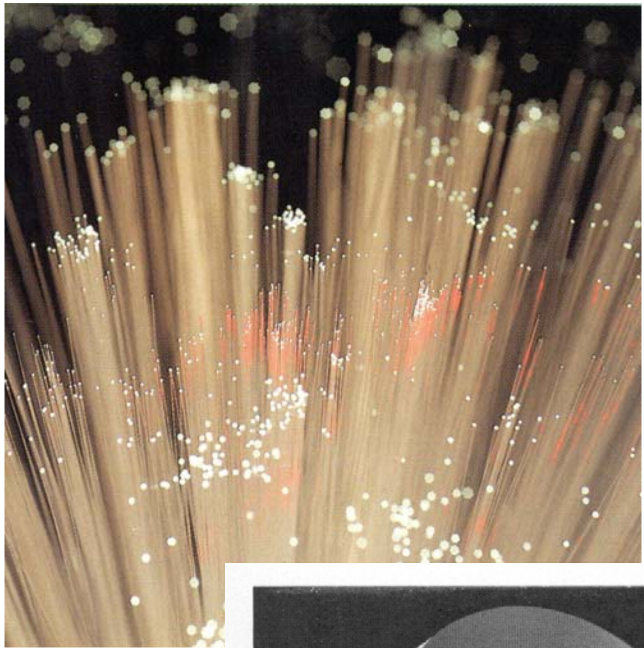
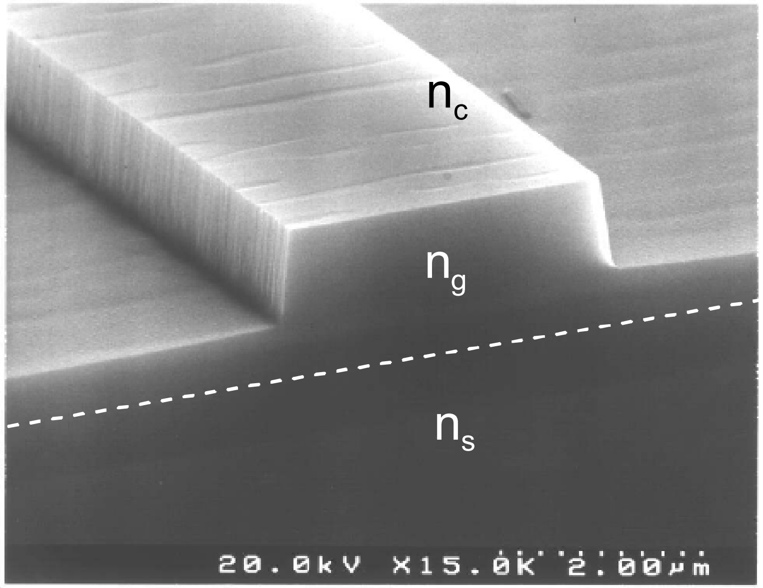
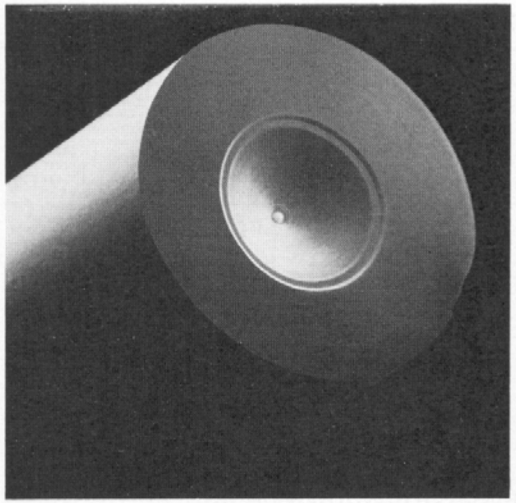
3 Guided waves in optical waveguides

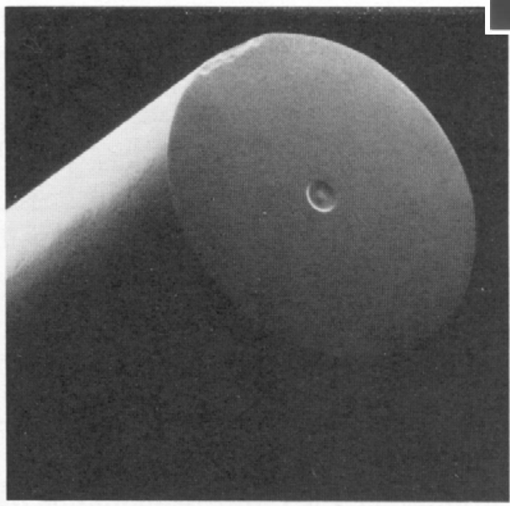
19/02/2010 rep





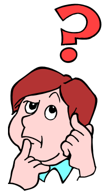
Ridge (rib) Waveguide





Multi-Mode Graded-Index Fiber (MMF)   Single-Mode Step-Index Fiber (SMF)

Goals of the chapter:



- **Investigate dielectric structures to guide and propagating waves and confine them transversally**
- **Description of light propagation in dielectric waveguide structures by mode fields** $\vec{E}_k(\vec{r},t);\vec{H}_k(\vec{r},t)$ **and propagation constant** $\beta_k(\omega)$ **assuming frequency independent dielectrics**
- **Relation of the guided wave properties, mode fields and propagation constant of the k**th **mode** $\vec{E}_k(\vec{r},t);\vec{H}_k(\vec{r},t)$ and $\beta_k(\omega)$ **to the geometric and dielectric structure of the waveguide**
- **Typical properties of dielectric waveguide structures for optical communication**
- **Frequency dependence of** $\beta_k(\omega)$ **and related dispersion effects (pulse broadening)**

Methods for the Solution:

- **Propagation of "classical" light is described as a electromagnetic (EM) wave obeying Maxwell field and material equations**
- **Solve Maxwell's equation with lateral dielectric boundary conditions of the waveguide using the Helmholtz equation** (eigenvalue problem) **and longitudinal and transversal decomposition** (variable reduction)

  ➡ **modes are eigenfunctions (time- and space dependent EM vector fields** $\vec{E}_k(\vec{r},t);\vec{H}_k(\vec{r},t)$ **) and the propagation constant** $\beta_k(\omega)$ **the corresponding eigenvalue**
- **Find modal dispersion** $D_{mode,k}(\omega)$ **from** $\beta_k(\omega)$ **for harmonic waves and pulse broadening effects**

Remark: Repetition

All material of Chap.3 on Maxwell's equations and dielectric waveguides has been treated in "Fields and Components II" by Prof. R.Vahldieck / Dr. P.Leuchtmann **(p.147-162) !!!**

3 Guides Waves in optical Waveguides

In chapter 2 we considered unguided, unconfined plane-waves in homogeneous dielectrics influenced by the carrier dynamics of dipoles.

Communications needs **longitudinally guided** and **transverse separated** (confined) waves in loss less dielectrics.

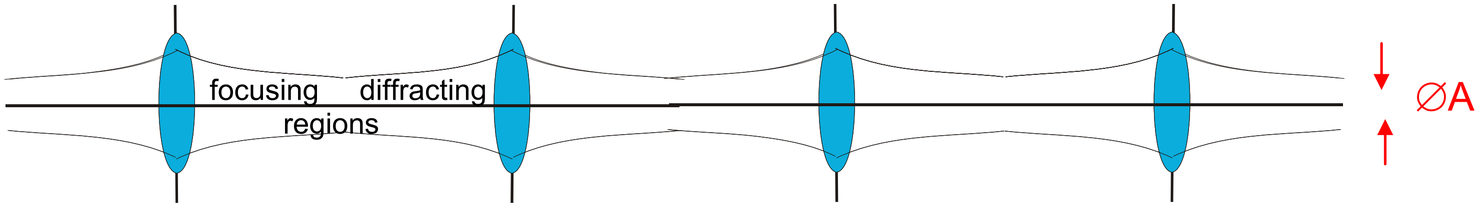
Laterally confined plane waves without dielectric guiding broaden laterally by **diffraction** !

3.1 Guiding Lightwaves – Historical Overview

Highly directed transport of light in free space is limited by attenuation and **beam broadening** due to **diffraction** and source spatial coherence.

1) Lens waveguides: **(Gobau 1960)**

Light beams can be formed and propagated by lens and mirror systems counteracting transversal diffraction



➡ but, light beam in free space are broadened by **diffraction** (beam widening) and need to be **periodically refocused** by lenses. Diffraction effects in light beams increase with decreasing beam diameter A.

2) metallic waveguides:

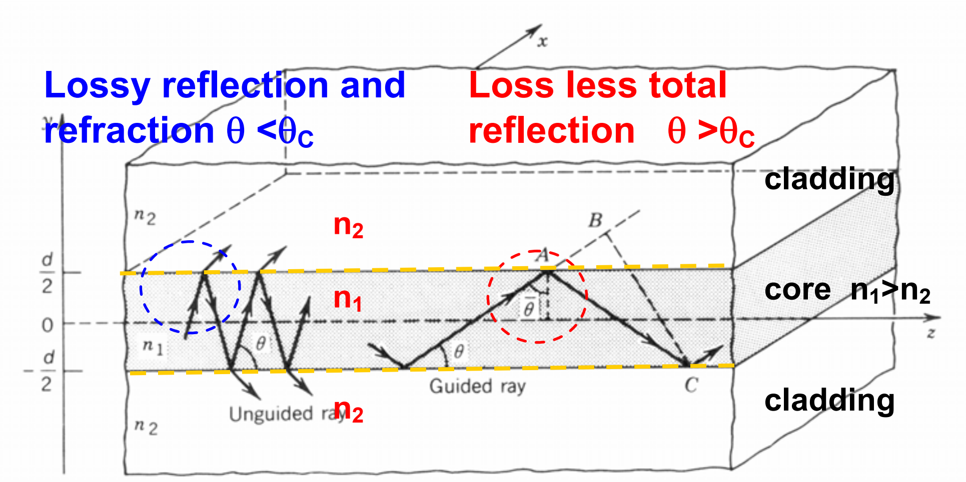
Possible conceptually, but free carrier losses in metals at optical frequencies are too high for long distances. Waveguide dimensions on the μm-scale are a technological challenge.

3) dielectric waveguides: (1966 – today)

Very low absorption and scattering losses achieved in ultra pure glasses as dielectrics
Fabrication of km-long wave guides with dimension ~ the optical wavelength λ ~1μm is feasible

Conceptual idea of light guiding by total reflection in dielectric structures:
(ray optic and total reflection picture)
use lossless **total reflections** at interfaces of 2 dielectrics with refractive indices n_2 and n_1 where

a) planar dielectric Film WG **(1D-guiding)**



Zig-zag ray propagation by total reflection requires $\mathbf{n_1 > n_2}$ (for details see chap.3.2)

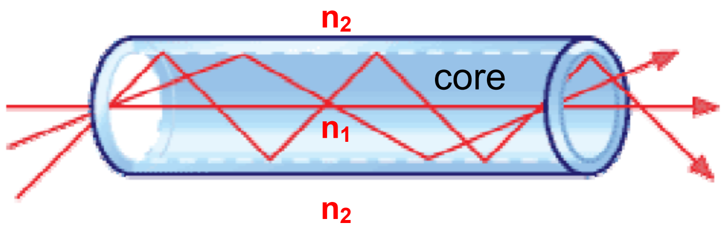
Length difference of different zig-zag paths creates substantial modal pulse broadening at the fiber end

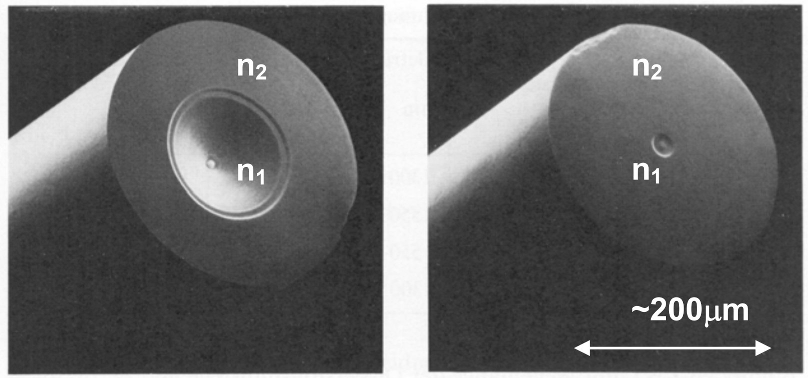
Thin core and small critical angle of total reflection θ_C resp. ($n_1 \sim n_2$) reduces dispersion effects

b) cylindrical glass fiber WG **(2D-guiding)**

- Glass fibers (after 1970)

cladding



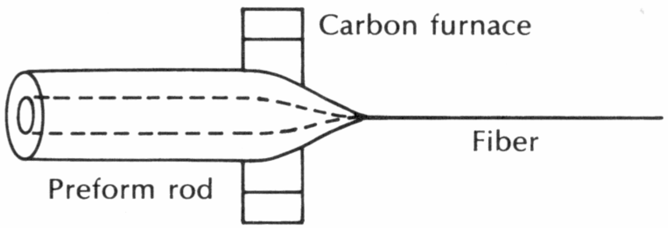


cladding Multi-Mode Fiber $\varnothing_{core}$~50µm Single-Mode Fiber $\varnothing_{core}$~8µm

$n_1 > n_2$

Glass fiber fabrication: drawing process

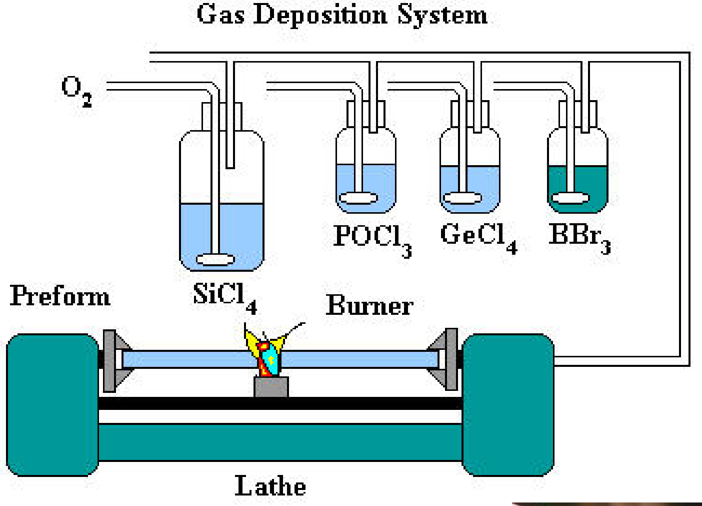
draw large diameter preforms into small fibers by local heating to the glass transition

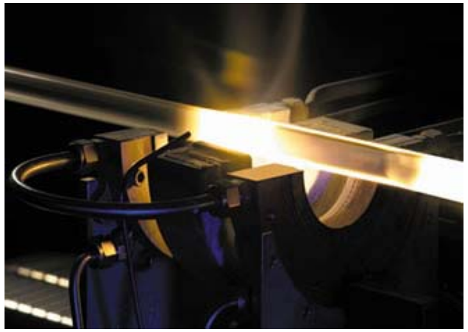


(c)

Principle: Collapse a large perform by "softening" (heating to glass transition temp.) **and drawing a 1000 x**

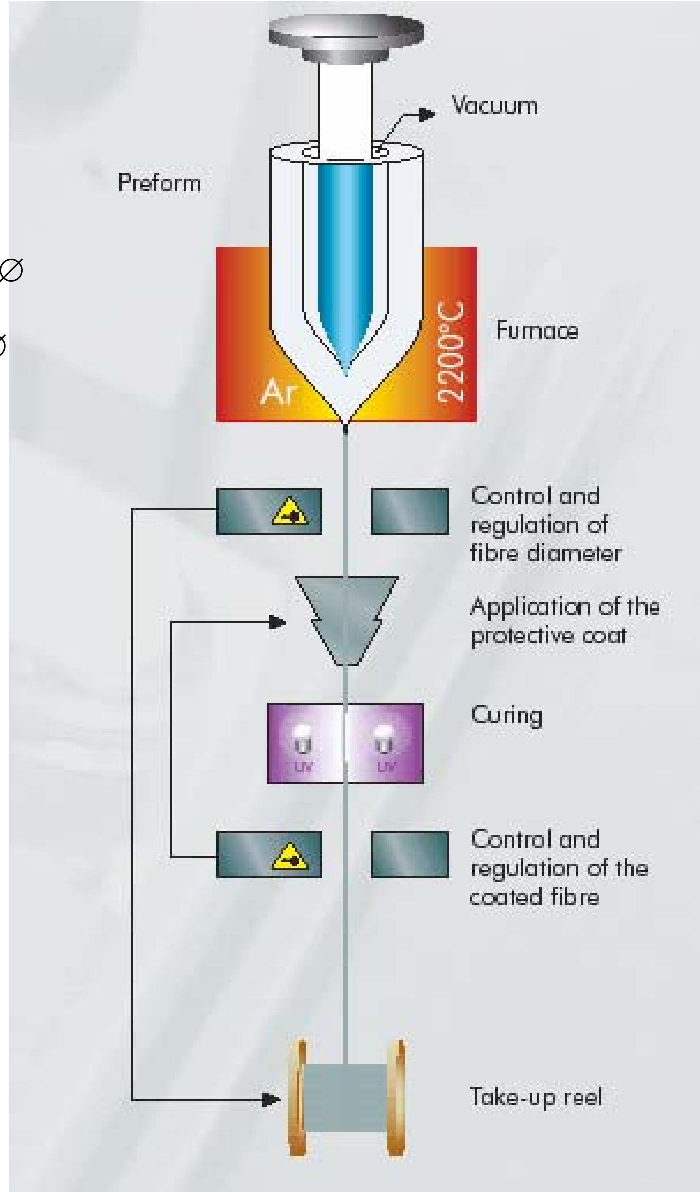
1) Preform fabrication: ∅ 10cm, L~1m ⟶ ∅ 250µm, L~ km **2) Fiber drawing and coating**





collapse heated perform, cm ∅

to fiber dimension, ~100um ∅

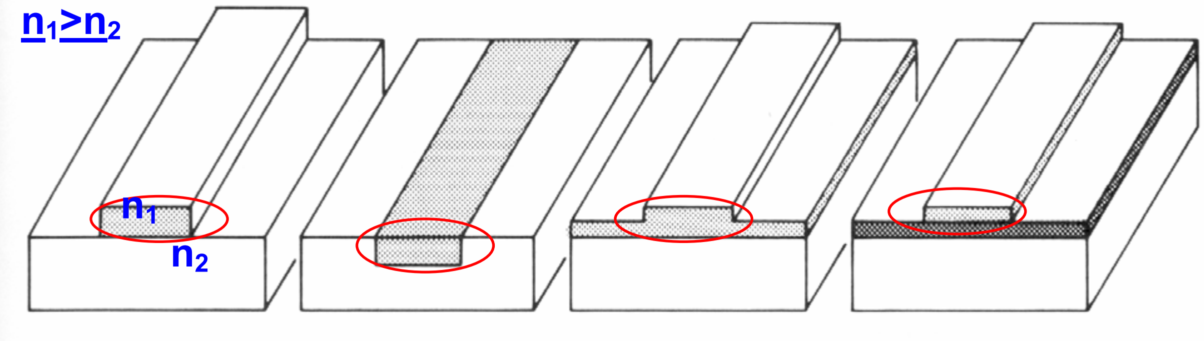


- Ultra pure materials for low light absorption below – 0.2 dB/km (glass)
- Precise geometry control low 0.1µm of ~5-10µm core diameter
- Homogenous and precise material composition
- High interface quality, low interface roughness, low scattering

Overview of different types of technical dielectric waveguides:

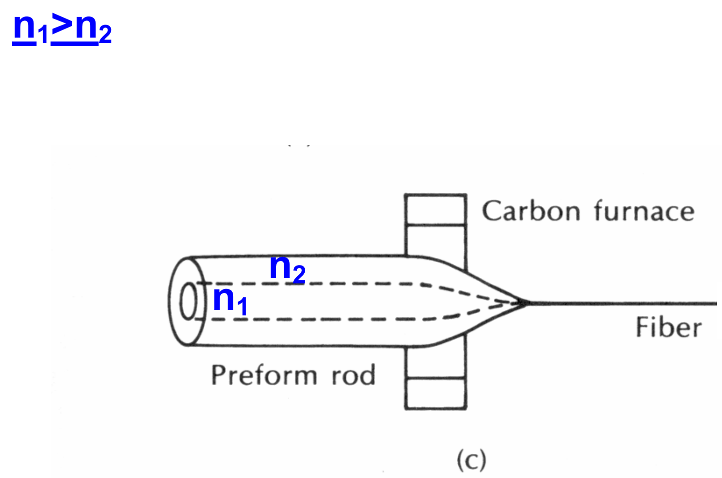
For fabrication technical reasons dielectric optical waveguides are often realized by:

a) **planar deposition** (evaporation, spinning, sputtering, epitaxial growth) of **dielectric films (glass, SC) on a substrate**:



Lateral structuring by etching, local diffusion etc.

b) **Extrusion (collapsing) of a dielectric fiber** from a heated layered cylindrical perform:
Very complex quasi-cylindrical fiber cross-sections are possible.



