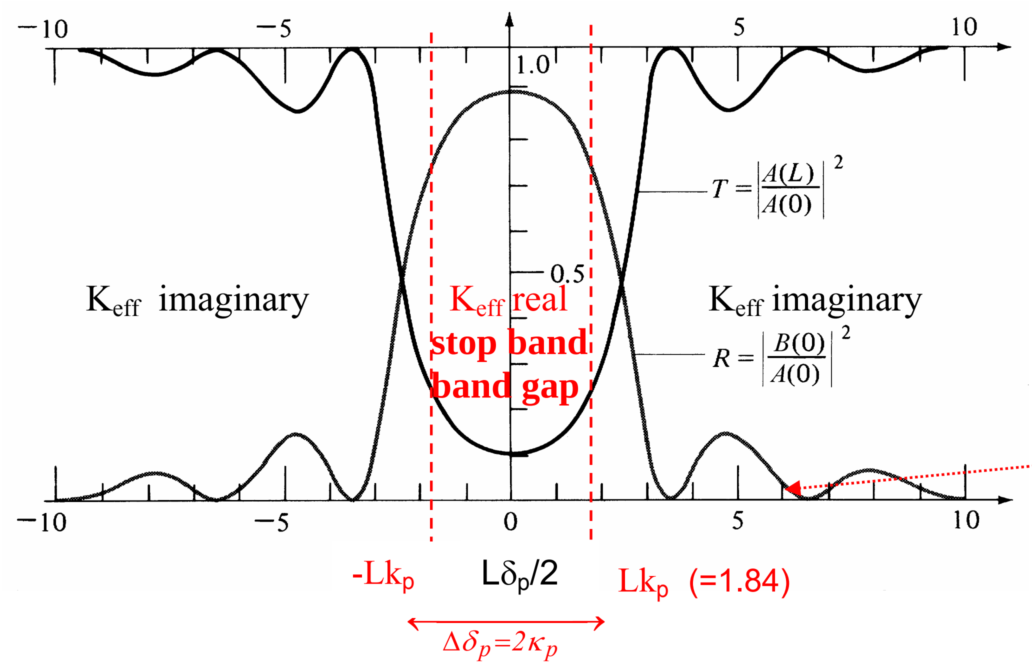
- Strong coupling κ (large corrugation) allows **short length L**
- **Narrow bandwidth** $\Delta\omega$ and **low losses** can be achieved by small κ
- High total reflections are possible with small reflections form small corrugations or dielectric contrasts between sequences of of different materials
- Bragg-mirrors are widely used in planar single-frequency laser diodes, VCSELs and anti-reflection coatings

The previous discussion of the transmission/reflection properties of the DBR with the propagation constant $\beta(\omega)$ of the two counter propagating wave is only very qualitatively and not sufficient for any filter design.

➡ **detailed discussion of R(δ_p), R(ω) resp. T(δ_p), T(ω) is required.**

Spectral dependency of the intensity reflection coefficient R(δ_p): Bandstop-Characteristic, frequency selective mirrors

Bragg mirrors show a high reflectivity at the Bragg-resonance (stop-band), but are otherwise almost transparent (pass-bands).



Numerical Simulation of the Spectrum of the power reflection factor R(ω) and the power transmission factor T(ω)=1-R(ω)

of a Bragg-grating with $/\kappa_p /L \approx 1.84$ versus detuning $L\cdot\delta_p$.

At the Bragg-resonance δ_p=0, resp. $\omega=\omega_B$

➡ ideal for mode-selection filter in lasers¨

undesired side-lobe

passband	**stopband**	**passband**
transparent	phase coherent constructive reflections	quasi-random destructive reflections

Transformation of detuning δ_p into optical frequency ω or wavelength λ:

$$\underline{\delta_p(\omega)} = 2\cdot\beta_m(\omega) - p\cdot K_G = \frac{4\pi n_{m,eff}}{\lambda_{o,m}} - p\frac{2\pi}{\Lambda} = \frac{2n_{m,eff}}{c_0}\omega - \frac{2n_{m,eff}}{c_0}\omega_B = \underline{\frac{2n_{m,eff}}{c_0}(\omega - \omega_B)}$$

For strong (rectangular, no side lobes) bandpass filtering Characteristic we see by directly going back to the transmission matrix T:

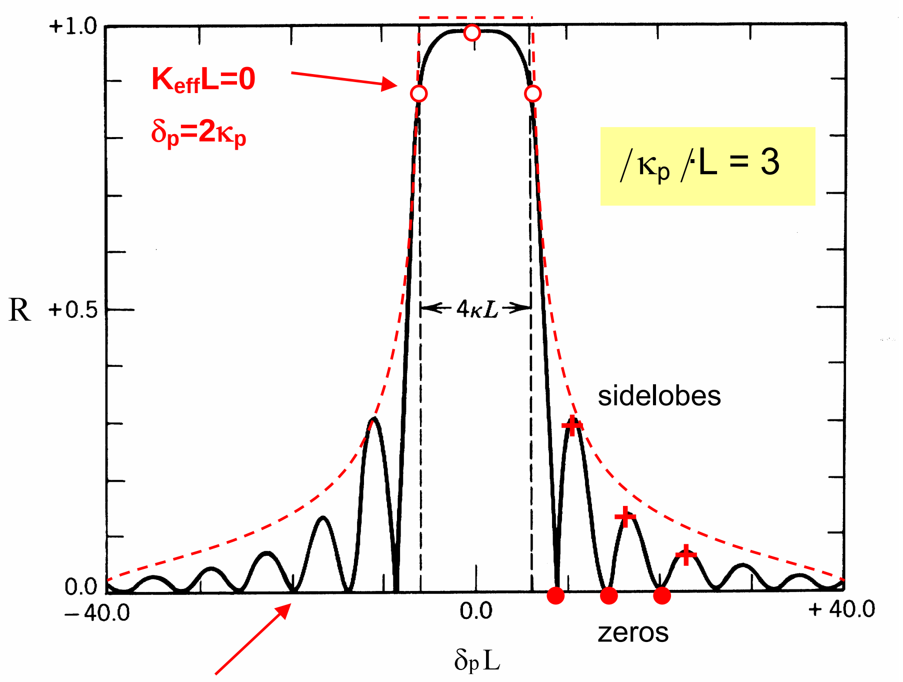
a necessary condition is: $|\kappa_p|\cdot L >> 1$

- ➡ strong coupling (large κ_p) and large length L for large reflection R
- ➡ strong coupling (large κ_p) for large **filter bandwidth** $\Delta\omega$ (independent of L !!!)

$$4\kappa_p L = \delta_p(\Delta\omega)L = \frac{2n_{eff}}{c_0}(\omega_{-1db}-\omega_B)L = \frac{2n_{eff}}{c_0}(\Delta\omega)L \quad \rightarrow$$

$$\underline{Bandwidth\ B = 2\Delta\omega = 4\kappa_p\frac{c_0}{n_{eff}} \neq f(L)}$$

Bandpass filter characteristic (large $|\kappa_p|\cdot L$)



Zero:
sinh(K_{eff}L)=0
for complex K_{eff}

Trade-off: large $|\kappa_p|\cdot L$ –values produce large side-lobe amplitudes close to the main-lobe.

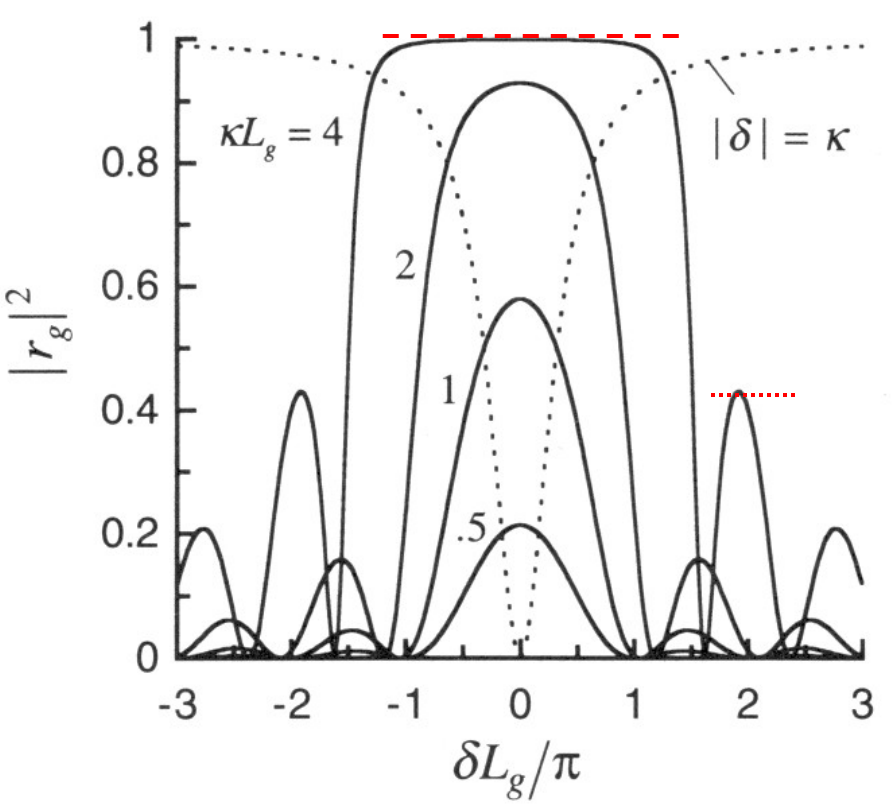
2) Transmission coefficient

The power transmission coefficient T can be calculated from R by applying the energy conservation argument:

$$T = t\cdot t^* = 1 - R \qquad t = \frac{1}{T_{22}(L)}\cdot\left\{T_{11}(L)\cdot T_{22}(L) - T_{12}(L)\cdot T_{21}(L)\right\}$$

Stopband-Charcteristics vers. ($|\kappa_p|\cdot L$):

Stopband-Flatness (desirable) for large $|\kappa_p|\cdot L$, but high side-lobe reflection (undesirable → filter x-talk) for large $|\kappa_p|\cdot L$



large $|\kappa_p|\cdot L$ –values produce large side-lobe amplitudes close to the main-lobe

Basic properties of Bragg-Bandpass filters: **(self-study)**

1) Reflections:

For large products $|\kappa_p|\cdot L = 3$ the Bragg-mirror reflects strongly in the stop-band. In the middle of the stop band $R = tanh^2(\,|\kappa_p|\cdot L\,) = tanh^2(3)\sim 0.99$.

$\kappa_{eff}(\omega)=\sqrt{\kappa_p\kappa_p^*(\omega)-\left(\frac{\delta_p(\omega)}{2}\right)^2} = \frac{1}{2}\cdot\sqrt{4\cdot|\kappa_p|^2-\delta_p^2}$ has zeros at $\delta_p = \pm 2\cdot|\kappa_p|$ (bandwidth). Therefore $\kappa_{eff}(\omega)$ for $|\delta_p| > 2\cdot|\kappa_p|$ becomes **imaginary in the pass band**, resulting in a **decaying oscillatory behaviour** of $R(\delta_p)$ (side-lobes).

We approximate the bandwidth $\Delta\omega=B_{\delta p}$ by the first two zeros of $\kappa_{eff}(\omega) \rightarrow \kappa_{eff}(\omega)=0 \rightarrow$ R=0

➡ Filterbandwidth: $B_{\delta p}\sim 4\cdot|\kappa_p|$ (independent of L ! as discussed qualitatively from $\beta(\omega)$)

➡ Reflections coefficient at Filterbandwidth edges: $\delta_p = 2\cdot|\kappa_p|$

$$\lim_{\kappa_{eff}\to 0}\left\{R\left(\delta_p=\pm 2\kappa_p\right)\right\} \simeq \frac{\left(|\kappa_p|\cdot L\right)^2}{1+\left(|\kappa_p|\cdot L\right)^2} \quad (4.78).$$

2) Spectral properties:

a) Bandwidth: inspecting the expression for the field reflection coefficient r

$$r = \frac{B(0)}{A_0} = -\frac{T_{21}(L)}{T_{22}(L)} = -\frac{i\cdot\kappa_p\cdot sinh\left(\kappa_{eff}L\right)}{\kappa_{eff}\cdot cosh\left(\kappa_{eff}L\right)+\frac{i\delta_p}{2}\cdot sinh\left(\kappa_{eff}L\right)}$$

we see that the function has a first zero at

$$\kappa_{eff} = \sqrt{\kappa_p \kappa_p^* - \left(\frac{\delta_p}{2}\right)^2} = \frac{1}{2}\cdot\sqrt{4\cdot\left|\kappa_p\right|^2 - \delta_p^2} = 0 \quad \rightarrow \quad \delta_p = \pm 2\cdot\left|\kappa_p\right|$$