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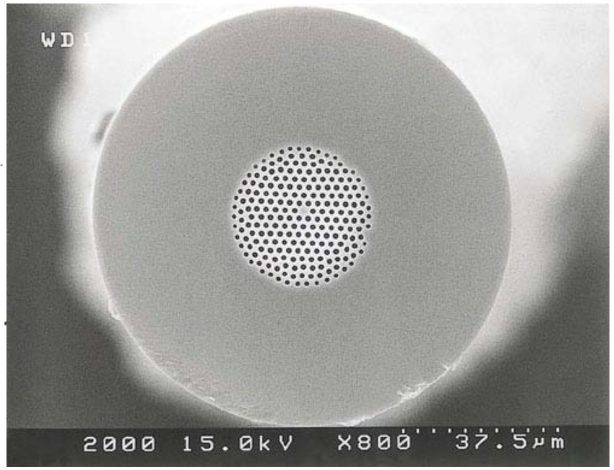
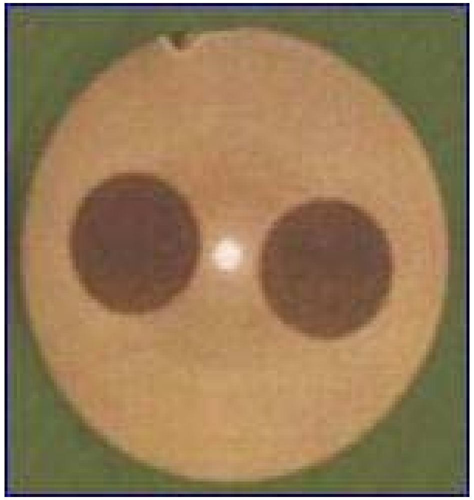
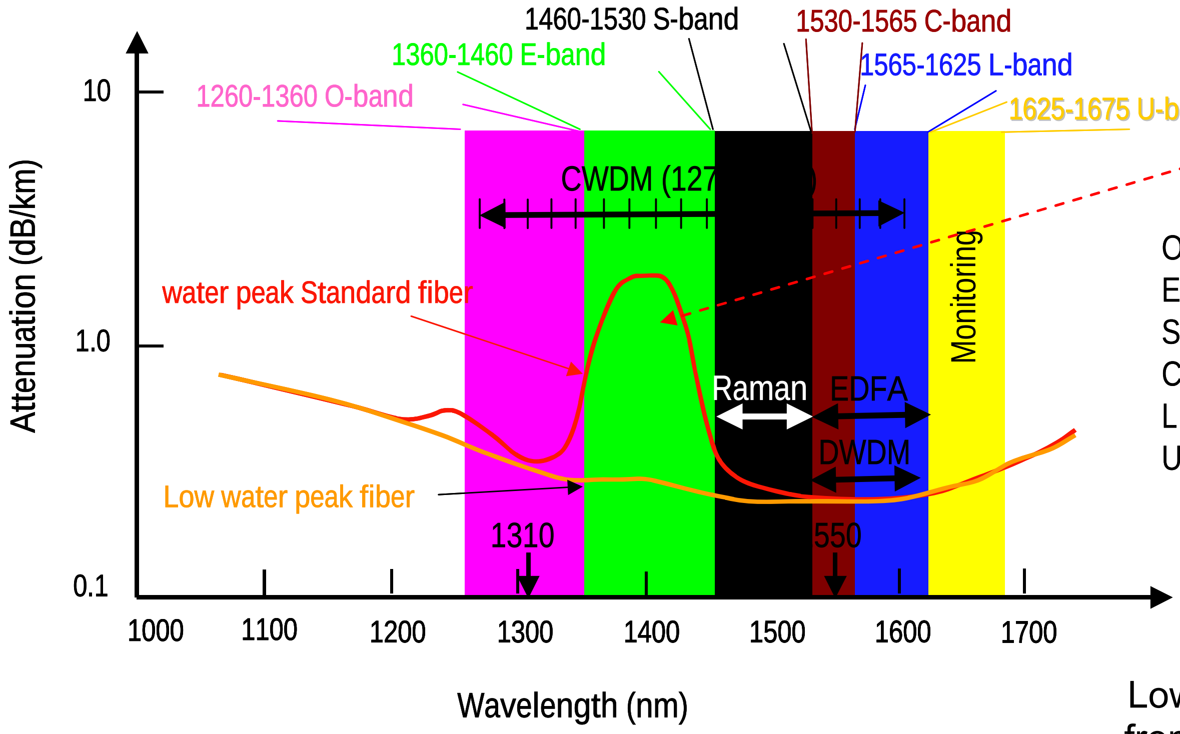


Photonic Crystal Fiber


Polarization Maintaining Fiber

Loss mechanisms in silica optical glass fibers:



Low water content fibers have a continuous band from 1200-1700nm (communication bands)

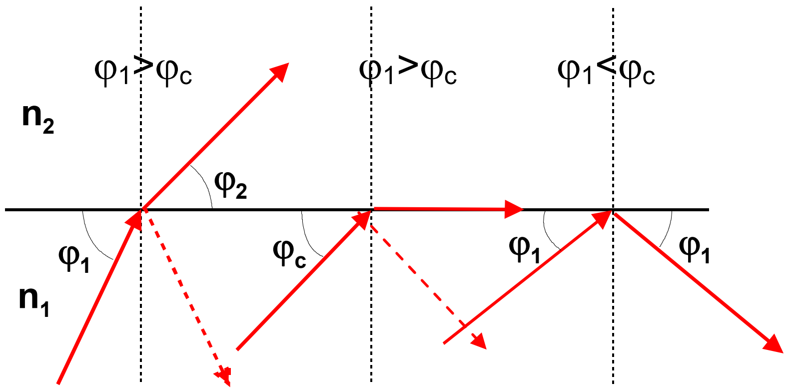
Absorption and loss mechanisms:

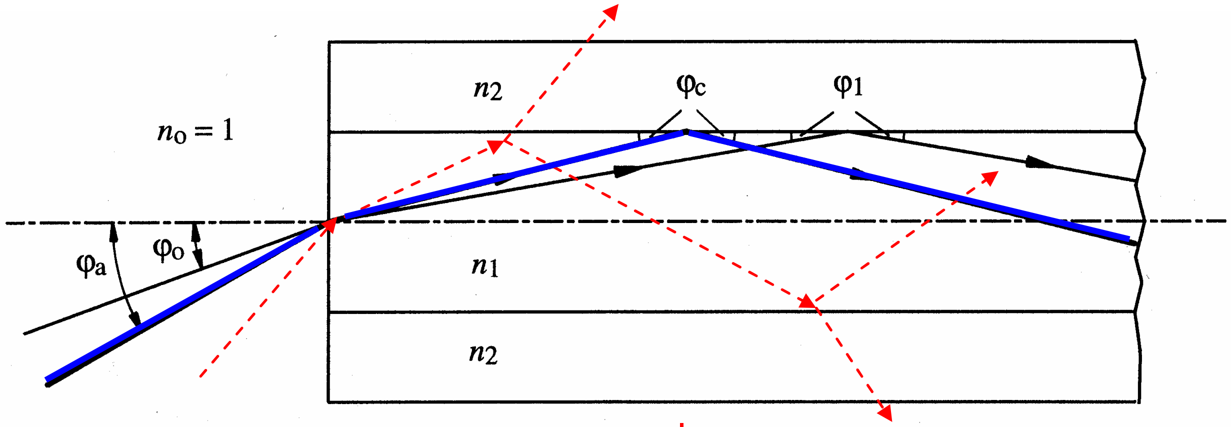
- **Absorption by impurities**, mainly OH-radicals at 0.95, 1.23 and 1.39μm wavelength
- Sub-wavelength **density fluctuations** ($\Delta l<\lambda$) ➨ Raleigh-Scattering ~1/λ
- **UV-Absorption** by electron excitation in the SiO_2-complex at ~0.3μm
- **IR-absorption** by Si-O-vibrations at ~5μm
- Geometrical **form fluctuations** ($\Delta l>\lambda$), Mie-Scattering, microbending

3.2 Ray Optics of total reflection

Light propagation in fibers with core diameters d much larger than the optical wavelength λ ($d>>\lambda$) can be approximated by the **propagation and refraction of light beams** (rays) at a dielectric interface n_1/n_2.

Total Reflection at the interface between fiber core (n_1) and cladding (n_2): ➡ lossless Zig-Zag-Transmission





Snell's Law of refraction and reflection:

$$\cos(\varphi_2)/\cos(\varphi_1) = n_1/n_2 > 0 \quad \text{and } n_1/n_2 < 1 \;➡ \qquad \text{(observe angle convention of } \varphi\text{ ! } \angle\text{surface - beam)}$$

Critical Angle φ_c for total reflection (**φ_2=0**, $\cos\varphi_2=1$) at a dielectric interface with refractive indices $n_1>n_2$:

$$\cos(\varphi_c) = \frac{n_2}{n_1} < 1 \quad ; \quad \varphi_c = \arccos\left(\frac{n_2}{n_1}\right) \xrightarrow[n_1 \sim n_2]{} \varphi_c \sim \left(\frac{n_1 - n_2}{n_1}\right) = \frac{\Delta n}{n_1}$$

$$\varphi_1 < \varphi_c \rightarrow \text{total reflection} \quad (\text{no refractionlosses})$$
$$\varphi_1 > \varphi_c \rightarrow \text{reflection and refraction}$$

Acceptance Angle φ_a at the critical angle for total internal reflection $\varphi_1=\varphi_c$

At the entrance interface (n_o, n_1) of the fiber the incoming beams (φ_0) are mainly refracted according to Snells-Law: (reflection at the air/glass interface is only ~4%). n_0 is the refractive index of the medium at the entrence.

$$\underline{n_o \sin(\varphi_a) = n_1 \sin(\varphi_c)} = n_1\sqrt{1-(n_2/n_1)^2} = \sqrt{n_1^2 - n_2^2}$$

(limiting situation for total reflection $\varphi_0=\varphi_a$)

(angle convention: $\angle$surface normal – beam)

<u>All beams with an entrance angle</u> $\varphi_0<\varphi_a$ are propagated lossless by total reflections through a <u>straight</u> fiber.

Beams with $\varphi_0>\varphi_a$ suffer refraction losses into the cladding and are attenuated by refraction losses.

For fiber characterization the numerical aperture NA is defined as a figure of merit:

$$NA = \sin(\varphi_a) = \frac{1}{n_0}\sqrt{n_1^2 - n_2^2} \xrightarrow{n_0=1} \approx \frac{n_1}{n_0}\sqrt{2\Delta} \quad \text{with} \quad \Delta = \frac{n_1 - n_2}{n_1}$$

Δ relative refractive index difference between core and cladding

➡ small Δn gives a small NA which is more difficult to couple light into, but modal dispersion from zig-zag propagation is less (trade-off)

Conclusions from the simple ray-model of the optical fiber:

- All light beams entering the straight fiber within the cone ($\varphi<\varphi_a$) defined by the NA are transmitted lossless by total reflections and exit the fiber within the NA-cone
- Large index differences Δ between core and cladding result in large φ_C and NAs and high coupling efficiency between light source and fiber.

  Time delays Δt (dispersion) between the different Zig-Zag-paths becomes large ➡ trade-off Δ, d ↔ Δt

  Multimode (MM) step index fiber: Δ = 1 – 3% NA~0.4
  Single mode (SM) step index fibers: Δ = 0.2 – 1% NA=0.1-0.2, φ_a=12.2^o mit n_1=1.5 and Δ=1%
- Strong bending of the fiber can result in a violation of the total reflection condition at the bends and the light beams can exit the fiber core (bending losses)

The ray model fails if λ~d (modes in SM-fibers) and does not provide the light intensity distributions (mode intensity profiles) correctly and also the longitudinal propagation constant becomes wrong.

➡ **needed: vector wave description of light propagation in cylindrical or rectangular dielectric structures governed by Maxwell's equations.**

3.3 Wave propagation in cylindrical optical waveguides

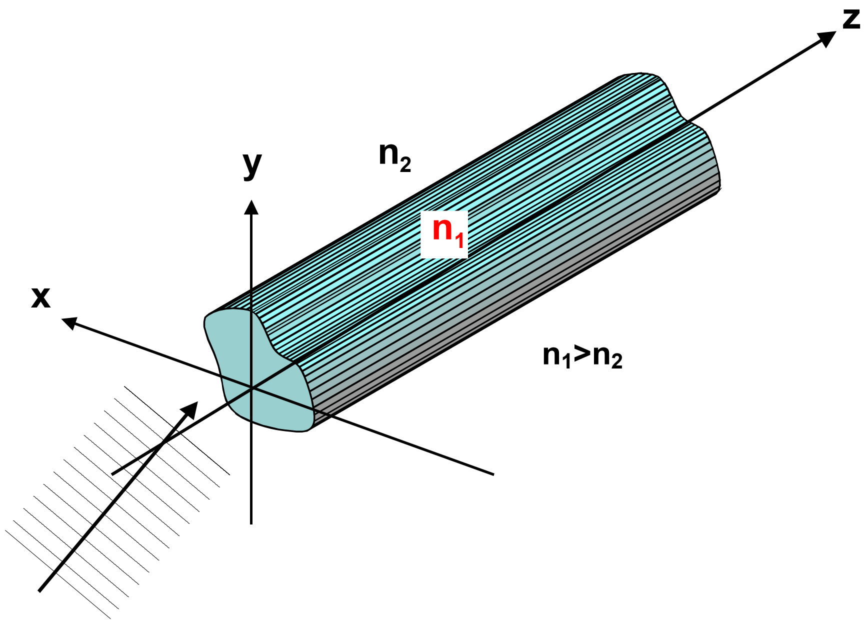
Waveguides for signal transmission must **propagate** a wave **longitudinally** in the z-direction ($\beta_Z(\omega)$) and should **confine** the wave (resonance-like) in the **transverse** T-direction (x,y-plane).

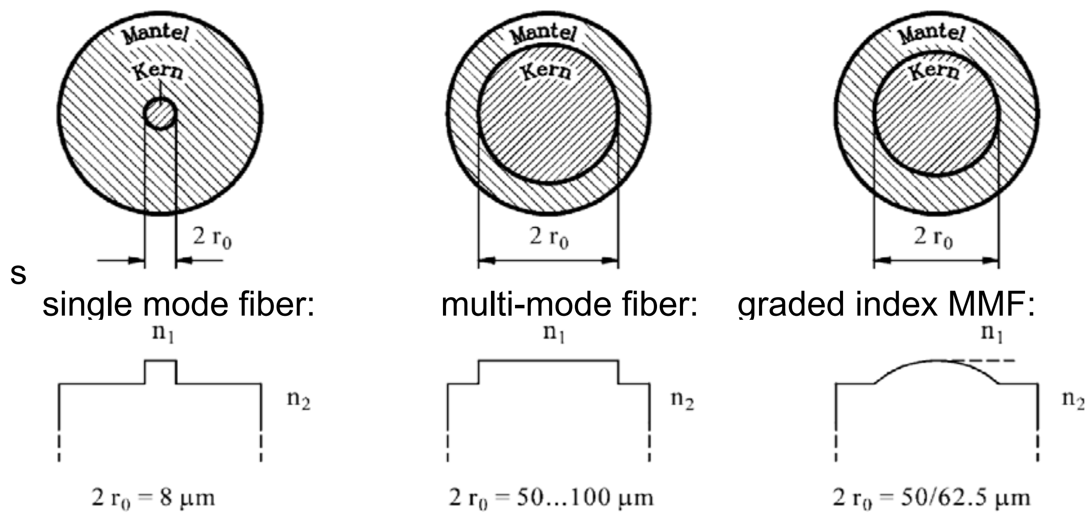
Question: how will the **transverse confinement** (n_1,n_2) of the wave influence the propagation in the z-direction $\beta_Z(\omega$, diel. geometry) ?

➨ dielectric optical fibers have often a **cylindrical structure**, with

1) a homogenous refractive index n in the **longitudinal** z direction $\rightarrow$ $n(x,y) \neq n(z)$.

2) a inhomogeneous refractive index profile in the **transverse** plane n(x,y) for lateral confinement (high index core).





Goal:

find all EM-modes $\vec{E}_i(\vec{r},t), \vec{H}_i(\vec{r},t)$ and their propagation constant β_i at a given frequency ω supported by the cylindrical WG-structure with a transverse index profile n(x,y) by solving Maxwell's equations.

Concept of analysis procedure: what do we want to achieve ?

We are considering lossless dielectric structures where the refractive index distribution n(x,y,z) does not change in the propagation direction z but only has a distribution in the transverse directions x,y ➨ n(x,y)

The transverse distributions n(x,y) consists of areas where the refractive index is different but constant.

➨ it can therefore be expected that the transversal field profile does not change transversally
➨ we restrict our investigated mode solutions to only the z-guided modes and do not consider any other possible solutions of Maxwell's equations. The time dependence is assumed to be harmonic with ω.

$$\vec{E}(\vec{r},t)=\vec{E}(\vec{r})e^{j\omega t}=\vec{E}(\vec{r}_T)e^{-jk_z z}e^{j\omega t}=\vec{E}(\vec{r}_T)e^{j\omega t-jk_z z} \quad \left(right\ \ propagating\ \vec{E}-wave\right)$$

Next we will show that the 6 vector components of E and H are related and only 2 components are independent – we can restrict the solutions further by assuming that 1 component should be zero

Separating the space vector r and the field vectors E, H in longitudinal (z) and transverse components (T) we will show that the field in a homogeneous region of constant refractive index n obeys a **Helmholtz-Eigenvalue** equation with the **transverse propagation constant k_T as Eigenvalue**:

$$\left(\Delta-\mu\varepsilon\omega^2\right)\vec{E}(\vec{r},t)=\left(\Delta-k_0^2(\omega)n^2\right)\vec{E}(\vec{r},t)=0 \quad \text{Helmholtz}-\text{equation}$$

$$\left(\Delta_T+k_T^2\right)\vec{E}(\vec{r}_T)=0 \qquad \text{with} \quad k_T^2=k^2-k_z^2$$

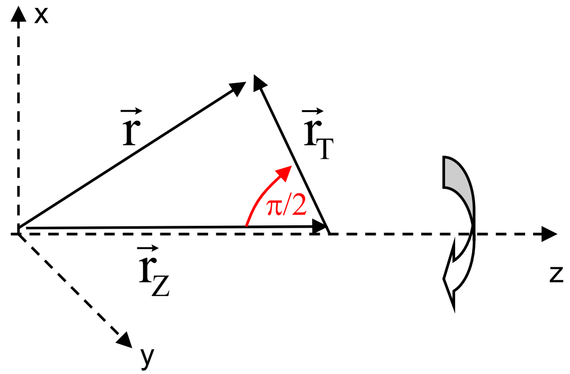
$$\left(\Delta_T+k_T^2\right)E_z(\vec{r}_T)=0$$

3.3.1 Maxwell's-Equations for EM-waves in cylindrical dielectric WGs:

Goal: Simplify Maxwell's equations for cylindrical coordinates by
→ transversal (x,y) – longitudinal (z) field decomposition,
→ use of minimum of independent vector components

Assumptions:

- dielectrics are free of fixed space charges, $\rho = 0$
- there are no convection currents flowing in the dielectric (isolator), $\vec{j} = \sigma\vec{E} = 0$
- the dielectrics are isotropic, $\varepsilon = \varepsilon_0\varepsilon_r\ ;\quad \varepsilon_r = \text{scalar}$
- the dielectrics are non-magnetic, $\mu = \mu_0\ ;\quad \mu_r = 1$
- consider only guided modes in the z-directions (incomplete set)



1) Separation of the longitudinal (z) and lateral (T), (x,y) geometry:

$$\vec{r} = (x,y,z) = \vec{r}_T + \vec{r}_z = \vec{r}_T + r_z\vec{e}_z$$

$$\vec{r}_T \text{ and } \vec{r}_z \text{ are orthogonal}: \vec{r}_T \bullet \vec{r}_z = 0$$

2) Maxwell Vector-Field Equations (MW) in a homogeneous dielectric:

$$\nabla\times\vec{E} = -\mu\frac{\partial}{\partial t}\vec{H} \quad ; \quad \nabla\times\vec{H} = -\varepsilon\frac{\partial}{\partial t}\vec{E}$$

$$\nabla\vec{E} = 0 \quad ; \quad \nabla\vec{H} = 0$$

with $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and $\vec{X} \equiv \vec{X}(\vec{r}, t)$

3) Materials Equations:

$$\vec{D} = \varepsilon \vec{E} = \varepsilon_0 \varepsilon_r \vec{E}$$

$$\vec{B} = \mu_0 \mu_r \vec{H} \underbrace{=}_{\mu_r = 1} \mu_0 \vec{H}$$

➡ 6 field variables: $E_x(\vec{r},t), E_y(\vec{r},t), E_z(\vec{r},t)$ and $H_x(\vec{r},t), H_y(\vec{r},t), H_z(\vec{r},t)$

Question: how many are independent ?

Elimination of one field variable from MW's equations leads to the

Homogeneous Vector Wave Equations: (derivation see Dr. Leuchtmann: F&K I)

$$\left(\Delta - \mu\varepsilon \frac{\partial^2}{\partial t^2}\right)\vec{E}(\vec{r},t) = 0 \quad ; \quad \left(\Delta - \mu\varepsilon \frac{\partial^2}{\partial t^2}\right)\vec{H}(\vec{r},t) = 0 \quad \text{with} \quad \Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = \nabla^2 \qquad \mu, \varepsilon \neq f(r)$$

Assuming that the fields are excited by sources with a **harmonic time dependence** $e^{+i\omega t}$ leads to

4) Harmonic field solutions with a separation of space $\vec{r}$ and time t dependence:

$$\vec{E}(\vec{r},t) = Re\left\{\vec{E}(\vec{r})e^{i\omega t}\right\} \equiv \frac{1}{2}\vec{E}(\vec{r})e^{i\omega t} + \frac{1}{2}\vec{E}^*(\vec{r})e^{-i\omega t} \quad ; \quad \vec{X}(\vec{r}) = \text{complex,}\underline{\text{spatial only}}\text{ Phasor of the vectorfield } \vec{X}(\vec{r},t)$$

$$\vec{H}(\vec{r},t) = Re\left\{\vec{H}(\vec{r})e^{i\omega t}\right\} \equiv \frac{1}{2}\vec{H}(\vec{r})e^{i\omega t} + \frac{1}{2}\vec{H}^*(\vec{r})e^{-i\omega t} \quad ; \quad * = \text{conjugate complex, cc.}$$

5) Homogeneous Helmholtz Equation (eigenvalue equation) for the <u>spatial</u> Functions $\vec{E}(\vec{r}); \vec{H}(\vec{r})$:

For the harmonic time dependence $e^{i\omega t}$ the time-derivation operators transform as

$$\frac{\partial}{\partial t} \longrightarrow +i\omega \quad ; \quad \frac{\partial^2}{\partial t^2} \longrightarrow -\omega^2 \;\Rightarrow$$

$$\left(\Delta + \mu\varepsilon\omega^2\right)\vec{E}(\vec{r}) = 0; \quad \left(\Delta + \mu\varepsilon\omega^2\right)\vec{H}(\vec{r}) = 0 \xrightarrow{\text{Def.: } \mu\varepsilon\omega^2 = k(\omega)^2 = k_0(\omega)^2 n^2} \boxed{\left(\Delta + k(\omega)^2\right)\vec{E}(\vec{r}) = 0 \quad ; \quad \left(\Delta + k(\omega)^2\right)\vec{H}(\vec{r}) = 0}$$

Helmholtz Equation (eigenvalue equation)

Eigenvalue equation with the eigenvalue k and the eigenfunction $\vec{E}(\vec{r})$, *resp.* $\vec{H}(\vec{r})$ and a generic

plane wave solution: $\vec{E}(\vec{r}) \sim e^{-j\vec{k}\vec{r}}$ with the **propagation vector** $\vec{k}$ and $|k| = 2\pi/\lambda$

Spherical waves would be an other simple solution.

Question: do we have to solve the Helmholtz-Equation for all 6 field component ?

For a simplification the cylindrical geometry (homogeneous in the z direction) suggests a formal decomposition of the vector operators into **spatial z-** (longitudinal) **and T -** (transversal, x,y**) operators and vectors:**

Definition: $\Delta = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + \left(\frac{\partial^2}{\partial z^2}\right) = \Delta_T + \Delta_z$

$$\left(\Delta_T + \Delta_z + k^2\right)\vec{E} = 0 \quad ; \quad \left(\Delta_T + \Delta_z + k^2\right)\vec{H} = 0$$

formal decomposition : $k^2 = k_T^2 + k_z^2 = \mu\varepsilon\omega^2 = \text{scalar} \qquad (k_T, k_z \textit{ is not defined yet})$

$$\left(\Delta_T + \Delta_z + k_T^2 + k_z^2\right)\vec{E}(\vec{r}) = 0 \quad ; \quad \left(\Delta_T + \Delta_z + k_T^2 + k_z^2\right)\vec{H}(\vec{r}) = 0$$

Solution-"Ansatz" for the Helmholtz-Equation for a z-guided wave:

Transverse field pattern is propagated in the z-direction

Guided wave "Ansatz": $\underline{\vec{E}(\vec{r},t) = \vec{E}(\vec{r})e^{j\omega t} = \vec{E}(\vec{r}_T)e^{-jk_z z}e^{j\omega t} = \vec{E}(\vec{r}_T)e^{j\omega t - jk_z z} \quad (right\ propagating)}$

Inserting the guided wave into the Helmholtz-equations:

$$\left(\Delta + k^2\right)\vec{E}(\vec{r}) = 0 \quad ; \quad \left(\Delta + k^2\right)\vec{H}(\vec{r}) = 0$$

$$\left(\Delta_T + \Delta_z + k^2\right)\vec{E} = 0 \quad \textit{dropping } e^{j\omega t} \textit{ and using the assumption } \vec{E}(\vec{r}) = \vec{E}(\vec{r}_T)e^{-jk_z z}$$

$$\left(\Delta_T\vec{E}(\vec{r}_T)e^{-jk_z z} + \Delta_z\vec{E}(\vec{r}_T)e^{-jk_z z} + k^2\vec{E}(\vec{r}_T)e^{-jk_z z}\right) = \left(\Delta_T\vec{E}(\vec{r}_T)e^{-jk_z z} - k_z^2\vec{E}(\vec{r}_T)e^{-jk_z z} + k^2\vec{E}(\vec{r}_T)e^{-jk_z z}\right) = 0$$

$$\left(\Delta_T + k^2 - k_z^2\right)\vec{E}(\vec{r}_T)e^{-jk_z z} = \left(\Delta_T + \left(k^2 - k_z^2\right)\right)\vec{E}(\vec{r}) \underbrace{=}_{\text{Def.: } k^2 - k_z^2 = k_T^2} \left(\Delta_T + k_T^2\right)\vec{E}(\vec{r}) = 0$$

longitudinal (z) and Transverse (T) Helmholtz-Equation for cylindrical waveguide:

(similar procedure for the H-field)

$$\boxed{\begin{aligned} &\left(\Delta_T + k_T^2\right)\vec{E}(\vec{r}) = 0 \\ &\left(\Delta_T + k_T^2\right)\vec{H}(\vec{r}) = 0 \\ &k^2 - k_z^2 = k_T^2 \end{aligned}}$$

Eigenvalue problem for k_T and k_z with

the Eigenfunctions $\vec{E}(\vec{r},\vec{k})$, resp. $\vec{H}(\vec{r},\vec{k})$

$$\boxed{\Rightarrow \quad \begin{aligned} &Solution: \quad \vec{E}(\vec{r}_T) \sim e^{i\vec{k}_T\vec{r}_T} \\ &nontrivial \end{aligned}}$$

The Eigenfunctions $\vec{E}(\vec{r},\vec{k})$, resp. $\vec{H}(\vec{r},\vec{k})$ **are called the modes of the field.**

Resp. :

$$\left(\Delta_T + k_T^2\right)\vec{E}\left(\vec{r}_T\right) = 0$$

$$\left(\Delta_T + k_T^2\right)\vec{H}\left(\vec{r}_T\right) = 0 \quad \text{with} \quad k^2 - k_z^2 = k_T^2$$

For the z-dependence we have assumed: $\Rightarrow \quad \boxed{\vec{E}\left(\vec{r}\right) \sim e^{ik_z z}}$

$$\text{with } \left|\vec{k}\right|^2 = \omega^2 \mu\varepsilon\left(\vec{r}_T\right) = k_T^2 + k_z^2$$

Solution space of the eigenvalues k_T and k_z for a given $k(\omega)$:

k is per definition real and positive !

1) k_z = real ➡ z-propagating wave (desired)
k_T real or imaginary ➡ transverse oscillatory or decaying wave

2) k_z = complex ➡ z-decaying wave
k_T real or imaginary ➡ transverse oscillatory or decaying wave

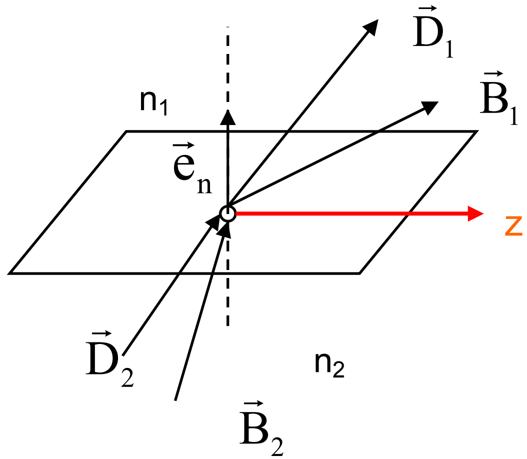
3) general case: k_z and k_T are imaginary fulfilling $k(\omega)^2 = k_z^2 + k_T^2$

See the discussion for planar film WG in chap.3.4.2.

The eigenvalue equation for H and E are decoupled, but the 2-fields are related by boundary conditions !

In addition the solutions must fulfill from Maxwell's eq. the transversal

boundary continuity conditions at the transverse interfaces



a) Normal components : $\vec{B}_{\perp};\vec{D}_{\perp}$ are continuous

$$\vec{e}_n\left(\vec{B}_2-\vec{B}_1\right)=0 \quad ; \quad \vec{e}_n\left(\vec{D}_2-\vec{D}_1\right)=\sigma_F=0$$

b) Tangential components : $\vec{E}_{-};\vec{H}_{-}$ are continuous

$$\vec{e}_n\times\left(\vec{E}_2-\vec{E}_1\right)=0 \quad ; \quad \vec{e}_n\times\left(\vec{H}_2-\vec{H}_1\right)=\vec{j}_F=0$$

(no proof)

➨ Total guided z-propagating plane wave solution:

The total time- and spatial dependent solutions for the E (H)-field of cylindrical, transverse inhomogeneous WGs from the solution of the Helmholtz-equation becomes:

Right (left) propagating wave: $\vec{E}(\vec{r}\,)e^{+i(\omega t)}=\vec{E}(\vec{r}_T)e^{i\left(\omega t\mp\vec{k}(\omega)\vec{r}\right)}=\vec{E}e^{i\vec{k}_T\vec{r}_T}\,e^{i\left(\omega t\mp\vec{k}_z(\omega)z\right)}$

with the phase velocity $v_{ph,z}(\omega)=\omega/k_z(\omega)$

standing transverse wave like
Eigenfunction (transverse mode)

z-propagating wave

The eigenvalues k_T and k_Z are not independent but coupled to $k(\omega)$ for a given ω.

Reminder:

The above plane wave solutions with standing waves in the transverse direction do not represent not all possible wave solutions in the WG structure.
We assumed guided waves along the z-axis in the "solution-Ansatz" reducing the solution space of the problem artificially.

Interpretation of the solutions:

- **the transverse Helmholtz-Eq. defines an Eigenvalue-problem for the propagation vector $k_T(\omega)$ resp. $k_z(\omega)$**
- **k_Z describes the spatial dependence in the z- , k_T the transverse direction**
- **the longitudinal propagation constant $k_z(\omega)$ is influenced by the transversal solutions k_T , resp. by the transverse dielectric WG structure, because $k^2 = k_T^2 + k_z^2$ must hold.**

  But $k(\omega)$ is also a material property.
- **$k_z(\omega)$** and $k_T(\omega)$ **will define the frequency dependence of the propagation properties**

  ➡ mode-dispersion relation $k_z(\omega)$

3.3.2 Separation of Longitudinal and Transversal Field components:

The next step is to see if we have to solve the Helmholtz equations for all 6 vector components or if there are relations between them reducing the number independent variables.

The solution of the EM-vector field has 6 vector components (E_x, E_y, E_z, H_x, H_y, H_z), which are not all independent.

$$\left(\Delta_T + k_T^2\right)E_i(\vec{r}) = 0 \quad ; \quad \left(\Delta_T + k_T^2\right)H_i(\vec{r}) = 0 \qquad i = x,y,z$$

$$\left(\Delta_z + k_z^2\right)E_i(\vec{r}) = 0 \quad ; \quad \left(\Delta_Z + k_z^2\right)H_i(\vec{r}) = 0$$

Helmholtz-equations

Question: Possibility to solve for a fraction of components (eg. 2 out of 6) and derive the rest by mutual relations ?

What is the **minimum number of independent field components** ?

Without prove (appendix 3 B) we state that any vector field $\vec{v}$ satisfies the following 2 universal vector relations 1) , 2):

$$\vec{v} = \vec{v}_T + \vec{v}_Z = \vec{v}_T + v_Z \vec{e}_z \quad \Rightarrow$$

$$1) \quad \vec{v}_T = \vec{e}_z \times \vec{v} \times \vec{e}_z$$

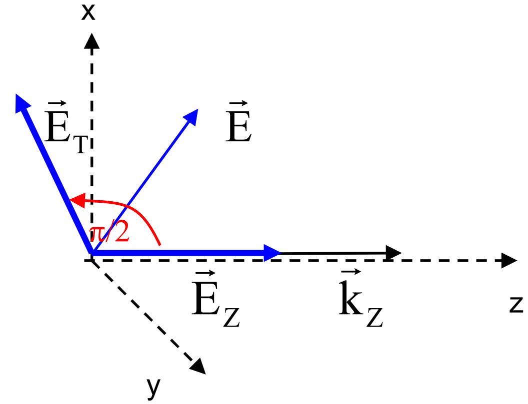
$$2) \quad \vec{e}_z \, v_z = \vec{e}_z \cdot \vec{v} \cdot \vec{e}_z$$

The equations 1) and 2) define relations between longitudinal and transversal field components $\vec{v} \rightleftarrows \vec{v}_T, \vec{v}_Z$ allowing the reduction of the number of independent field components.