➡ $\underline{\boldsymbol{B_{\delta p} = 4\cdot|\ \kappa_p|}}$

Bandwidth $\sim\Delta\delta_p = 4\cdot\left|\kappa_p\right|$ (detuning at the bandedges)

using the relation:

$$\delta_p = \frac{2n_{m,eff}}{c_0}(\omega_m - \omega_B) \rightarrow \Delta\delta_p = \frac{2n_{m,eff}}{c_0}\Delta\omega_m \rightarrow 2\kappa_p = \frac{n_{m,eff}}{c_0}B_{\delta p}$$

$$\boxed{B_{\delta p} = \frac{2c_0}{n_{m,eff}}\kappa_p}$$

DBR Frequency-Bandwidth $\neq$f(L), depends only on κ_p

b) Side-lobe Maxima and Reflection Zeros:

For filters with low crosstalk the out-of-band reflection should be very low , the side-lobes must be small.

Reflection Maxima:

Most filter applications require low side-lobes (small cross-talk)

Investigating the expression for r($\kappa_{eff}\cdot L$) we find (without prove) the reflection maxima at imaginary $\kappa_{eff}\cdot$ (!)

Maxima-requirement: $\kappa_{eff}\cdot L = i\cdot(q + ½)\cdot\pi, \quad \forall\ q = 1, 2,\ldots$

expressed in detuning, leads to: $\delta_p = \pm\, 2\cdot\{|\ \kappa_p\ |^2 + (q + ½)^2\cdot(\pi/L)^2\}^{½}$

➡ **qth Power-Reflection-Maxima $\boldsymbol{R = r\cdot r^*}$:** $R_q = \dfrac{\left(\left|\kappa_p\right|\cdot L\right)^2}{\left(q+\frac{1}{2}\right)^2\cdot\pi^2 + \left(\left|\kappa_p\right|\cdot L\right)^2} \quad \sim\kappa_p \ ; \ \sim\dfrac{1}{q^2}$ with $q = 1, 2,\ldots$

for large q (large side lobe order) R_q~1/q^2

R_1<0.1 if $|\ \kappa_p\ |\cdot L < \pi\ /2$ (without proof)

Reflection Zeros:

Zero-requirement: $\delta_p = \pm 2 \cdot \{|\kappa_p|^2 + (q \cdot \pi / L)^2\}^{½}, \quad \forall\, q = 1, 2, \ldots$

for large q the zeros occur at $\delta_p = \pm 2\,(q \cdot \pi / L)$, resp. at $L\delta_p = \pm 2\pi\, q$

Envelope of R: $R_{enveloppe} = \begin{cases} \frac{4 \cdot |\kappa_p|^2}{\delta_p^2} & \forall\ \frac{\delta_p}{2} > |\kappa_p| \quad \text{Passband} \\ 1 & \forall\ \frac{\delta_p}{2} < |\kappa_p| \quad \text{Stopband} \end{cases}$ (red curve in Fig. on p4-34)

Design procedure: Trade-off for Bragg-Grating design:

1) if the grating κ_p is given, then the bandwidth $B_{\delta p}$ is determined independent of L

2) long length L increases the stopband reflection

3) long length L increases the density of the passband maxima, therefore the hight of the first sidelobe tends to increase too

→ reduced first side-lobe suppression (trade-off)

Field distribution and Dispersion Characteristics of Bragg-Gratings:

Inside the Bragg-grating we have a superposition of a forward and backward propagating wave forming a standing wave.

From the solution of the transmission matrix of the coupled mode equation we get for the **field envelop functions A(z) and B(z)** in general:

$$\boxed{\begin{aligned}\frac{A(z)}{A_0} &= \frac{1}{T_{22}(L)}\cdot\left\{\, T_{11}(z)\cdot T_{22}(L) - T_{12}(z)\cdot T_{21}(L)\,\right\}\\ \frac{B(z)}{A_0} &= \frac{1}{T_{22}(L)}\cdot\left\{\, T_{21}(z)\cdot T_{22}(L) - T_{22}(z)\cdot T_{21}(L)\,\right\}\end{aligned}} \qquad (4.81).$$

Close to the *Bragg*-Resonance $\omega \sim \omega_B$ in the middle of the stop band $(\delta_p \to 0)$ the above equations for A and B simplify to hyperbolic functions:

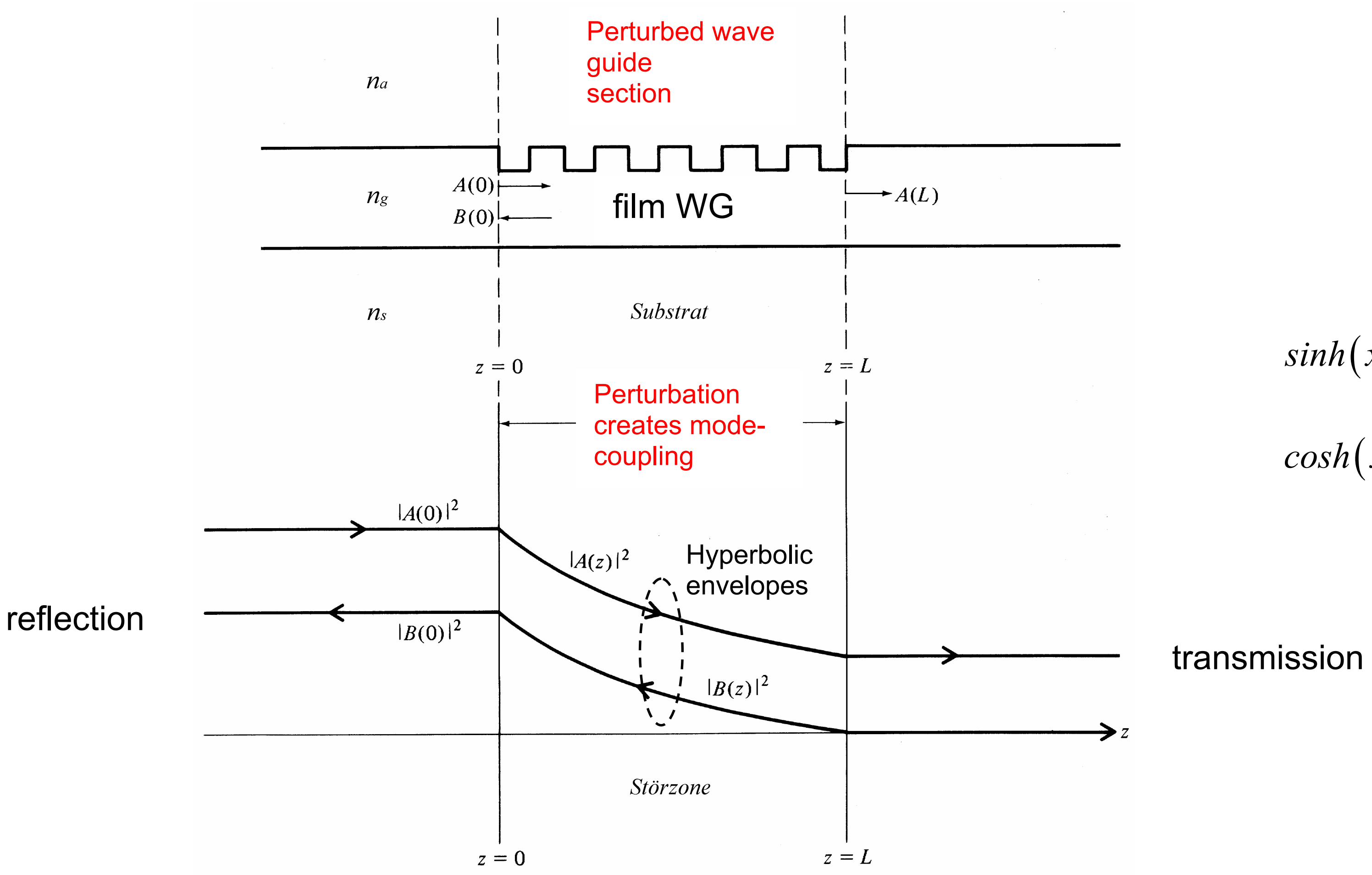
wave envelopes:

right propagating wave: left propagating wave:

$$\boxed{\frac{A(z)}{A_0} = \frac{cosh\left(\left|\kappa_p\right|\cdot[L-z]\right)}{cosh\left(\left|\kappa_p\right|\cdot L\right)} \quad ; \quad \frac{B(z)}{A_0} = -i\cdot\frac{sinh\left(\left|\kappa_p\right|\cdot[L-z]\right)}{cosh\left(\left|\kappa_p\right|\cdot L\right)}}$$

for $\omega \sim \omega_B$, $\lambda \sim \lambda_B$

Envelope-Field distribution A(z) and B(z) close to the Bragg-resonance:



Perturbed wave guide section

Perturbation creates mode-coupling

Hyperbolic envelopes

$$sinh(x) = \frac{e^{+x} - e^{-x}}{2}$$

$$cosh(x) = \frac{e^{+x} + e^{-x}}{2}$$

Dispersion relation β(ω) in periodic structures

Generic prototype for Bragg-gratings, Photonic Crystals and Electrons in atomic crystals

For the complete spatial field amplitudes $E_m(z)$ and $E_{-m}(z)$ we include the eliminated spatial carriers $e^{-i\cdot\beta_m\cdot z}$ and get for β(ω):

Stopband propagation (low detuning):

$$E_m\left(z\right)= A\left(z\right)\cdot e^{-i\cdot\beta_m\cdot z} = A\left(0\right)e^{-i\cdot\left(\frac{\delta_p}{2}-\kappa_{eff}+\beta_m\right)\cdot z} = A\left(0\right)e^{-i\cdot\left(\frac{p\cdot\pi}{\Lambda}\cdot\pm\frac{i}{2}\sqrt{4\kappa_p^2-\left(\frac{2n_{eef}}{c_0}\right)^2\left(\omega-\omega_B\right)^2}\right)z} \xrightarrow[\propto]{\left|\kappa_p\right|\cdot L\to gross} e^{-i\cdot\left(\frac{p\cdot\pi}{\Lambda}-i\cdot\left|\kappa_p\right|\right)\cdot z}$$

$$E_{-m} = B\left(z\right)\cdot e^{i\cdot\beta_m\cdot z} = B\left(z\right)\cdot e^{i\cdot\frac{p\cdot K_G}{2}\cdot z} = B\left(z\right)\cdot e^{i\cdot\frac{p\cdot\pi}{\Lambda}\cdot z} \xrightarrow[\propto]{\left|\kappa_p\right|\cdot L\to gross} e^{i\cdot\left(\frac{p\cdot\pi}{\Lambda}+i\cdot\left|\kappa_p\right|\right)\cdot z} \qquad (4.84)$$

damped

oscillatory

In the band center δ=0 $\beta = \frac{p\cdot\pi}{\Lambda}+i\left|\kappa_p\right|$ contains an imaginary (damping) and a real (oscillatory) part.

Passband propagation (strong detuning):

β(ω) can be real (propagating wave, band) or complex (attenuated wave, bandgap)