T- and z-decomposition of the vector-fields:

$$\vec{E}(\vec{r}) = \vec{E}_T(\vec{r}) + \vec{E}_Z(\vec{r}) = \left(E_x, E_y, 0\right) + \left(0,0,E_z\right)$$

$$\vec{H}(\vec{r}) = \vec{H}_T(\vec{r}) + \vec{H}_Z(\vec{r})$$



T- and z-decomposition of vector-operations:

Expressing the vector-operations in Maxwell's-equations for a field decomposed into transversal and longitudinal components $\vec{v} = \vec{v}_T + \vec{v}_Z = \vec{v}_T + v_Z \vec{e}_z$: (without proof, appendix 3 B)

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \nabla_T + \nabla_Z = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0 \right) + \left(0, 0, \frac{\partial}{\partial z} \right)$$

$$\overrightarrow{grad}: \quad \nabla s = \nabla_T s + \vec{e}_z \cdot \frac{\partial}{\partial z} s$$

$$div: \quad \nabla \cdot \vec{v} = \nabla_T \cdot \vec{v}_T + \frac{\partial}{\partial z} v_z$$

$$\overrightarrow{rot}: \quad \nabla \times \vec{v} = \left(\nabla_T \cdot \left(\vec{v}_T \times \vec{e}_z \right) \right) \cdot \vec{e}_z + \left(\nabla_T v_z - \frac{\partial}{\partial z} \vec{v}_T \right) \times \vec{e}_z$$

1) replacing the vector operators $\nabla = \nabla_T + \nabla_Z$ in Maxwell's-equations, eg. for $\vec{E}$ leads to:

$$\nabla \times \vec{E} = \left(\nabla_T \cdot \left(\vec{E}_T \times \vec{e}_z \right) \right) \cdot \vec{e}_z + \left(\nabla_T E_z - \frac{\partial}{\partial z} \vec{E}_T \right) \times \vec{e}_z = -i \omega \mu \vec{H} = -i \omega \mu \left(\vec{H}_z + \vec{H}_T \right)$$

2) separating into transversal (T) and longitudinal (z) components:

$$\left(\nabla_T E_z - \frac{\partial}{\partial z} \vec{E}_T \right) \times \vec{e}_z = -i \omega \mu \vec{H}_T \qquad (*t)$$

$$\nabla_T \cdot \left(\vec{E}_T \times \vec{e}_z \right) = -i \omega \mu H_z \qquad (**l)$$

3) vector multiplying $\boldsymbol{e}_z \times (*)$ and using $\boldsymbol{e}_z \times \boldsymbol{v}_T \times \boldsymbol{e}_z = \boldsymbol{v}_T$ and applying $\partial/\partial z \to -i\, k_z$ gives:

$$k_z \vec{E}_T - i \nabla_T E_z = -\omega \mu \left(\vec{e}_z \times \vec{H}_T \right)$$

4) from the second equation (**) by using $\boldsymbol{v}_T \times \boldsymbol{e}_z = -\boldsymbol{e}_z \times \boldsymbol{v}_T$, we obtain:

$$\nabla_T \cdot \left(\vec{e}_z \times \vec{E}_T \right) = i\omega\mu H_z$$

Applying the same transforms to the $\boldsymbol{H}$-field results in the

Maxwell's-equation separated by transversal and longitudinal vector-operators:

Vector relations transversal and longitudinal field components

$$\boxed{\begin{aligned} k_z\vec{E}_T - i\nabla_T E_z &= -\omega\mu\left(\vec{e}_z \times \vec{H}_T\right) && eq.1 \\ k_z\vec{H}_T - i\nabla_T H_z &= \omega\varepsilon\left(\vec{e}_z \times \vec{E}_T\right) && eq.2 \\ \nabla_T \cdot \left(\vec{e}_z \times \vec{E}_T\right) &= i\omega\mu H_z && eq.3 \\ \nabla_T \cdot \left(\vec{e}_z \times \vec{H}_T\right) &= -i\omega\varepsilon E_z && eq.4 \end{aligned}}$$

Interpretation:

- 4 relations between transversal and longitudinal field components

  4 relations and 6 variable

  ➡ **only 2 independent field variables**
- the 4 other dependent field variables can be derived from the 2 independent ones

  ➡ **we chose the <u>longitudinal components Ez, Hz</u> as <u>independent</u> field variables**

Solution-Procedure for Maxwell's equations:

Concept:

Solve Helmholtz-Eigenvalue equations (if possible) for the longitudinal E_Z- and H_Z-components and the determine the transversal components E_T and H_T by the relations (eq.1-4).

1) Elimination of the 4 transversal field components (E_T, H_T) ➡ **E_z and H_z are independent field variables** (no proof)

$$\boxed{\begin{aligned} \left(\Delta_T + k_T^2\right) E_z\left(\vec{r}_T\right) &= 0 \\ \left(\Delta_T + k_T^2\right) H_z\left(\vec{r}_T\right) &= 0 \end{aligned}}$$

2-dimensional Helmholtz-Equation for longitudinal E_Z, H_Z (eigenvalue equation for k_T)

2) Solve for E_Z, H_Z and $k_T(\omega)$ → k_T , E_X, H_X, E_Y, H_Y from the eigenvalue equation obtained from the matching of the boundary continuity conditions

3) Express **transversal** field components E_T, H_T by the **longitudinal E_z, H_z**:

$$\boxed{\begin{aligned} \vec{E}_T &= \frac{1}{ik_T^2}\left\{ k_z \cdot \nabla_T E_z - \omega\mu\left(\vec{e}_z \times \nabla_T\right) H_z \right\} \\ \vec{H}_T &= \frac{1}{ik_T^2}\left\{ k_z \cdot \nabla_T H_z + \omega\varepsilon\left(\vec{e}_z \times \nabla_T\right) E_z \right\} \end{aligned}}$$

longitudinal z (k_Z, E_Z, H_Z) → transversal T Transform (k_T, E_T, H_T)

4) Using the relation between k, k_T, k_Z provides the calculation of the longitudinal propagation constant k_Z:

Making use of the continuity equations for the transversal and longitudinal fields gives an Eigenvalue equation for:

$$\boxed{k_T^2 = k^2 - k_z^2 = \omega^2\mu\varepsilon(\vec{r}_T) - k_z^2} \quad (3.30) \quad ➡ \quad k_z\left(\omega\right) = \sqrt{k\left(\omega\right)^2 - k_T^2}$$

5) matching the source boundary conditions will provide the absolute values for H_z and E_z

Categories of Wave Solutions (Modes):

- ***TM-Wave*** **(*transverse magnetic wave*) or *E-wave*** $E_z \neq 0, H_z \equiv 0$:

  Guided wave with only longitudinal E-field and a **purely transverse magnetic field**.

  For the solution we need only to solve the *Helmholtz-equation* for the E_z-component.

- ***TE-Wave*** **(*transverse electric wave*) or *H-wave*** $H_z \neq 0, E_z \equiv 0$:

  Guided wave with only longitudinal H-field and **a purely transverse electric field**. For the solution we need only solve the *Helmholtz-equation* for the H_z-component.

- ***Hybrid EH- or HE-Wave*** **(*transverse electric wave*) or *H-wave*** $E_z \neq 0, H_z \neq 0$:

  Guided wave with both longitudinal E- and H-fields (EH: E_z is dominating, HE: H_z is dominating). For the solution we need solve both *Helmholtz-equation* for the E_z- and H_z-components.

- ***TEM-Wave*** **(*transverse electromagnetic Wave*)** $E_z \equiv 0, H_z \equiv 0$:

  Guided wave with only transversal E- and H-fields. We can not solve the *Helmholtz-equation* for the E_z- and H_z-components. For *TEM-waves* we must directly solve the *Helmholtz-equation* for the *transverse* components.

  $$\left(\Delta_T + k_T^2\right)\vec{E}_T = 0$$

  TEM-wave often occur in weakly guiding WGs with small index difference between core and cladding.

Summary: Solutions for cylindrical waveguides

- For waveguide the Helmholtz-equations define the eigenvalue-problem with eigenvalue k_T resp. k_z, being the transverse and longitudinal propagation constants and the Eigenfunction of the transversal field distribution $\boldsymbol{E}(r_T)$, $\boldsymbol{H}(r_T)$.
- The **longitudinal** Helmholtz-equations for the longitudinal components E_z and/or H_z are formulated for the reduction of field variables. Using the field boundary conditions the solutions are evaluated.
- Depending on the selection of the field components – only H_z, only E_z or E_z and H_z combined – the corresponding mode-type is determined (*TE*-, *TM*-, or hybrid *HE*- bzw. *EH*-modes).
- The transversal components $\boldsymbol{E}_T$ and $\boldsymbol{H}_T$ are calculated from the primary solutions of E_Z and H_Z.
- Enforcing the boundary conditions for the longitudinal z- and for the corresponding transversal T-components provides the necessary Eigenvalue-equation for the propagation constants k_T and k_z.
- *TEM*-Waves are solutions of the simple transversal potential problem alone.

  This type of wave modes occurs in dielectric waveguides with very small index differences between core and cladding or in metallic multi-conductor waveguides.
- Hybrid modes are the most general solution for transverse inhomogeneous dielectric waveguides.
- Each transverse inhomogeneous, dielectric waveguide has transverse Eigensolutions, *TE*- or *TM*-Modes (no proof).

Observe that we only considered the the z-guided, confined modes by the chosen solution-Ansatz.

Concept of analysis procedure: what do we want to achieve ?

Before we derived the Helmholtz-equation of the Eigenvalue-type for a homogeneous region

In the following we will match the Eigenfunctions of the Helmholz-equations of the different dielectric regions i of a given waveguide structure for a common z-propagation vector k_Z at all interfaces of all regions.

The matching conditions provide a nonliner **eigenvalue equation** for the common propagation constant $k_Z(\omega)$ for a given ω.

The in general nonlinear function **$k_Z(\omega)$ is the dispersion relation of the WG** describing the influence (modification) of the geometrical dielectric guiding structure on the linear material dispersion relation $k(\omega)$.

The simplest wave structure is the symmetric planar film waveguide n_1-n_2-n_1 which we solve in detail to demonstrate the solution procedure and the classification of the different modal solutions.

3.4 Planar Film Waveguides (Repetition 4.sem F&K II)

For optoelectronic devices fabricated by planar IC-processes the generic *dielectric planar **slab waveguide*** consists of a 2-D **core slab** of a high refractive index n_1 and thickness 2d covered in the x-direction by two “infinitely” thick **cladding layers** of refractive index n_2, n_3.

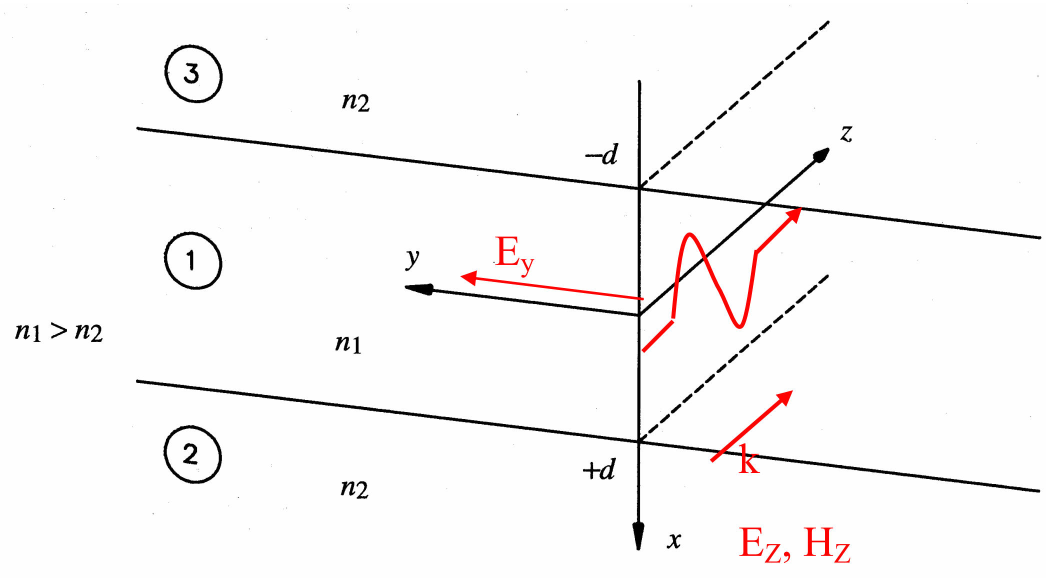
Wave guiding occurs only in the yz-plane, - the wave is confined only in the transverse x-direction.

We choose the z-direction as the propagation direction. The problem is homogeneous in the y-direction: $\partial / \partial y = 0$

3.4.1 Symmetric planar slab (film) waveguide , $n_2=n_3$

For a lossless propagating wave $e^{\pm j\beta z}$ (mode)
$k_z=\beta$ and k_i must be real with the restriction:

$$k_{T,i}^2 + k_{z,i}^2 = k_{T,i}^2 + \beta_i^2 = k_i^2(\omega)$$



➡ $\mathbf{k_{T,i}}$ can be **real** (oscillatory solution in the core) or **imaginary** (decaying solutions in the cladding)

a) Guided TE-Modes $\boldsymbol{E_z \equiv 0}$ (assumption), $\boldsymbol{H_z \neq 0}$

1) solve the 1D-Helmholtz-Equation for the $H_z(x)$-component in the loss-less medium:

$$\left(\Delta_T + k_T^2\right) H_z = \left(\frac{\partial^2}{\partial x^2} + \underbrace{k_i^2 - \beta^2}_{k_{T,i}^2} \right) H_z = 0$$

with the longitudinal propagation constant $k_z=\beta$ and n_i ($\forall\ i = 1, 2$)

$$k_i = \omega \cdot \sqrt{\mu \varepsilon_i} = \frac{2\pi}{\lambda_i} = \frac{n_i}{c}\omega = k_0 n_i > 0 \quad ; \quad k_0 = \omega \cdot \sqrt{\mu_0 \varepsilon_0} = \frac{2\pi}{\lambda_0}$$ (vacuum propagation constant)

1) Solve the Helmholtz eq. for the core and cladding layers separately and match boundary conditions at interfaces
2) $H_z(x)$ must be symmetric or anti-symmetric in the x-direction leading to harmonic solutions of the *Helmholtz-equation (plane* waves) of the form $e^{\mp j k_{Ti} x}$ with the transverse wave number $k_{Ti}^2 = k_i^2 - \beta^2$ resp. $\beta^2 = k_i^2 - k_{Ti}^2$ for the medium i. β must be the same for all layers (core and cladding).
3) Useful optical wave for signal propagation require a **transverse confined mode** therefore k_T must be imaginary in the cladding for a **decaying** transverse cladding field $H_z(\pm\infty)=0$.

   The field in the core can be oscillatory and k_T real.

   ➡ eigenvalue $k_z(\omega)=\beta(\omega)$ must fulfill the interval inequality (solution space):

$$k_1 = k_0 \cdot n_1 > \beta(\omega) > k_2 = k_0 \cdot n_2 \qquad \text{resp.} \qquad \omega / c_0 n_1 > \beta(\omega) = k_z(\omega) > \omega / c_0 n_2$$

decaying cladding wave   symmetrie condition

Interpretation: the resulting mode field distribution must “balance” the different phase velocities of core and cladding.

Solutions of the transverse Helmholtz-equation:

- ***core area ①:*** $n = n_1, \ k_1 > \beta, \ |x| < d$ ***oscillatory solutions***

$$H_z(x) = A \cdot \begin{Bmatrix} \sin\left(x \cdot \sqrt{k_1^2 - \beta^2}\right) \\ \cos\left(x \cdot \sqrt{k_1^2 - \beta^2}\right) \end{Bmatrix} \quad ; \quad E_Z = 0$$

A, B are arbitrary unknown amplitude constants

- <u>***Top cladding ②:***</u> ***$n = n_2$, $k_2 < \beta$, $x > d$ exponential decaying solution***

$$H_z(x) = B \cdot e^{-(x-d)\cdot\sqrt{\beta^2 - k_2^2}} \quad ; \; E_Z = 0 \qquad \text{field decay constant } \; 1/\sqrt{\beta^2 - k_2^2}$$

- <u>***Bottom cladding ③:***</u> ***$n = n_2$, $k_2 < \beta$, $x < -d$*** (+ for the *cos*-solution in the core) ***exponential decaying solution***

$$H_z(x) = -(+)B \cdot e^{(x+d)\cdot\sqrt{\beta^2 - k_2^2}} \quad ; \; E_Z = 0 \qquad (3.58).$$

***2)* Boundary conditions at the core-cladding-interface** requires the **continuity of the tangential field** components, that is the continuity of H_z and also for E_y with $E_Z=0$.

These equations couple the core and cladding field together (constants a A and B). The transverse field components are obtained by the relations with $E_Z=0$:

$$\vec{E}_T = \frac{1}{ik_T^2}\left\{ \cancel{k_z \cdot \nabla_T E_z} - \omega\mu\left(\vec{e}_z \times \nabla_T\right) H_z \right\}$$

$$\vec{H}_T = \frac{1}{ik_T^2}\left\{ k_z \cdot \nabla_T H_z + \omega\varepsilon\left(\cancel{\vec{e}_z \times \nabla_T}\right) E_z \right\}$$

$$\nabla_T = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} = 0, \frac{\partial}{\partial z} = 0 \right)$$

➡ (1-D and E_Z=0) ➡

$$\boxed{\begin{aligned} E_y &= \frac{i\omega\mu}{k_i^2 - \beta^2} \cdot \frac{\partial}{\partial x} H_z \quad ; \quad E_X = 0 \quad ; \quad E_z = 0 \\ H_x &= \frac{-i\beta}{k_i^2 - \beta^2} \cdot \frac{\partial}{\partial x} H_z \quad ; \quad H_y = 0 \quad ; \quad \leftarrow H_Z(x) \neq 0 \end{aligned}}$$

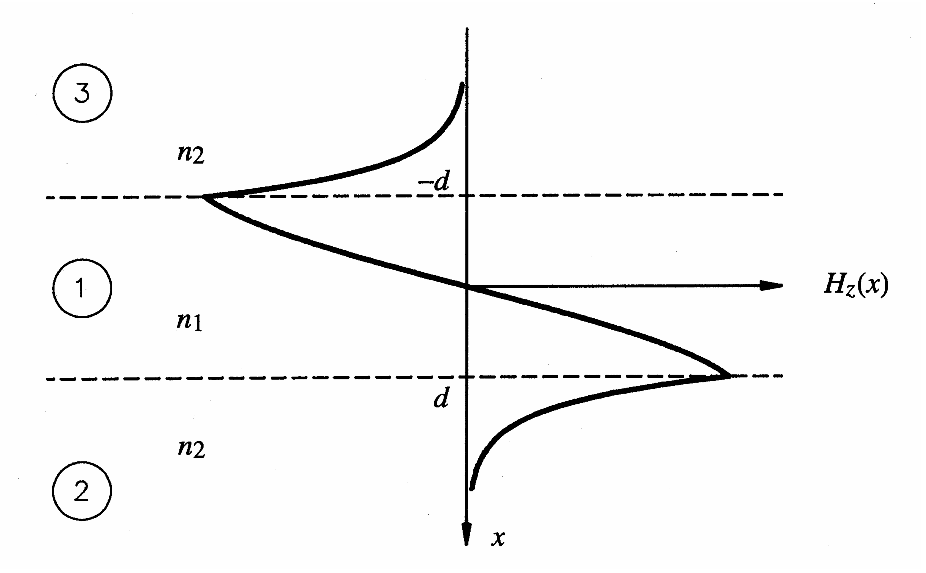
Using the basic <u>assumption</u> for the TE-mode $E_z \equiv 0$ leads to the conclusion that E_x- , resp. the H_y-component vanish because the y-components are constant ($\partial / \partial y = 0$).

The ***E***-field of the *TE-mode* has only a E_y-component and the ***H*** -field has only a H_z-and a H_x-component.

Remark:

It can be shown, that the continuity of E_y also fulfills the continuity of H_z by using the relation:

$$H_X(x) = -\frac{j\beta}{k^2 - \beta^2} \frac{\partial}{\partial x} H_Z(x)$$

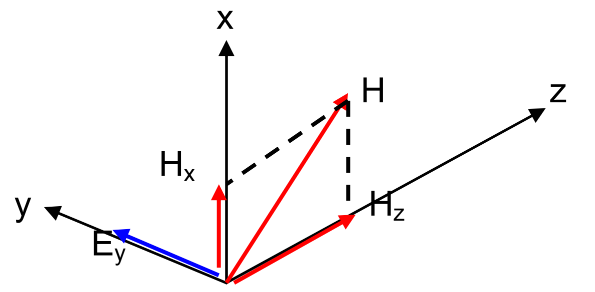


H_z -field distribution of the fundamental mode of the symmetric slab waveguide:

decaying

oscillatory

decaying



Tangential continuity conditions for H_z at the interfaces x= ±d relates A and B: ((+) for the cos-solution)

$$H_z(\pm d,\beta) = A\cdot\begin{Bmatrix} \pm\sin\left(d\cdot\sqrt{k_1^2-\beta^2}\right) \\ \cos\left(d\cdot\sqrt{k_1^2-\beta^2}\right)\end{Bmatrix} = \pm(+)B \qquad \text{(c1)}. \qquad B=B(A)$$

Tangential continuity conditions for E_y at the interfaces x= ±d: ((±) for the cos-solution H_z)

$$E_y(\pm d,\beta) = +\frac{i\omega\mu_1\cdot A}{\sqrt{k_1^2-\beta^2}}\cdot\begin{Bmatrix} \cos\left(d\cdot\sqrt{k_1^2-\beta^2}\right) \\ \mp\sin\left(d\cdot\sqrt{k_1^2-\beta^2}\right)\end{Bmatrix} = +(\pm)\frac{i\omega\mu_2\cdot B}{\sqrt{\beta^2-k_2^2}} \qquad \text{(c2)}$$

3) Eigenvalue-Equation for the longitudinal propagation constant β:

the continuity equations (c1) and (c2) must be fulfilled simultaneously and for non-magnetic dielectrics $\mu_1 = \mu_2 = \mu$. Eliminating A and B by a division of eg.(c1)/(c2) leads to the eigenvalue-eq. for β:

For the **sin**-function (symmetric, even for E_y) in the core:

$$\boxed{\tan\left(d \cdot \sqrt{k_1^2 - \beta(\omega)^2}\right) = \frac{\sqrt{\beta(\omega)^2 - k_2^2}}{\sqrt{k_1^2 - \beta(\omega)^2}}} \qquad (3.63)$$

Transcendental Eigenvalue Equation for β(ω)= ?

and for the cos-function (anti-symmetric, odd for E_y) in the core:

$$\boxed{-\cot\left(d \cdot \sqrt{k_1^2 - \beta(\omega)^2}\right) = \frac{\sqrt{\beta(\omega)^2 - k_2^2}}{\sqrt{k_1^2 - \beta(\omega)^2}}} \qquad (3.64)$$

Transcendental Eigenvalue Equation for β(ω)= ?

which are only relations between:

- the propagation constants β(ω) and $k_i(\omega)$ (containing ω and n_i) and
- geometrical parameters (d).

➡ Solutions for <u>real</u> β (undamped propagation in in z-direction) exist only if $\boxed{\mathbf{k_2 < \beta < k_1}}$

For a <u>graphical solution</u> of the Eigenvalue-equations we substitute the functions ξ, η:

$$\xi(\omega) = d \cdot \sqrt{k_1^2 - \beta^2} = d\sqrt{k_{T1}^2} > 0$$
$$\eta(\omega) = d \cdot \sqrt{\beta^2 - k_2^2} = d\sqrt{k_{T2}^2} > 0 \qquad (3.65)\text{-}(3.66)$$

solutions for η,ξ>0 are in the 1.quadrant of the η-ξ-plane

$\xi(\omega)/d = \sqrt{k_1^2 - \beta^2} = k_{T1}$ is the transverse wave number k_{T1} of the core n_1

$\eta(\omega)/d = \sqrt{\beta^2 - k_2^2} = k_{T2}$ is the inverse field decay constant in the claddings n_2

The **transcendental Eigenvalue-equations for $\beta(\omega)$** have the generic forms:

$$\boxed{\sin: \quad \eta(\omega) = \xi(\omega)\cdot\tan\big(\xi(\omega)\big) \quad ; \quad \cos: \quad \eta(\omega) = -\xi(\omega)\cdot\cot\big(\xi(\omega)\big)} \quad \text{(3.67} \qquad \text{1. equation containing } \eta, \zeta$$

For the **graphical solution** we define the **structure parameter $V(\omega)$**:

$$V(\omega) = \sqrt{\xi^2 + \eta^2} = d\cdot\sqrt{k_1^2 - k_2^2} = k_0 d\cdot\sqrt{n_1^2 - n_2^2} = 2\pi d/\lambda_0 \cdot\sqrt{n_1^2 - n_2^2} = \omega/c\cdot\sqrt{n_1^2 - n_2^2} \quad \text{(3.69).} \qquad \text{2. equation containing } \eta, \zeta$$

V combines only the structural parameters of the waveguide ***d*, *n_1*,** and ***n_2*** with the wavelength λ_0, resp. ω of the EM wave

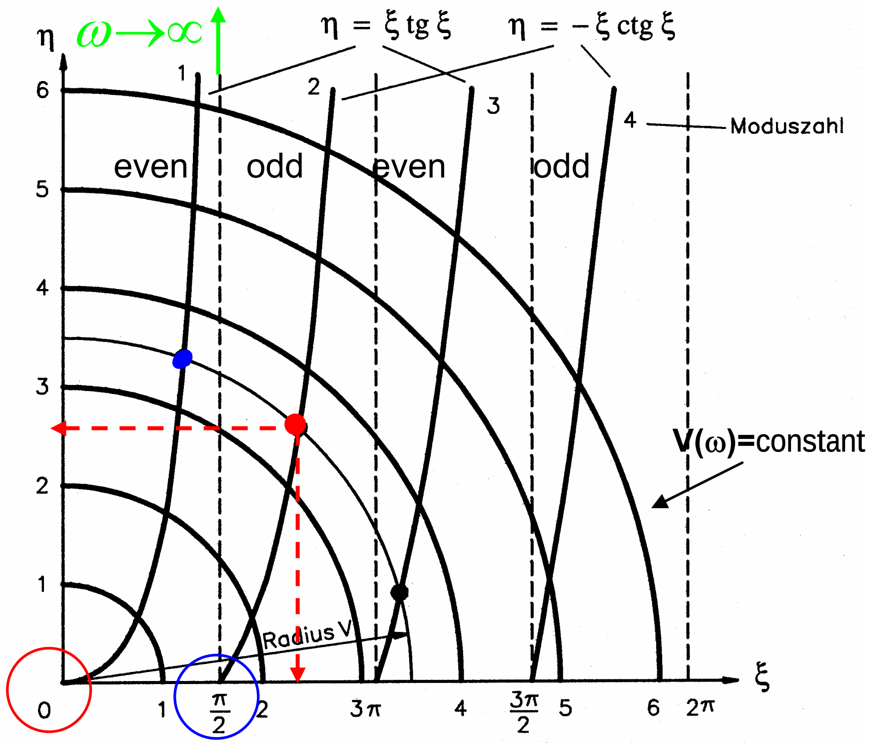
For a given optical frequency ω (resp. wavelength λ_o) 1) the eigenvalue equations and 2) the condition $V(\omega)$ have to be fulfilled simultaneously in the $\xi - \eta$ -plane:

1) $\eta = \xi\cdot\tan(\xi) \qquad \eta = -\xi\cdot\cot(\xi) \quad \rightarrow \quad \eta(\xi)$-tan or cot-curves

2) $V(\omega) = \sqrt{\xi^2 + \eta^2} = \frac{\omega}{c}\cdot\sqrt{n_1^2 - n_2^2} \quad \rightarrow$ circles with radius $V(\omega)$

(tan and cot are periodic in p !)

Graphical representation of the two eigenvalue equations and the circles for constant V(ω):



SM !

Each intersection ($\bullet$; $\bullet$) represents the η,ξ-solution values for a particular odd or even **propagating TE-mode** at a given frequency ω.

$$\beta(\omega)=\beta(\eta,\xi)=\sqrt{k_1^2-\left(\frac{\xi}{d}\right)^2}=\sqrt{k_2^2+\left(\frac{\eta}{d}\right)^2}$$

Dispersion-Relation

zero-frequency cut-off !

finite frequency cut-off

even symmetry mode $\bullet$

odd symmetry mode $\bullet$

Graphic solution: ➡ $\xi(\omega)$ *and* $\eta(\omega)$ ➡ $\beta(\omega)$

- For **large** ω resp V **many** odd and even modes exist - multimode operation

- For **small** ω resp V **only one even** modes exist - singlemode (SM) operation !

- small core diameter d or small index difference favour singlemode operation

- cut-off: $\eta=0$ $\beta=k_2=n_2k_0$ (“cladding mode”)

- asymptotic behaviour: $\omega\to\infty$ $V\to\infty$ $\xi(\omega)/d=\sqrt{k_1^2-\beta^2}=\mathit{finite}\to$ $\beta=k_1=n_1k_0$ (“core mode”)

What do we learn: Existence and Propagation of Modes

- Guides modes are **z-propagating wave with planar phase fronts in the transverse x-y-plane** and a **confined** field variation in the transverse x-direction (the y-direction is homogenous)

- The **field distribution in the transverse direction (x)** must be such that a **real propagation constant** $\beta(\omega)$ results, which is the "same" for core and claddings (qualitative).

- **The structure parameter V(ω) determines**

  1) if there **exists no, one or multiple modes** (solutions, intersections= number of modes) and
  2) the values of the corresponding **propagation constants** $\beta(\omega)$.

  Large values of V, resp ω often allow multiple modes to coexist (depending if they are excited by a source)

  For small values of V<<1, resp. ω (small radius of V) only one mode can propagate, the TE_1-mode, resp. H_1-mode. This mode exists even for $\omega \to 0$ (mode without a cut-off)

- **Dispersion of the modes:**

  Modal dispersion: $\beta_i(\omega)$

  if V(ω), resp. ω changes then the propagation constant $\beta(\omega)$ and the **phase propagation velocity** $v_{ph}(\omega) = \omega / \beta(\omega)$ varies, - a particular mode can travel at different velocities, depending on its frequency ω.
  If $\beta \sim \omega$ the phasevelocity is constant (dispersion free, no modal pulse broadening)

  Intermodal dispersion: $\beta_i(\omega) \neq \beta_k(\omega)$

  if multiple modes i coexist with different propagation constants $\beta_i(\omega)$ then each mode may propagate at a different phase velocity $v_{ph,I}$ and the total field of all modes may show large dispersion.
  (excitation of multiple modes)

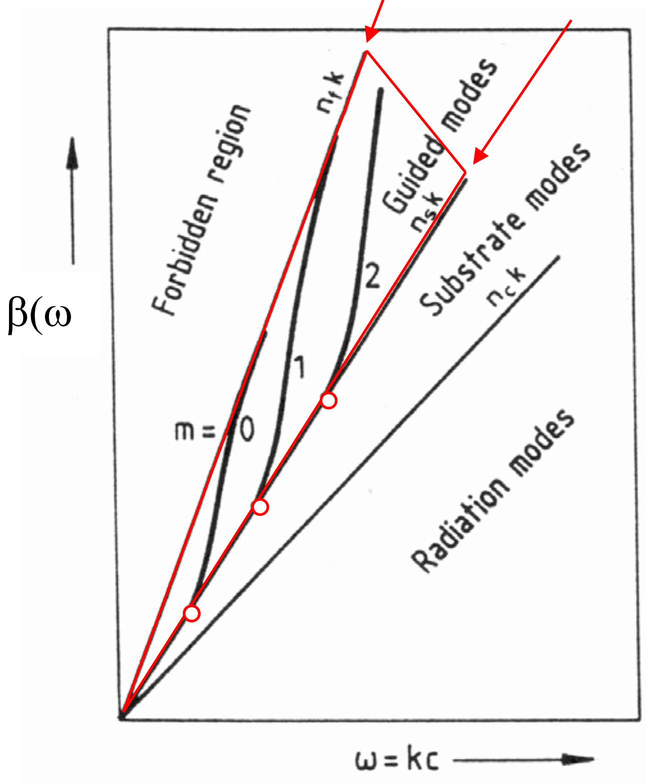
Dispersion curves of modes: $\beta(\omega)$ and $n_{eff}(\omega)$

Definition: of the **modal effective index of refraction n_{eff}:**

$$\beta(\omega) = k_{eff}(\omega) = \frac{2\pi}{\lambda_0} n_{eff}(\omega) \quad \rightarrow \quad n_{eff}(\omega) = \beta(\omega)\frac{\lambda_0}{2\pi}$$

core: $\beta=k_1(\omega)=n_1k_0$

cladding: $\beta=k_2(\omega)=n_1k_0$



Transformation:

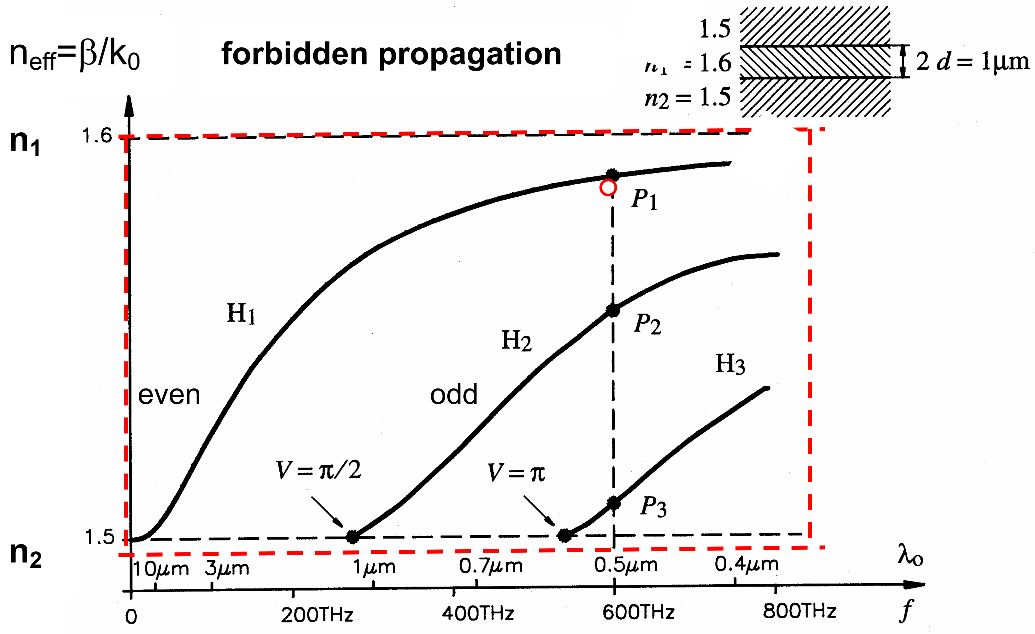
$\beta(\omega)$ ➡ $n_{eff}(V)$

$V \sim \omega$

$\mathbf{n_{eff}} = \beta(\omega)/\omega c_0$

➡

E



Schematic propagation constant $\beta(\omega)$ versus optical frequency ω (Dispersion relation)

$\beta(\omega)$ ➡ provides information about dispersion

Effective index of refraction $n_{eff}(f)$ of the symmetric slab WG vers. optical frequency f for the modes TE_1 (H_1), TE_2 (H_2) and TE_3 (H_3)

$n_{eff}(\omega)$ ➡ provides information about dispersion

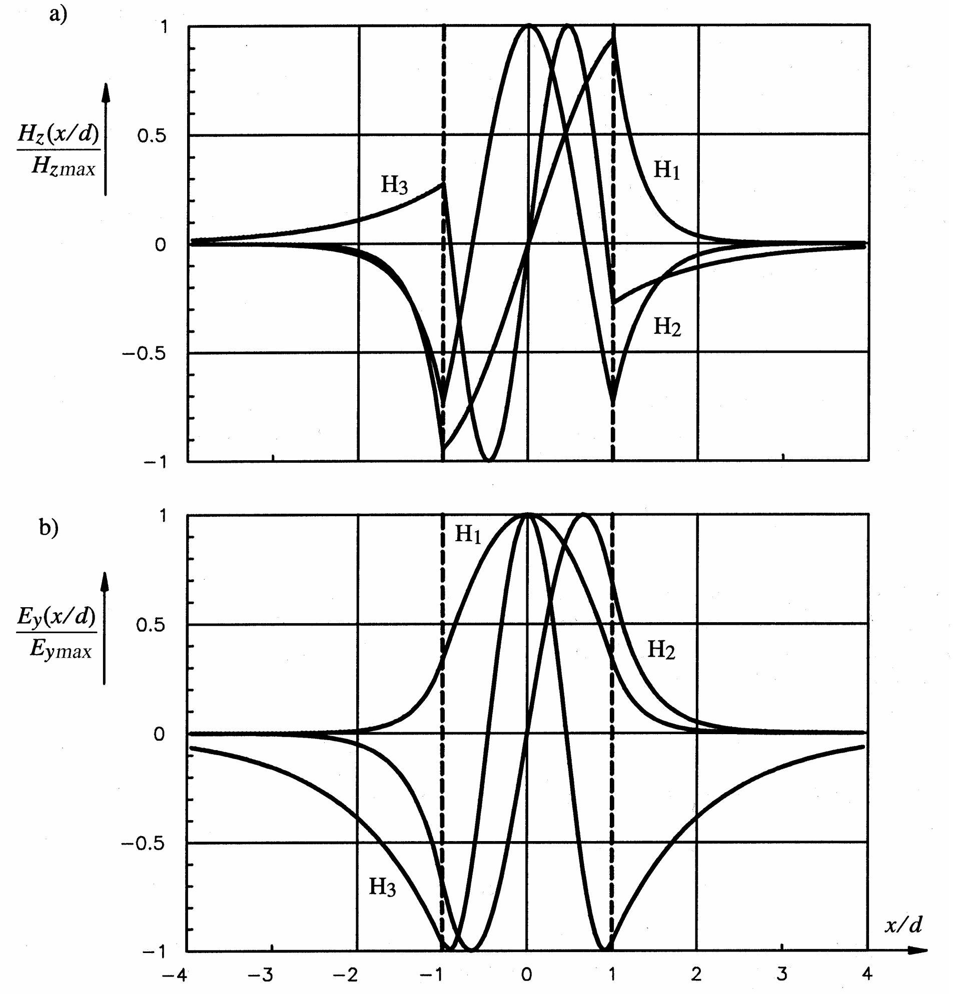
(corresponding mode fields at 600 THz see next foil).