Nonlinear Dispersion-Relation β(ω) and bandgap formation (stopband)

$$\beta(\omega) \simeq \underbrace{\frac{K_G}{2}}_{\text{real, constant}} \pm \frac{i}{2}\underbrace{\sqrt{4|\kappa_p|^2 - \left(\frac{2n_{eff}}{c_0}\right)^2(\omega-\omega_B)^2}}_{\text{real or imaginary}}$$

if the second term under the square root is smaller than the first

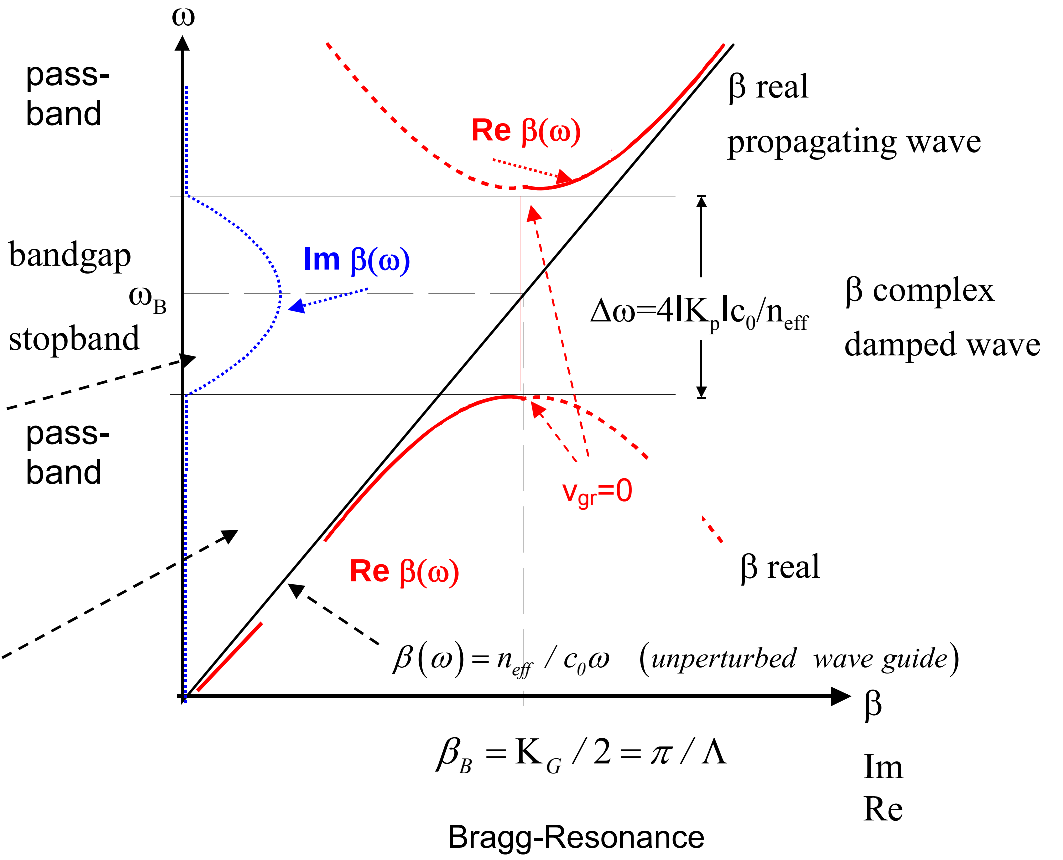
$$2|\kappa_p| > \left(\frac{n_{eff}}{c_0}\right)(\omega-\omega_B) \;\rightarrow\; \beta \text{ complex, } \textbf{Stopband}$$

$$\mathrm{Re}\,\beta = \frac{K_G}{2} \quad ; \quad \mathrm{Im}\,\beta = \frac{1}{2}\sqrt{4|\kappa_p|^2 - \left(\frac{2n_{eff}}{c_0}\right)^2(\omega-\omega_B)^2}; \quad \omega$$

if the second term under the square root is lager than the first

$$2|\kappa_p| < \left(\frac{n_{eff}}{c_0}\right)(\omega-\omega_B) \;\rightarrow\; \beta \text{ real , } \textbf{Passband}$$

$$\mathrm{Re}\,\beta = \frac{K_G}{2} \pm \frac{1}{2}\sqrt{\left(\frac{2n_{eff}}{c_0}\right)^2(\omega-\omega_B)^2 - 4|\kappa_p|^2} \quad ; \quad \mathrm{Im}\,\beta = 0 \quad ; \quad \omega$$



Interpretation:
Creation of Propagation Bandgaps (stop band) by Bragg-Resonance

- Bragg resonance (strong synchronization) creates a **photonic band gap** (propagation stop band) $\Delta\omega$

  $\rightarrow$ strong reflection

- The stronger the coupling κ_p, the wider the bandgap (stopband) $\Delta\omega$, independent of length L

  Inside the stop band the wave envelop decays exponentially $\sim e^{-|\kappa_p|z}$ by coupling to the reflected wave. The fast field amplitude oscillated with K_G (no propagation).

- In the transmission bands, far from the band gap the wave propagates unattenuated as in the unperturbed film waveguide with almost the same dispersion characteristics $\beta(\omega)$, resp. n_{eff}.

  The back-reflected wave disappears and shows only some small oscillations of the envelop (loss of synchronization).

- At the band edge the group-velocity $v_{gr} = \left(\frac{\partial \beta}{\partial \omega}\right)^{-1}$ becomes zero, meaning the envelope signal does not propagate

  ➨ “slow light effects”, stopping of light

- This formation of a **photonic stop-band** for wave-propagation in periodic structures is of generic interest, because the matter waves of electrons in a periodic atomic 3D-crystal exhibit a similar characteristic for the **electronic band gap**.

Comparison Photonic Crystal and Solid State Crystal:

Bragg-Gratings behave like a 1-dimensional dielectric crystal for photons (EM-waves) similar to the 1-dimensional atomic crystal lattice for electrons (matter waves).

Photonic Crystal:		Atomic Crystal Lattice
Optical Frequeny ω	⬅➡	Energy $E=\hbar\omega$
Propagation vector β	⬅➡	Momentum vector k
Grating periode Λ	⬅➡	Lattic constant a
Dielectric constant $\varepsilon(z)=n^2(z)$	⬅➡	Potential V(z)

Conclusions and summary:

The mode coupling analysis is an approximation in many respects:

- Our analysis is a scalar field representation, neglecting the vector field characteristics (the vector analysis can be included by modifications of the scalar formalism (4.46)).
- The mode coupling analysis does not include any boundary conditions of the field components.
- The mode coupling analysis is a relative analysis as a function of detuning $\delta_{\ell m}$.
- The mode coupling analysis is very efficient due to the modest mathematical theory in comparison to a full field calculation!
- Due to the assumption of the perturbation calculation of a small disturbance the dielectric variations can not be too strong violating the normal mode decomposition. Small disturbances provide better, resp. a more precise analysis.
- Nevertheless, as demonstrated empirically, the mode coupling analysis is rather robust even for strong (grating) perturbations (δn^2~20%).
- Mode coupling theory plays an important role in the following applications:
  - narrow band optical filters and reflectors
  - multi-layer optical coatings
  - Single frequency laser design (chap.6)

Literature

[26] L. A. Coldren, S. W. Corzine. *Diode Lasers and Photonic Integrated Circuits*. New York: John Wiley & Sons, 1995.