Interpretation of n_{eff} and β:

- Dispersion curves show that modes exist in general in a frequency range from $\omega_{min,i}$ (cut-off frequency) to infinity $\omega \to \infty$.

- At cut-off β approaches k_2 , resp. n_{eff} approaches n_2 of the cladding of the cladding – the decay of the cladding field becomes small and the mode field mostly propagates in the cladding.

  At high frequencies above cut-off, $\beta \rightarrow k_1$ and $n_{eff} \rightarrow n_1$ the decay of the cladding field is strong and the mode is almost completely confined to the core.



Normalized transversal field distributions (a) $H_z(x)$ and (b) $E_y(x)$ or the modes TE_1 (H_1), TE_2 (H_2) und TE_3 (H_3) at 600 THz.

- $n_{eff} \in [n_2, n_1]$,

$$k_2 \leq \beta \leq k_1$$

$$k_0 \cdot n_2 \leq \beta \leq k_0 \cdot n_1$$

$$n_2 \leq \beta/k_0 \leq n_1 \rightarrow n_{eff} = \beta/k_0$$

- The eigenvalue equation in the $\eta-\xi$-plane show that for $\xi=j\pi/2$, j=0, 1,2, 3, ... $\rightarrow$ $\eta=0$.

  These are the **cut-off points** where $V(\omega)= \xi=j\pi/2$ and β**=**k_2**.**

  p=even (p=0, 2,4, ..) ➨ **symmetric even modes, H_{p+1}**

  p=odd (p=1, 3, 5, ..) ➨ **asymmetric odd modes H_{p+1}**

  p is the mode index (see next section)

  (p-1) is the number of nodes of the mode E-field

Mode classification (mode number) and mathematical formulation of the eigenvalue problem:

a) Guided TE-Modes $E_z \equiv 0$, $H_z \neq 0$

Replacing η by $V(\omega)$ in the eigenvalue equation we obtain for the variable $\xi(\omega)$:

$$\sqrt{V^2 - \xi^2} = \begin{cases} \xi \cdot \tan(\xi) \\ -\xi \cdot \cot(\xi) \end{cases} \qquad (3.71).$$

Using the addition theorem for tan- / cot-functions (periodicity of tan, cot: π) :

$$\tan\left(\alpha - \frac{p \cdot \pi}{2}\right) = \begin{cases} -\cot(\alpha) & \forall\, p : odd \\ \tan(\alpha) & \forall\, p : even \end{cases} \qquad (3.72)$$

p is the mode index

we eliminate the cot-function in the eigenvalue equation and combine both equations to.

$$\sqrt{V^2 - \xi^2} = \xi \cdot \tan\left(\xi - \frac{p \cdot \pi}{2}\right) \qquad p=0,\ 1,\ 2,\ \ldots \qquad (3.73),$$

Solving the eigenvalue equation for the eigenvalue ξ :

$$\boxed{\xi = \operatorname{Arctan}\left(\sqrt{\frac{V^2}{\xi^2} - 1}\right) + \frac{p \cdot \pi}{2}}$$

p=0, 1, 2, ... = mode index alternative form: $C(\beta,\omega)=0$

➡ p serves as a counting index to classify the modes TE_{p+1}- resp. H_{p+1}-modes.

With $\xi(\omega)$ we calculate $\beta(\omega)$ using $\xi(\omega) = d \cdot \sqrt{k_1^2(\omega) - \beta^2(\omega)}$

Repeating the previous procedure we can investigate also the TM_{p+1}- resp. E_{p+1}-modes.

b) Guided TM-Modes $H_z \equiv 0, E_z \neq 0$ (self-study)

Repeating the Helmholz-equation for the longitudinal E_z-field component $E_z(x) \neq 0$, $H_z \equiv 0$:

$$\left(\frac{\partial^2}{\partial x^2} + k_i^2 - \beta^2\right) E_z = 0 \quad \rightarrow \quad E_z(x)$$

Using the similar formal solutions for the longitudinal component $E_z(x)$, we can determine the transverse field components from the derivation of E_z:

$$E_x = \frac{-i\beta}{k_i^2 - \beta^2} \cdot \frac{\partial}{\partial x} E_z$$

$$H_y = \frac{-i\omega\varepsilon}{k_i^2 - \beta^2} \cdot \frac{\partial}{\partial x} E_z$$

$E_z(x) \rightarrow E_x(x), H_y(x)$

We assumed already $H_z \equiv 0$ and again the E_y-, resp. the H_x-components vanish, because the field components in the y-direction are constant, resp. $\partial / \partial y \rightarrow 0$.

➡ The ***H***-field of the *TM-m*odes has only one H_y-component and the corresponding ***E***-field is composed of only the an E_z- and E_x-component.

The field continuity boundary conditions between core and cladding lead again to the formulation of the eigenvalue equation:

- **Continuity of the E_z-component:** ((+)-sign for the cos-solution)

$$E_z(\pm d) = A \cdot \left\{ \begin{matrix} \pm \sin\left(d \cdot \sqrt{k_1^2 - \beta^2}\right) \\ \cos\left(d \cdot \sqrt{k_1^2 - \beta^2}\right) \end{matrix} \right\} = \pm(+) B \qquad (3.78).$$

- **Continuity of the H_y -component:** ((±) –sign for the *cos*-solution for E_z)

$$H_y(\pm d) = -\frac{i\omega\varepsilon_1 \cdot A}{\sqrt{k_1^2-\beta^2}} \cdot \begin{Bmatrix} \cos\left(d\cdot\sqrt{k_1^2-\beta^2}\right) \\ \mp\sin\left(d\cdot\sqrt{k_1^2-\beta^2}\right) \end{Bmatrix} = -(\pm)\frac{i\omega\varepsilon_2 \cdot B}{\sqrt{\beta^2-k_2^2}} \qquad (3.79).$$

For the TM-modes $\varepsilon_1 = n_1^2 \neq \varepsilon_2 = n_2^2$ we write the eigenvalue equation slightly in a different way as for the TE-mode:

$$\tan\left(d\cdot\sqrt{k_1^2-\beta^2}\right) = \frac{\varepsilon_1}{\varepsilon_2}\cdot\frac{\sqrt{\beta^2-k_2^2}}{\sqrt{k_1^2-\beta^2}}$$

$$-\cot\left(d\cdot\sqrt{k_1^2-\beta^2}\right) = \frac{\varepsilon_1}{\varepsilon_2}\cdot\frac{\sqrt{\beta^2-k_2^2}}{\sqrt{k_1^2-\beta^2}} \qquad (3.80)\text{-}(3.81).$$

Using the same substitutions for $\xi(\omega)$ and $\eta(\omega)$ and introducing υ we obtain:

$$\xi = d\cdot\sqrt{k_1^2-\beta^2} \qquad \eta = d\cdot\sqrt{\beta^2-k_2^2} \qquad \vartheta = \frac{\varepsilon_1}{\varepsilon_2} = \frac{n_1^2}{n_2^2} \qquad (3.82)\text{-}(3.84)$$

eliminating η by using the definition of $V(\omega)$ results in the **eigenvalue equation for the TM_{p+1}- resp. the E_{p+1}-modes**:

$$\boxed{\xi = \operatorname{Arctan}\left(\vartheta\cdot\sqrt{\frac{V^2}{\xi^2}-1}\right) + \frac{p\cdot\pi}{2}} \qquad (3.85),$$

- It can be shown that only one eigenvalue exist for each TM-mode if $V > p\,\pi/2$
- *TE*- and *TM*-modes are degenerated for *symmetric* slab waveguides at cutoff, meaning that a TM and TE-solution with the same β exist at the cutoff-frequency !

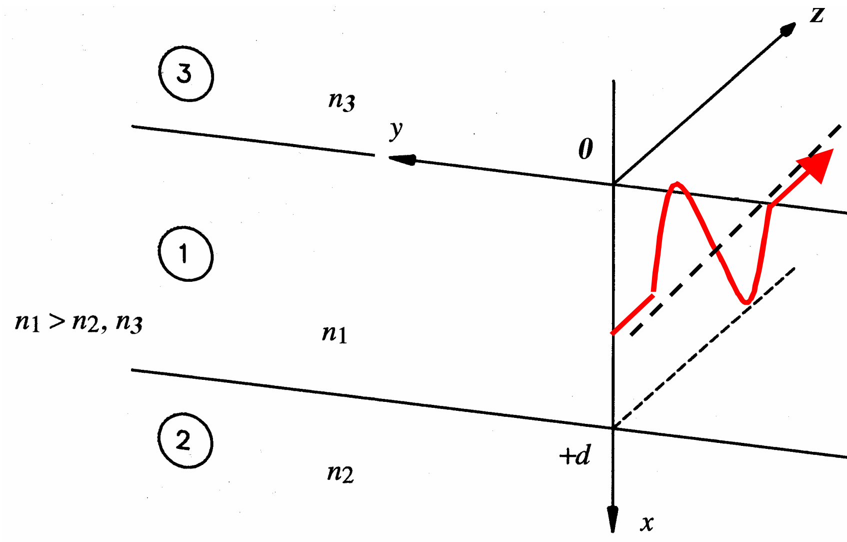
3.4.2 Asymmetric planar slab waveguide (self-study, exercise)

Real film waveguides are often asymmetric in the sense that the top-cladding has a refractive index n_3 which is different from the index n_2 of the bottom-cladding (eg. substrate).

The waveguide core has an index $\mathbf{n_1 > n_2, n_3}$ and a thickness d (not 2d as before !).

Assumptions:

- Propagating guided wave in z-direction
- Layer structure in the x-direction
- Homogeneous in the y-direction $\frac{\partial}{\partial y} = 0$



$k_3 = k_0 n_3$ top cladding

$k_1 = k_0 n_1$ core

$k_2 = k_0 n_2$ bottom cladding (eg. substrate)

Asymmetric plane film waveguide .($n_2 \geq n_3$) with propagation direction z.

Applying the same formalism as for the symmetric WG it is useful to define **2 structure parameters** V(ω) and $\tilde{V}(\omega)$ because the asymmetric slab-waveguide has 2 different dielectric interfaces n_1-n_3 and n_1-n_2 (2 different conditions for total reflection).

$$V(x) = k_0 d \cdot \sqrt{n_1^2 - n_2^2} \quad ; \quad \tilde{V}(x) = k_0 d \cdot \sqrt{n_1^2 - n_3^2} \qquad (3.86)\text{-}(3.87).$$

1-2 Interface $\qquad$ 1-3 Interface

a) Guided TE-Modi $E_z \equiv 0, H_z \neq 0$

The Helmholz-equation for the H_z –component becomes:

$$\left(\frac{\partial^2}{\partial x^2}+k_i^2-\beta^2\right)H_z=\left(\frac{\partial^2}{\partial x^2}+k_{Ti}^2\right)H_z = 0 \qquad \text{with} \quad k_{Ti}^2=k_i^2-\beta^2 \qquad \text{i =, 1, 2, 3}$$

The general harmonic solutions in the transverse direction x have the form: $e^{\pm i(k_{Ti}x-\Psi)}$

The new phase parameter ψ takes into account that the solution may be asymmetric in the x-direction

Typ of transverse mode solutions:

Depending on the value of β relative to k_i , resp. on the value of k_{Ti} we distinguish

- k_{Ti} = real ($\beta < k_i$, eg. in the core with $k_1=k_0\, n_1$) ➡ undamped oscillatory solution (oscillatory standing transverse wave)
- k_{Ti} = imaginary ($\beta > k_i$, eg. in the claddings with eg. $k_2=k_0\, n_2$) ➡ damped exponential (decaying standing transverse wave)

For guided (confined in the x-direction, decaying in the cladding) waves the eigenvalues β must lay in the intervall
$max(k_2, k_3) < \beta < k_1$

To characterize the refractive index properties at the two interfaces we define again as before:

$$\vartheta = \frac{\varepsilon_1}{\varepsilon_2} = \frac{n_1^2}{n_2^2} \quad ; \quad \tilde{\vartheta} = \frac{\varepsilon_1}{\varepsilon_3} = \frac{n_1^2}{n_3^2} \qquad \text{(3.92)-(3.93).}$$

Again we introduce the abbreviations ξ, η and $\tilde{\eta}$ for the eigenvalue equations:

$$\xi = d\cdot\sqrt{k_1^2-\beta^2} \quad ; \quad \eta = d\cdot\sqrt{\beta^2-k_2^2} \quad ; \quad \tilde{\eta} = d\cdot\sqrt{\beta^2-k_3^2} \qquad \text{(3.94)-(3.96),}$$

$k_{T1} = \sqrt{k_1^2 - \beta^2}$; $k_{T2} = \sqrt{\beta^2 - k_2^2}$; $k_{T3} = \sqrt{\beta^2 - k_3^2}$ (Observe the interchanged definition of k_{T2} and k_{T3} compared to k_{T1}!)

Transverse H_z-field profile of z-propagating *TE*-Modes (without proof):

From the solution of the Helmholtz-equation:

- <u>*Core ①:*</u> $n = n_1$, $k_1 > \beta$, $x < d$

$$H_z(x) = A \cdot \begin{Bmatrix} \sin(k_{T1} \cdot x - \psi) \\ \cos(k_{T1} \cdot x - \psi) \end{Bmatrix} \quad (3.97);$$

oscillatory harmonic function

- <u>*Bottom Cladding ②:*</u> $n = n_2$, $k_2 < \beta$, $x > d$

$$H_z(x) = A \cdot \begin{Bmatrix} \sin(\xi - \psi) \\ \cos(\xi - \psi) \end{Bmatrix} \cdot e^{-k_{T2} \cdot (x-d)} \quad (3.98);$$

decaying (oscillatory) exponential

- <u>*Top Cladding ③:*</u> $n = n_3$, $k_3 < \beta$, $x < 0$

$$H_z(x) = A \cdot \begin{Bmatrix} \sin(\psi) \\ \cos(\psi) \end{Bmatrix} \cdot e^{k_{T3} \cdot x} \quad (3.99);$$

decaying (oscillatory) exponential

Appling the boundary conditions (continuity eq.) at the two interfaces for the H_z - and the transverse E_y -components leads to the
eigenvalue equation for the TE_{p+1} - resp. H_{p+1} -mode:

$$\boxed{\xi = \mathrm{Arctan}\left(\sqrt{\frac{V^2}{\xi^2} - 1}\right) + \mathrm{Arctan}\left(\sqrt{\frac{\tilde{V}^2}{\xi^2} - 1}\right) + p \cdot \pi} \quad (3.100)$$

mode index: p=0, 1, 2, ...alternative form: $C(\beta,\omega)=0$

Solving for $\xi(\omega)$ we get finally the propagation constant $\beta(\omega)$ of mode p+1:

$$\Rightarrow \quad \beta(\omega,p) = \sqrt{k_1^2 - \left(\frac{\xi(\omega)}{d}\right)}$$

Cut-off-Condition: ($\eta(\omega)=0$)

In view that we have 2 structure parameters $V(\omega)$ and $\tilde{V}(\omega)$ for the top- and bottom core-cladding interface it is obvious that the interface with the smaller index difference violates the total reflection condition first (one-sided leakage of the mode into the corresponding cladding).

The detailed discussion of the ***cutoff*-condition V_p for *TE_{p+1}* - ** resp. ***H_{p+1}* -modes** becomes (without proof)

$$V > V_p = \operatorname{Arctan}\left(\sqrt{\frac{1}{\tilde{\vartheta}} \cdot \frac{\tilde{\vartheta}-\vartheta}{\vartheta-1}}\right) + p \cdot \pi \qquad (3.101).$$

Asymmetric film waveguides with $\vartheta \neq \tilde{\vartheta}$ and subsequently $V > V_p$>0 can <u>not guide</u> the fundamental mode p=0, TE_1 for ω=0. (for the symmetric $\vartheta = \tilde{\vartheta}$ case V_p=0 becomes possible)

The **transverse mode-shift** ψ in the core-solution is determined from the eigenvalue ξ, η as:

$$\tan(\xi - \psi) = \frac{\xi}{\eta} \qquad (3.102)$$

b) Guided TM-mode $H_z \equiv 0, E_z \neq 0$

Again the Helmholtz-eq. reads as: $\left(\frac{\partial^2}{\partial x^2} + k_i^2 - \beta^2\right) E_z = 0$ (3.103),

resulting in a similar eigenvalue equation:

$$\boxed{\xi = \operatorname{Arctan}\left(\vartheta \cdot \sqrt{\frac{V^2}{\xi^2} - 1}\right) + \operatorname{Arctan}\left(\tilde{\vartheta} \cdot \sqrt{\frac{\tilde{V}^2}{\xi^2} - 1}\right) + p \cdot \pi} \qquad (3.104),$$

The modified Cutoff relation becomes:

$$V > V_p = \operatorname{Arctan}\left(\sqrt{\tilde{\vartheta} \cdot \frac{\tilde{\vartheta} - \vartheta}{\vartheta - 1}}\right) + p \cdot \pi \qquad (3.105).$$

From the eigenvalue ξ we can calculate the *transverse mode shift* ψ as:

$$\vartheta \cdot \tan(\xi - \psi) = \frac{\xi}{\eta} \qquad (3.106).$$

The procedure for the eigenvalue and field calculation can be extended straight forward to more complex dielectric multi-layer structures with more than three layers.
Of course the analytical procedure becomes then rather lengthy and numerical methods are appropriate.