[27] A. Yariv. *Optical Electronics*. 3rd Ed., Fort Worth: Saunders College Publishing, 1985.

[28] T. Tamir. *Guided-Wave Optoelectonics*. Berlin: Springer-Verlag, 1988.

[29] K. J. Ebeling. *Integrierte Optoelektronik – Wellenleiteroptik, Photonik, Halbleiter*. Berlin: Springer-Verlag, 1989.

[30] Reinhard März. *Integrated Optics.* Boston: Artech House, 1994.

[31] D. Erni. *Periodische und nichtperiodische Wellenleitergitter- und Laserkavitätskonzepte*. Diss. ETH Nr. 11654, 19

Appendix 1: (self study)

Solution of the coupled mode equation for Bragg-Reflectors

Starting from the contradirectional coupled mode equation

$$\boxed{\begin{aligned}\frac{\partial A}{\partial z} &= -i\cdot\kappa_p^*\cdot B\cdot e^{i\cdot\delta_p\cdot z}\\ \frac{\partial B}{\partial z} &= i\cdot\kappa_p\cdot A\cdot e^{-i\cdot\delta_p\cdot z}\end{aligned}}$$

Using the variable transformation R(A, δ_p), S(B, δ_p) (phase shift by detuning along z):

$$\begin{aligned}R &= A\cdot e^{-i\cdot\frac{\delta_p}{2}\cdot z}\\ S &= B\cdot e^{i\cdot\frac{\delta_p}{2}\cdot z}\end{aligned} \quad \rightarrow \quad \begin{bmatrix}R\\S\end{bmatrix} = \left[\Gamma(z)\right]\cdot\begin{bmatrix}A\\B\end{bmatrix} \quad \text{with} \quad \left[\Gamma(z)\right] = \begin{bmatrix}e^{-i\cdot\frac{\delta_p}{2}\cdot z} & 0\\ 0 & e^{i\cdot\frac{\delta_p}{2}\cdot z}\end{bmatrix}$$

Introducing R(A) and S(B) in the coupled mode equation leads to:

$$\frac{\partial}{\partial z}\begin{bmatrix}R\\S\end{bmatrix} = \underbrace{\begin{bmatrix}-\frac{i\cdot\delta_p}{2} & -i\cdot\kappa_p^*\\ i\cdot\kappa_p & \frac{i\cdot\delta_p}{2}\end{bmatrix}}_{\substack{\text{A}\\ \text{no function of z}}}\cdot\begin{bmatrix}R\\S\end{bmatrix}$$

formal vector representation:

$$\frac{\partial}{\partial z}\vec{f}(z) = [A]\cdot\vec{f}(z)$$

$$\vec{f}_0 = \vec{f}(0) \quad \textit{boundary condition}$$

Spatial Laplace transformation L(s) of the system of differential equation:

$$\frac{\partial \mathrm{f}}{\partial \mathrm{z}} \underset{L^{-1}}{\overset{L}{\rightleftharpoons}} \mathrm{sf} + \mathrm{f}(0)$$

s=spatial frequency

$$\frac{\partial}{\partial z}\vec{f}(z) = [A]\cdot\vec{f}(z) \quad \overset{L}{\rightleftharpoons} \quad s\,\vec{f}(s)-\vec{f}(0)= [A]\cdot\vec{f}(s) \quad ; \quad \vec{f}(0)=\text{initial condition}$$

$$\vec{f}_0 = \vec{f}(0)$$

$$\{s[1]-[A]\}\cdot\vec{f}(s)=\vec{f}(0)$$

$$\rightarrow$$

$$\vec{f}(s)=\{s[1]-[A]\}^{-1}\vec{f}(0)=[\Phi(s)]\vec{f}(0) \qquad ; \quad \text{using} \quad [\Phi(s)]=\{s[1]-[A]\}^{-1}$$

As a next step we have to carry out an inverse L-trafo back into the spatial z-domain:

$$\vec{f}(s)=[\Phi(s)]\vec{f}(0) \quad \overset{L^{-1}}{\rightleftharpoons} \quad \vec{f}(z)=[\Phi(z)]\vec{f}(0)$$

resp.

$$[\Phi(s)]=[s\cdot[I]-[A]]^{-1} \quad \overset{L^{-1}}{\rightleftharpoons} \quad [\Phi(z)]=?$$

with the previous definition of [A]: $\begin{bmatrix} -\frac{i\cdot\delta_p}{2} & -i\cdot\kappa_p^* \\ i\cdot\kappa_p & \frac{i\cdot\delta_p}{2} \end{bmatrix}=[\mathrm{A}]$ we arrive at:

$$[s\cdot[I]-[A]] = \begin{bmatrix} s+\frac{i\cdot\delta_p}{2} & i\cdot\kappa_p^* \\ -i\cdot\kappa_p & s-\frac{i\cdot\delta_p}{2} \end{bmatrix}=[\Phi(s)]$$

Calculating the inverse of $[s\cdot[I]-[A]]$ for $[\Phi(s)]=[s\cdot[I]-[A]]^{-1}$ with the help of matrix relation $[M]^{-1} = adj[M] / det[M]$

Without going through the detailed calculation, we obtain:

$$\left[\Phi(s)\right]=\left[s\cdot[I]-[A]\right]^{-1} = \frac{1}{\underbrace{\left(s+\frac{i\cdot\delta_p}{2}\right)\cdot\left(s-\frac{i\cdot\delta_p}{2}\right)-\kappa_p\kappa_p^*}_{N(s)}}\cdot\begin{bmatrix} s-\frac{i\cdot\delta_p}{2} & -i\cdot\kappa_p^* \\ i\cdot\kappa_p & s+\frac{i\cdot\delta_p}{2}\end{bmatrix}$$

For $N(s)=\left(s+\frac{i\cdot\delta_p}{2}\right)\cdot\left(s-\frac{i\cdot\delta_p}{2}\right)-\kappa_p\kappa_p^* = s^2-\left\{\kappa_p\kappa_p^*-\left(\frac{\delta_p}{2}\right)^2\right\} = s^2-\kappa_{eff}^2$

Making use of the following elementary L-trafo-pairs:

$$\frac{1}{\kappa_{eff}}\cdot sinh\left(\kappa_{eff}\,z\right) \quad \xrightleftharpoons[L^{-1}]{L} \quad \frac{1}{s^2-\kappa_{eff}^2}$$

$$cosh\left(\kappa_{eff}\,z\right) \quad \xrightleftharpoons[L^{-1}]{L} \quad \frac{s}{s^2-\kappa_{eff}^2}$$

we get for [Φ(*z*)] in the spatial domain:

$$\left[\Phi(z)\right] = \begin{bmatrix} cosh\left(\kappa_{eff}\,z\right)-\frac{i\cdot\delta_p}{2\cdot\kappa_{eff}}\cdot sinh\left(\kappa_{eff}\,z\right) & -\frac{i\cdot\kappa_p^*}{\kappa_{eff}}\cdot sinh\left(\kappa_{eff}\,z\right) \\ \frac{i\cdot\kappa_p}{\kappa_{eff}}\cdot sinh\left(\kappa_{eff}\,z\right) & cosh\left(\kappa_{eff}\,z\right)+\frac{i\cdot\delta_p}{2\cdot\kappa_{eff}}\cdot sinh\left(\kappa_{eff}\,z\right)\end{bmatrix}$$

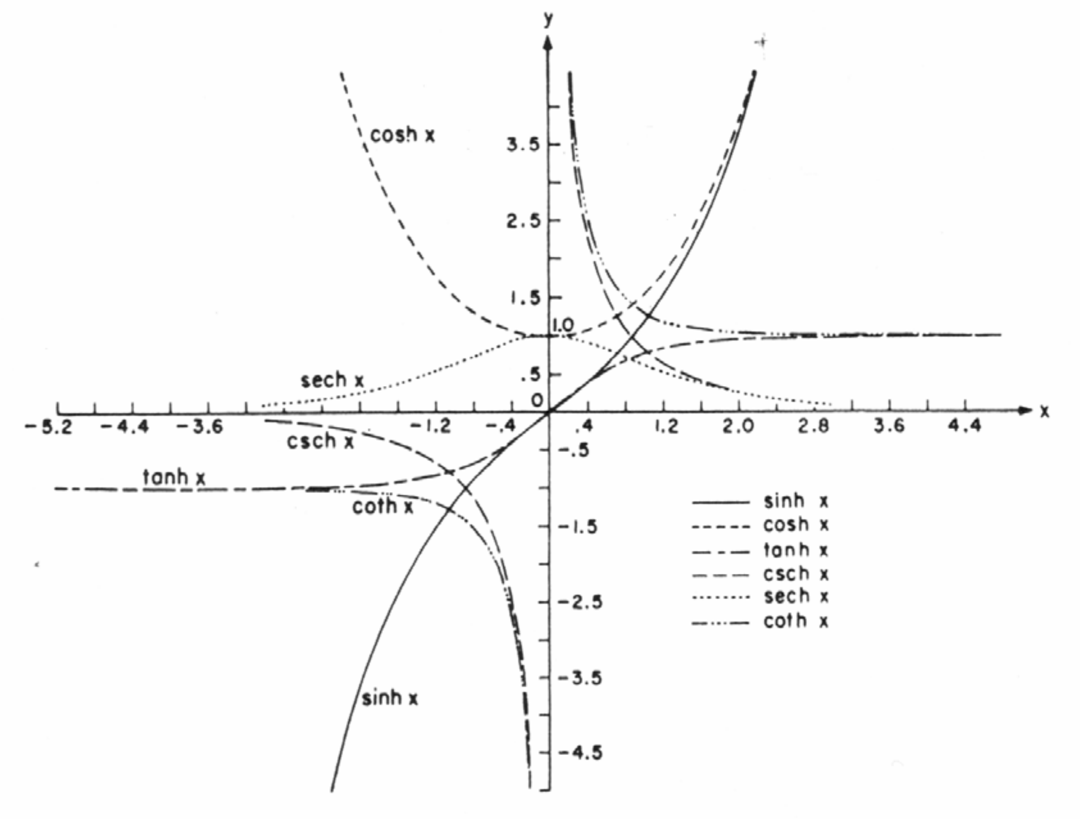
Using the inverse Γ^{-1} we get back to the original variable A(z) and B(z):

$$\begin{bmatrix} A(z) \\ B(z)\end{bmatrix} = \underbrace{\left[\Gamma(\mathrm{z})\right]\cdot\left[\Phi(\mathrm{z})\right]\cdot\left[\Gamma(0)\right]^{-1}}_{[T]}\cdot\begin{bmatrix} A(0) \\ B(0)\end{bmatrix}$$

$$\boxed{\begin{bmatrix} A(z) \\ B(z)\end{bmatrix} = \begin{bmatrix} T_{11}(z) & T_{12}(z) \\ T_{21}(z) & T_{22}(z)\end{bmatrix}\cdot\begin{bmatrix} A(0) \\ B(0)\end{bmatrix}}$$

Functions:

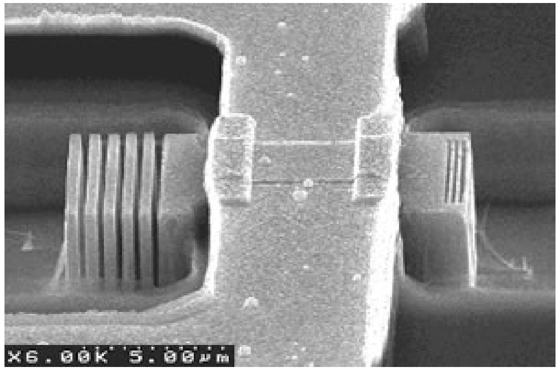
Hyperbolic functions:



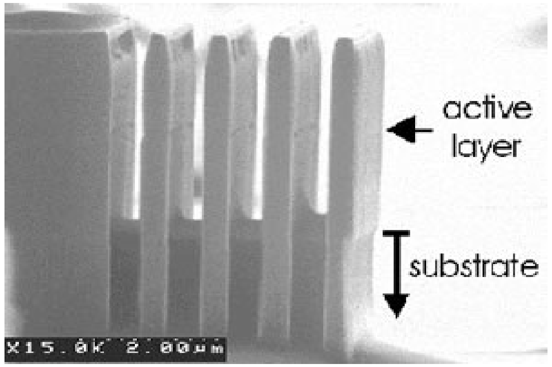
Technical examples of photonic devices based on coupled mode theory:

2 µm long Micro-Laser Diode with an etched single and a Bragg-reflector mirror

(Forchel et al)



3rd order Bragg mirror (air-semiconductor)



Wavelength Tunable laser diode at 1500nm with overgrown InGaAsP/InGaAs-Bragg-mirrors:

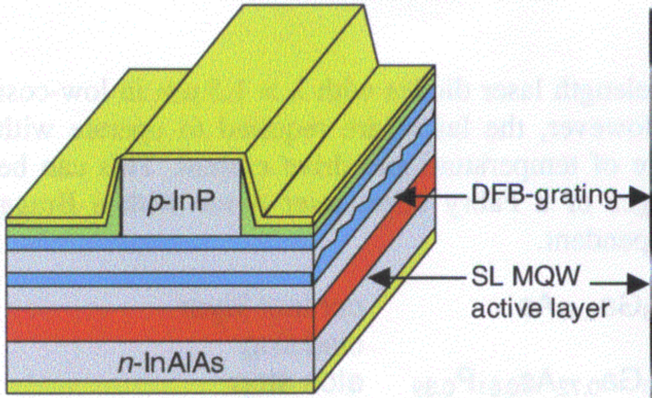


Fig. 2(a): Schematic structure of Distributed feedback laser

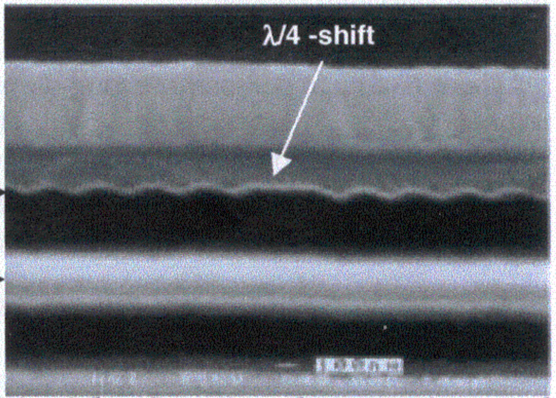
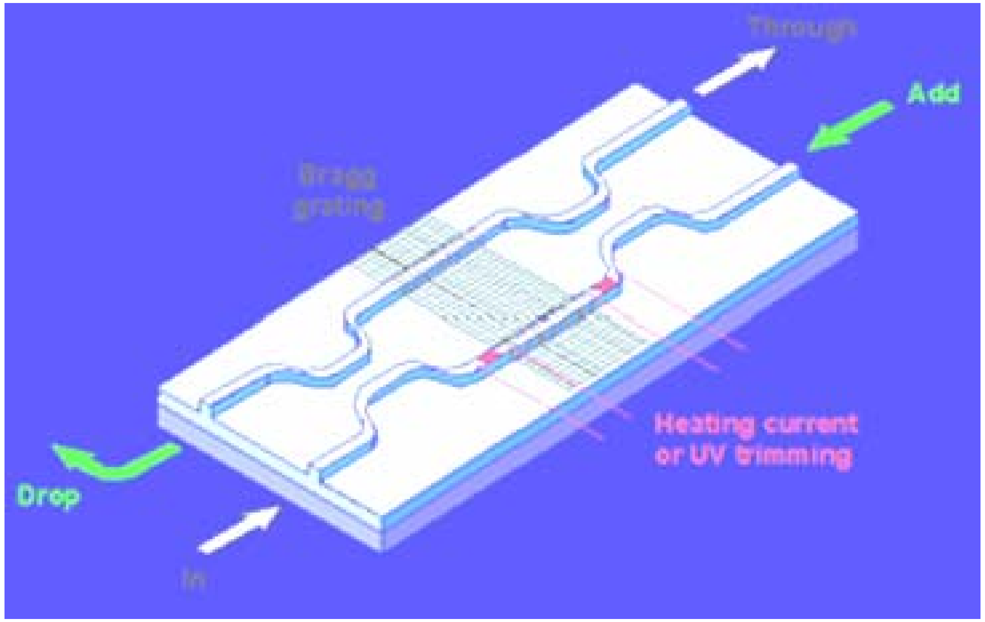
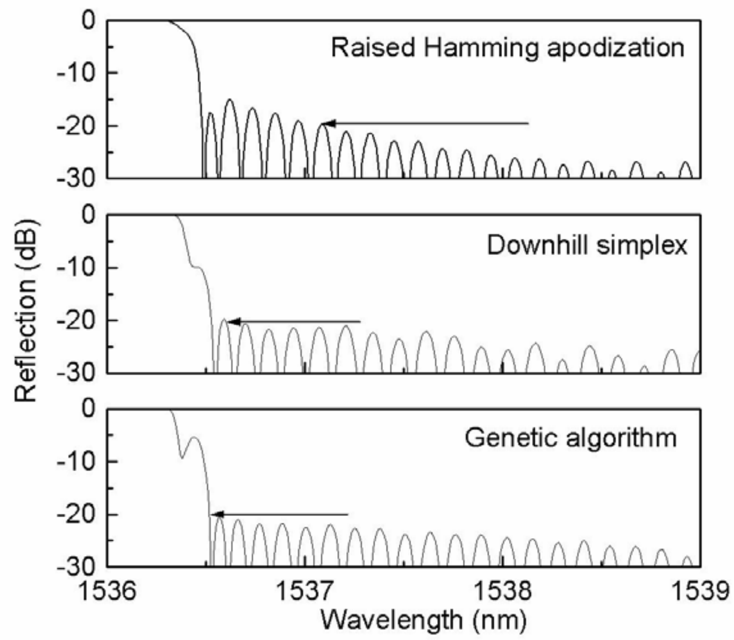


Fig. 2(b): Scanning electron micrograph of a λ/4-shifted corrugation

Add-drop Mach-Zehnder Interferometer with SiO_2/SiN_4-Bragg-gratings:





In order to reduce the height of the side-lobe maxima the Bragg-grating is apodized, meaning that the perturbations are periodic , but the strength of the perturbations is a spatial function of z.