3.4.3 Different Types of Modes:

In the previous chapter we restricted ourselves to special mode-solutions (z-propagating, xy-transverse confined modes)

1) **propagating** in the z-direction ➨ $\beta = k_Z = \text{real}$ and

2) where the field energy is **confined** to the core layer with the highest index of refraction, resp. where the field in the claddings decays to zero ➨ $k_T = \text{imaginary}$

Thus this set of solutions are probably not complete.

$$k_{Ti} = \sqrt{(k_0 n_i)^2 - \beta^2} = \textit{real in the core} \quad \rightarrow \textit{harmonic solution}$$

$$k_{Ti} = \sqrt{(k_0 n_i)^2 - \beta^2} = \textit{imaginary} > 0 \;\; \textit{or} \;\; k_{Ti} = \sqrt{\beta^2 - (k_0 n_i)^2} = \textit{reel} > 0 \;\; \textit{in the claddings} \quad \rightarrow \textit{decaying exponential solution}$$

With the definitions: $k_{T1} = \sqrt{k_1^2 - \beta^2}$ $\quad k_{T2} = \sqrt{\beta^2 - k_2^2}$ $\quad k_{T3} = \sqrt{\beta^2 - k_3^2}$

$$\xi = d \cdot \sqrt{k_1^2 - \beta^2} \qquad \eta = d \cdot \sqrt{\beta^2 - k_2^2} \qquad \tilde{\eta} = d \cdot \sqrt{\beta^2 - k_3^2}$$

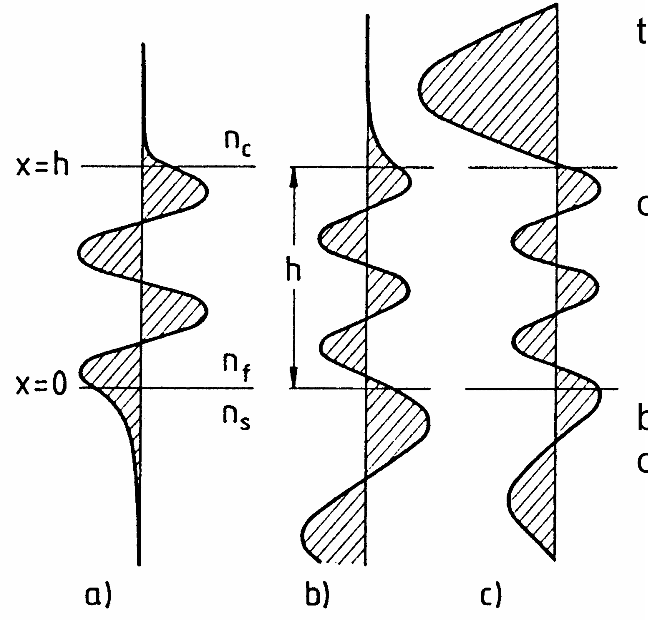
These requirements lead to the following restrictions of possible $\beta(\omega)$ with respect to $k_i = \omega / c n_i$ or n_i for a given ω:

$\boxed{k_0 n_{clad} < \beta < k_0 n_{core}}$ ➨ **propagating and confined modes (a)**

$\beta > k_0 n_{core}$ is not possible because β must be complex leading to non-propagating, decaying waves in the z-direction

$\beta < k_0 n_{clad}$ possible, but it leads to non-decaying cladding fields ➨ **propagating unconfined radiation modes (b,c)**

Confined and unconfined mode-profiles:



top cladding

core

bottom cladding

Mode profiles for
(a) confined propagating wave
(b) substrate wave (radiation into the substrate) and
(c) unguided wave (radiation into substrate and top-cladding).

Extension to non-propagating modes in the z-direction:

These previous conditions for eigenvalues are equivalent to searching for **real** eigenvalues ξ and η .

The eigenvalue equation are of the type:

$$w = -z \cdot \cot(z)$$

However this eigenvalue-equation can have in principle **complex** solutions

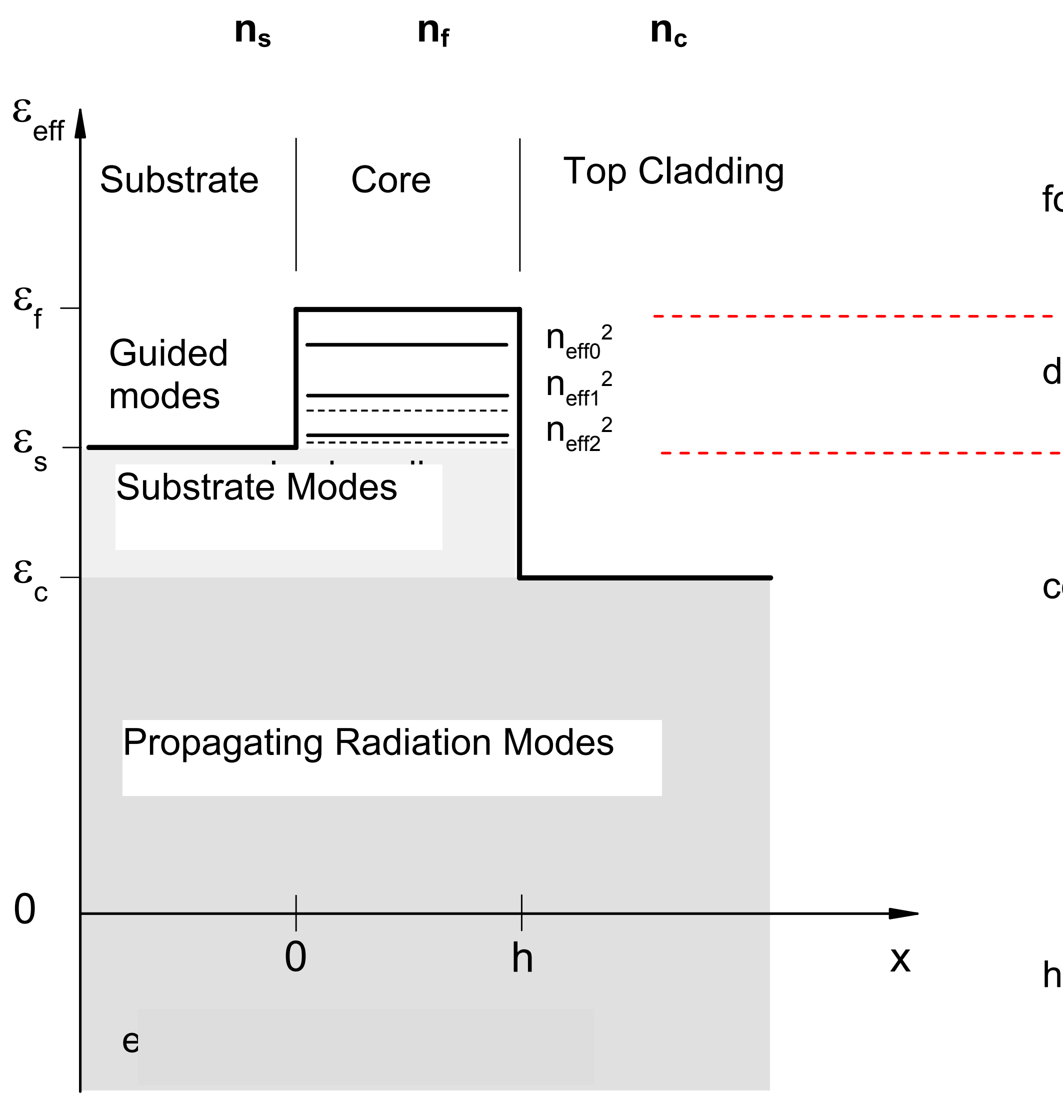
$w = u + i\,v$ and $z = x + i\,y$ with $V^2 = z^2 + w^2 = real.$

It can be shown these solutions lead to so called **leaky modes** (Leckwellen) appearing at frequencies **below the cut-off** of the waveguide.

Leaky modes are solutions that **decay in the propagation direction z , but grow exponentially in the cladding**. Leaky modes are interesting to describe out-coupling effects of waves from a waveguide.

Categories of propagation constants and effective refractive indices of different modes:

$n_{eff}^2 = \varepsilon_{eff}$



forbidden (diverging fields $x \rightarrow \pm\infty$)

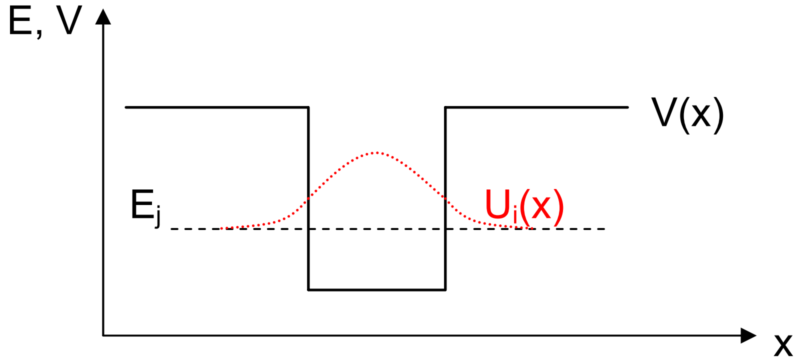
discrete, guided modes

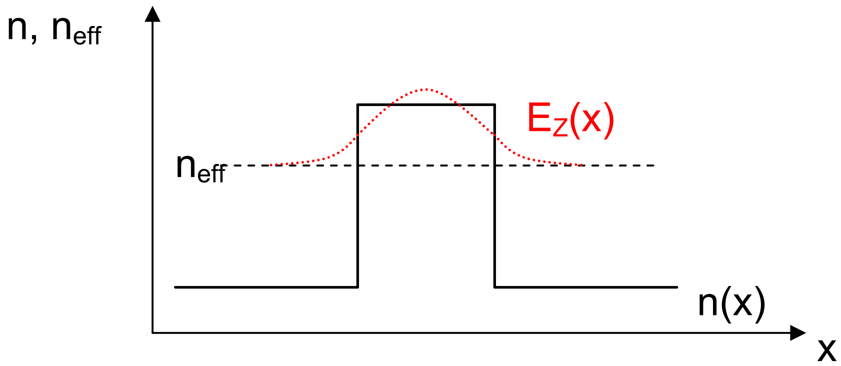
continuous, unguided modes

h=d

Analogy between Guided optical Modes and to Eigenstates in Quantum Mechanics:

There is a formal analogy between the wavefunctions $U_i(r)$ with the energy eigenvalues E_i of bound electrons in a rectangular potential well and the transverse wavefunction $E_i(r_T)$ of a guided (confined photon) mode i in a step-index dielectric waveguide.





Time-independent Schrödinger Equation: ⇔ Helmholtz-Equation:
(1-dimensional) (1-dimensional)

$$\left(\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+\underbrace{V(x)-E_i}\right)U_i=0$$

$$\left(\frac{\partial^2}{\partial x^2}+k_T^2\right)E_z=0=\left(\frac{\partial^2}{\partial x^2}+\omega^2\mu\varepsilon(x)-k_z^2\right)E_z=\left(\frac{\partial^2}{\partial x^2}+\underbrace{\omega^2\frac{n^2(x)}{c_0^2}-k_z^2}\right)E_z$$

Potential V(x): $|x|\le d \quad V(x)=V_1$; $|x|>d \quad V(x)=V_2$ $\qquad$ $V(x) \Leftrightarrow -n(x)^2$

Refractive index n(x): $|x|\le d \quad n(x)=n_1$; $|x|>d \quad n(x)=n_2$

Ansatz: $\Psi_i(x,t)=U_i(x)e^{-j\frac{E_i}{\hbar}t}$

Ansatz: $E_{z,i}(x,t)=E_{z,i}(x)e^{-j\omega t}$ (separation of variables)

Eigenvalue: E_i $\qquad$ $E_i=\hbar\omega \Leftrightarrow k_z^2$

Eigenvalue: $\beta=n_{eff}\,k_o$

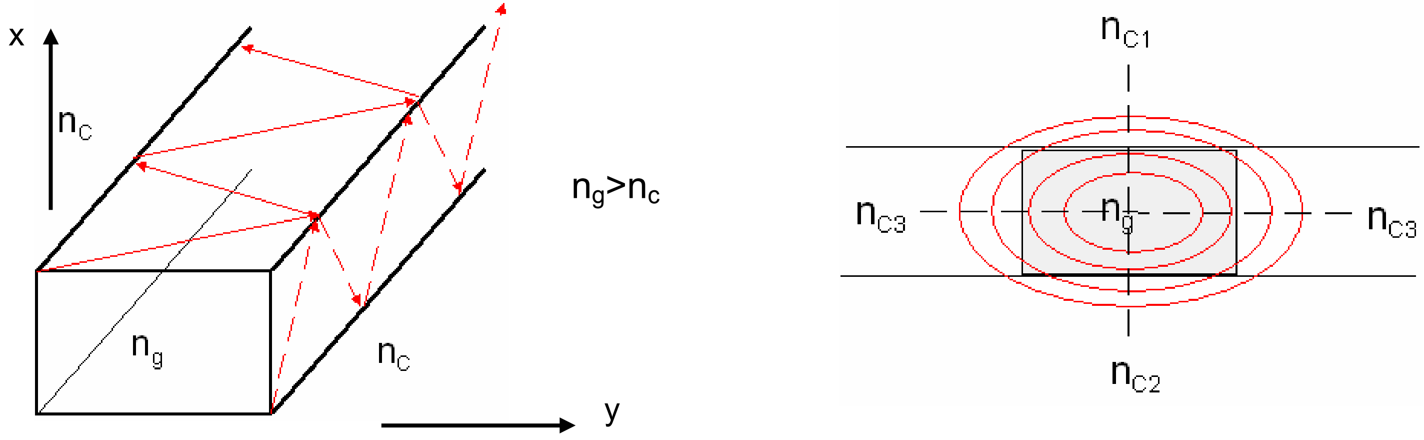
Transverse standing matter-wave for bound particle

Transverse standing EM-wave for confined photon

3.5 Ridge (Rib) Waveguides

Practical planar optical waveguides (confinement in x-direction) need an **additional lateral (y) confinement** to separate the optical channels from each other in the film plane.

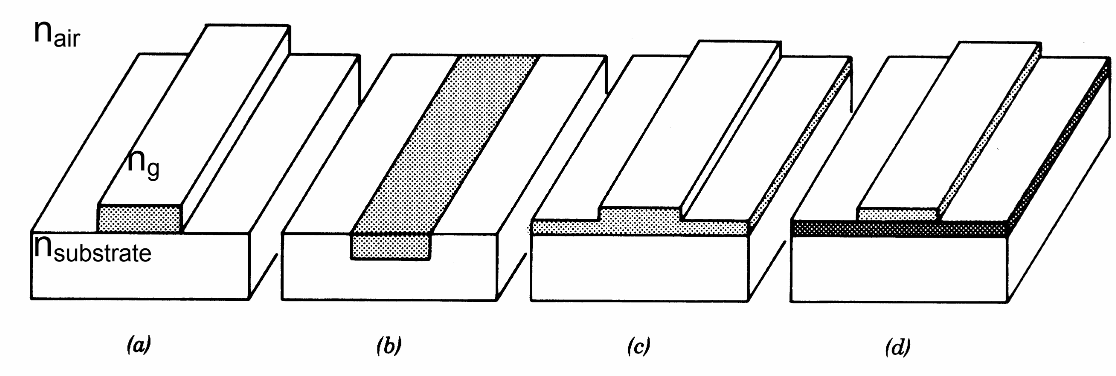
➡ **2-dimensional dielectric confinement in the 2 transverse directions x and y**



Applying the concept of total reflections in the x- and y-direction lead to the **requirements for a 2-D waveguide**:

- **The core (n_g) must be surrounded by claddings (n_c) of lower refractive index than the core $n_g > n_c$**

Technical realizations of 2-dimensional film waveguides (WG):

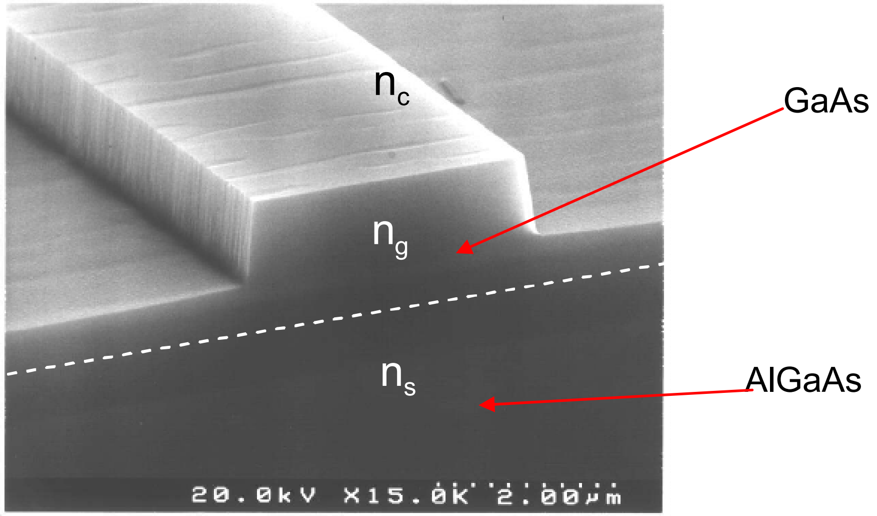


(a) strip WG (Streifenleiter),
(b) embedded strip WG (eingebetteter Streifenleiter),
(c) rib- or ridge WG (Rippenwellenleiter),
(d) loaded strip WG (aufliegender Streifenleiter)

Legend:
the darker the grey-scale, the larger the refractive index.

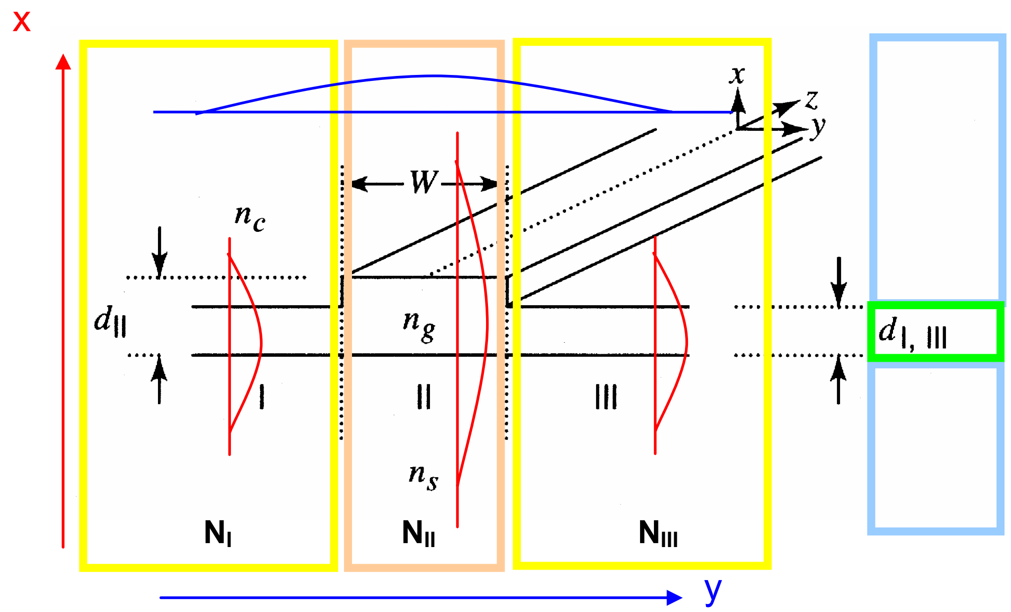
Approximation: Method of the effective refractive index (example rib waveguide)

In general no analytic solutions exist for these 2-dimensional WG ➡ numerical solution methods or approximations



Scanning Electronmicroscope (SEM) picture of a ridge waveguide crossection.

Ridge width is W = 2 µm and ridge height ~1 µm.



Effective index approximation: derive from the ridge structure in the xy-plane a 3 layer film WG in the y-direction with effective indices N_I, N_{II}, N_{III} of the vertical (x-direction) 3-layer WG with the real indices n_s, n_g, n_c.

Concept of effective refractive index:

- Separation of the 2-dimensional (x,y) cross-section into 2 orthogonal 1-dimensional (x) and (y) 3-layer waveguides 3-layer WG (I – III) in the y-direction are approximated by the corresponding eff. Indices N_I, N_{II}, N_{III}.
- Separation of the 2-dimensional mode profile into 2 1-dimensional mode profiles X(x) and Y(y):

$$\boxed{\phi(x, y) = X(x)\cdot Y(y) \quad \textit{Separation Ansatz (approximation)}}$$

Description of the effective index method:

1) X-layer profiles: separation of the rib geometry in the y-direction in to 3-lateral sections I, II, III.
Sections are described individually in the vertical x-direction by a 3-layer waveguides (n_s,n_g,n_c,d_c) with different d_i:

Sec. I: n_s, n_g / (d_I, n_s) ⇨ $X_I(x)$, $N_{eff,I}$ Sec. II: n_s, n_g / (d_{II}, n_s) ⇨ $X_{II}(x)$, $N_{eff,II}$ Sec. III: n_s, n_g / (d_{III}, n_s) ⇨ $X_{III}(x)$, $N_{eff,III}$

Each lateral layer structure in layers I, II, III is characterized by an effective index of refraction $N_{eff,i}=N_i=N_{eff,i}(d_i,n_g,n_s,n_c)$.

We assume in the following that we consider the **TE-solution** (x) in the 3 vertical sections.

2) Y-layer profile: the rib geometry in the y-direction is described by an "effective" 3-layer slab waveguide by the sections I, II, III and their effective indices $N_{eff,I}$, $N_{eff,II}$, $N_{eff,III}$

$N_{eff,I}$, $N_{eff,II}$ / (W, $N_{eff,III}$) ⇨ lateral solution: Y(y), $N_{eff,Y}$

In the lateral direction we must now consider the **TM-solution** (y) to be compatible to the above TE-assumption (x).

3) Solution:

section I: $\phi(x,y)=X_I(x)Y(y)$, section II: $\phi(x,y)=X_{II}(x)Y(y)$, section III: $\phi(x,y)=X_{III}(x)Y(y)$

3 layer WG problem in x-direction: Film I Film II Film III	➡	$N_{eff,I}$ $N_{eff,II}$ $N_{eff,III}$ W	➡	3 layer WG problem in y-direction: N_{eff}, W

Lateral weakly guiding approximation:

- The ratio *d/W* <<1 (small disturbance in the x-direction)
- $\frac{\Delta N_{eff,I-II}}{N_{eff,II}} = \frac{N_{eff,II} - N_{eff,I}}{N_{eff,II}} << 1 \quad ; \quad \frac{\Delta N_{eff,III-II}}{N_{eff,II}} = \frac{N_{eff,II} - N_{eff,III}}{N_{eff,II}} << 1$ weak lateral confinement

resp. $\sqrt{\varepsilon_g - max(\varepsilon_s, \varepsilon_c)} \approx 0.1\ldots1$ with the dielectric constants: $\varepsilon_i = n_i^2$ i= s, g , c

$$\{N_{eff}, R(r)\} = \mathrm{n}_{\mathrm{eff}}(n_c, n_i, n_s, d_i, \lambda_0, m_R, P_R)$$

1. step: vertica (x)l WGs

$$\{N_I, X_I(x)\} = N_{\mathrm{eff}}(n_c, n_g, n_s, d_I, \lambda_0, m_X, P_X)$$

$$\{N_{II}, X_{II}(x)\} = N_{\mathrm{eff}}(n_c, n_g, n_s, d_{II}, \lambda_0, m_X, P_X)$$

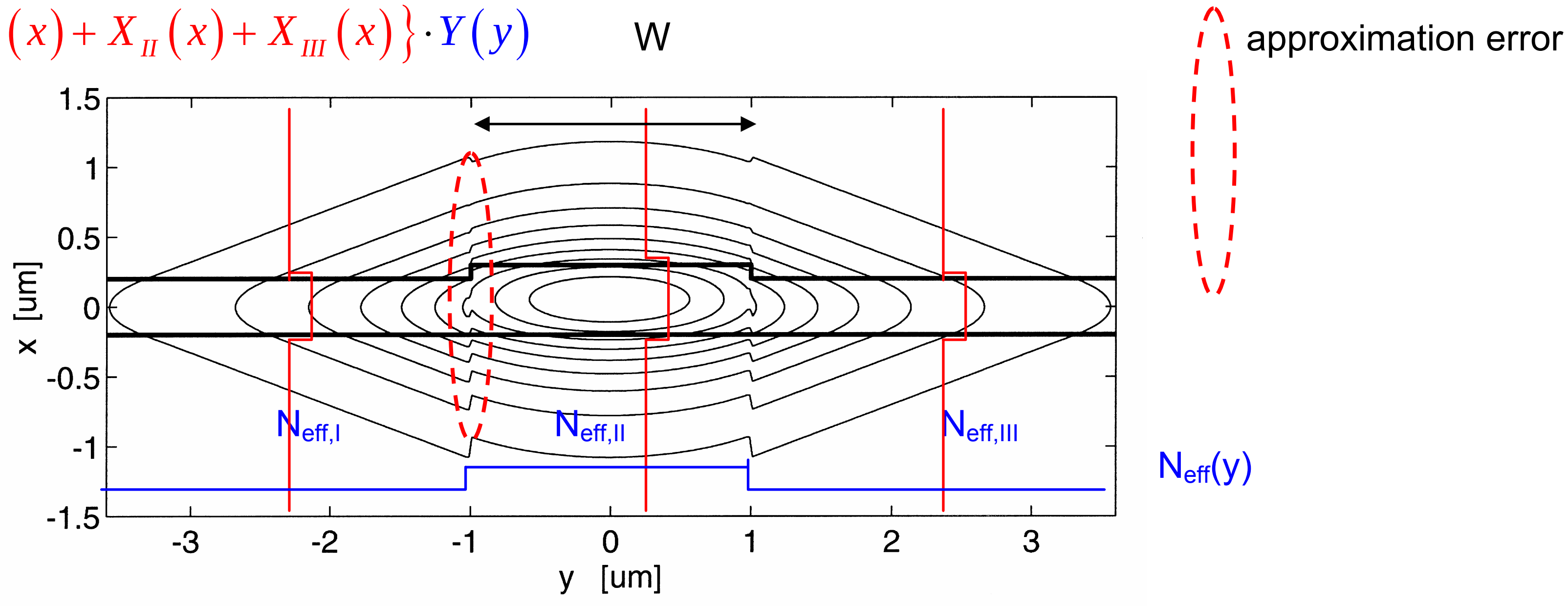
$$\{N_{III}, X_{III}(x)\} = N_{\mathrm{eff}}(n_c, n_g, n_s, d_{III}, \lambda_0, m_X, P_X)$$

vacuum wavelength λ_0 , Polarization P_X, m_x mode index

2. stept lateral (y) WG:

$$\{N_{eff}, Y(y)\} = N_{\mathrm{eff}}(N_I, N_{II}, N_{III}, W, \lambda_0, m_Y, P_Y)$$

➡ $\phi_{m_X, m_Y}(x, y) = \{X_I(x) + X_{II}(x) + X_{III}(x)\} \cdot Y(y)$



……… n(x), d_I n(x), d_{II} n(x), d_{III}

Concept of analysis procedure: what do we want to achieve ?

We translate the Helmholtz-equation to the cylindrical geometry of optical fibers by using cylindrical coordinates.

The solution procedure for the dispersion relation and the field eigenfunctions is identical to the 1-dimensional film WG except that the exponential functions have to be replaced by 2-dimensional cylindrical functions.

From the dispersion of relation $k_z(\omega)=\beta(\omega)$ we can determine frequency dependent group velocity $v_{ph}(\lambda)$ and the dispersion factor $D(\lambda)$ as a function of the waveguide geometry and refractive indices

3.6 Optical Glass Fibers (Repetition 4.sem F&K II)

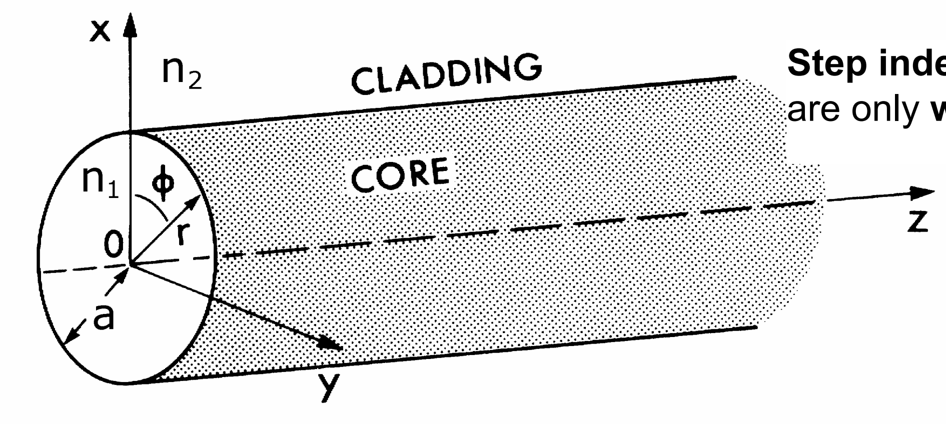
Optical glass fibers are the most important waveguides for long transmission distances in optical communication and therefore **attenuation** and **dispersion** effects limit the max. transmission distance L at a given bit-rate B.

Fiber fabrication uses a preform drawing process leading to **cylindrical wave guides** with a high index core cylinder n_1 (SMF: a~4μm, MMF: a~25-31μm) surrounded by a low index n_2 cylindrical cladding layer of ~250μm diameter.

➨ **cylindrical symmetry of the WG**

The **step index fiber** with an abrupt lateral index difference $\Delta n(a)=(n_2-n_1)$ is the simplest transverse index profile $n(r,\varphi)$.

For symmetry reasons a cylindrical coordinate system (z,r,φ) is the appropriate representation with z as the **longitudinal** propagation direction and r, φ as the **transverse** coordinates.



Step index glass fibers
are only **weakly guiding** with a small $\Delta n=n_1-n_2$~0.01

Assumption: n_2 and n_1 are homogeneous in the core and cladding sections.

For the formulation of the **Helmholtz-equations** we transform the vector operators into the cylindrical coordinates:

Coordinate transformation x, y, z ➡ r, φ, z:

The coordinate transform is straight forward but lengthy, so only the starting point is given.

$$x = r \cdot \cos\varphi \quad ; \quad y = r \cdot \sin\varphi \quad ; \quad z = z$$

$$r = \sqrt{x^2 + y^2} \quad ; \quad \varphi = \arctan(y/x) \quad ; \quad z = z$$

$$\Delta - \text{Operator transform}: \quad f(x,y) = f\left(x(r,\varphi), y(r,\varphi)\right)$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + \frac{\partial f}{\partial r}\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 f}{\partial \varphi^2}\left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{\partial f}{\partial \varphi}\frac{\partial^2 \varphi}{\partial x^2} \qquad \textit{analog for } \frac{\partial^2 f}{\partial y^2}$$

$$\rightarrow \boxed{\Delta_T = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \varphi^2}} \qquad \textit{transverse Laplace-operator in cylindric coordinates}$$

3.6.1 Vector field solutions for the step-index fiber

The step-index fiber has the only simple index profile where the field can be calculated analytically in terms of cylindric Bessel-functions.

1) Transformation of Helmholtz-equations (longitudinal components) into cylindrical coordinates:

$$\boxed{\begin{aligned} \left(\Delta_T + k_T^2\right) E_z(x,y) &= 0 \\ \left(\Delta_T + k_T^2\right) H_z(x,y) &= 0 \end{aligned}} \xrightarrow[\Delta_T = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \varphi^2}]{} \boxed{\begin{aligned} \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \varphi^2} + k_T^2\right) E_z(r,\varphi) &= 0 \\ \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2}{\partial \varphi^2} + k_T^2\right) H_z(r,\varphi) &= 0 \end{aligned}}$$

with the definition for the transverse wave number k_T :

2D eigenvalue differential equation

$$\boxed{k_T^2 = k^2 - \beta^2 = \omega^2 \mu \varepsilon(\vec{r}_T) - \beta^2}$$

As the following derivation is basically just an extension of the technique of the symmetric planar 3-layer WG to 2-dimensions we leave the conversion to cylinder-functions as **(self-study)**.

2) Solution by coordinate separation:

$$\left.\begin{matrix} E_z(r,\varphi) \\ H_z(r,\varphi) \end{matrix}\right\} = \left\{\begin{matrix} A \\ B \end{matrix}\right\} \cdot R(r)\cdot\phi(\varphi)$$

Solution-“Ansatz” with radial and azimuthal separation

Insertion of the “Ansatz” and separating into R(r) and $\phi(\varphi)$ leads to 2 second order, uncoupled differential equations:

$$\frac{\partial^2 R}{\partial r^2}\phi + \frac{1}{r}\cdot\frac{\partial R}{\partial r}\phi + \frac{R}{r^2}\cdot\frac{\partial^2\phi}{\partial\varphi^2} + k_T^2 R\cdot\phi = 0 \quad / \text{multiply on both sides} \bullet \frac{r^2}{R\phi} \rightarrow$$

$$\underbrace{\frac{r^2}{R}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R}\cdot\frac{\partial}{\partial r} + r^2 k_T^2}_{only\ f(r)} = \underbrace{-\frac{1}{\phi}\cdot\frac{\partial^2\phi}{\partial\varphi^2}}_{only\ f(\varphi)} = m^2 = \text{constant} \neq \text{f}(\text{r}) \neq \text{f}(\varphi)$$

$$\boxed{\begin{gathered}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial}{\partial r} + k_T^2 - \frac{m^2}{r^2}\right) R(r) = 0 \\ \left(\frac{\partial^2}{\partial\varphi^2} + m^2\right)\phi(\varphi) = 0\end{gathered}}$$

m=constant !

2 decoupled differential equations for R(r) and $\phi(\varphi)$

m is still an undefined constant.

3) Harmonic azimuthal solutions for $\phi(\varphi)$:

$$\boxed{\phi(\varphi) = \begin{cases} sin(\text{m}\cdot\varphi) \\ cos(\text{m}\cdot\varphi) \end{cases}}$$

➡ for symmetry reason: **m**=0, 1, 2, 3 integer are possible (azimuthal symmetry, node number)

The radial solutions must be radial periodic and symmetric with respect to the z-axis and 2m indicates the number of radial nodes of the field.

4) Bessel-functions for radial solutions for R(r):

The radial Bessel-solutions R(r) depend on k_T and m

function m= 0, 1, 2, 3, ...	physical interpretation	Cartesian correspondence
k_T : real $R(r)=J_m(k_T r)$ Zylinderfunktion 1. Art $\to J_m$: *Besselfunktion*	standing cylindrical wave	$\cos(k_x x)$
$R(r)=N_m(k_T r)$ Zylinderfunktion 2. Art $\to N_m$: *Neumannfunktion*		$\sin(k_x x)$
$R(r)=H_m^{(1)}(k_T r)=J_m(k_T r)+i\,N_m(k_T r)$ $R(r)=H_m^{(2)}(k_T r)=J_m(k_T r)-i\,N_m(k_T r)$ Zylinderfunktion 3. Art $\to H_m^{(1,2)}$: *Hankelfunktionen*	propagating cylindrical wave	$e^{+ik_x\cdot x}$ $e^{-ik_x\cdot x}$
$k_T \to -i\cdot k_T'$: imaginary $R(r)=I_m(k_T' r)=i^m\cdot J_m(-i k_T' r)$ $\to I_m$: *modifizierte Besselfunktionen*	growing cylindrical wave	$e^{k_x\cdot x}$
$R(r)=K_m(k_T' r)=\frac{\pi}{2}(-i)^{m+1}\cdot H_m^{(2)}(-i k_T' r)$ $\to K_m$: *modifizierte Hankelfunktionen*	decaying cylindrical wave	$e^{-k_x\cdot x}$

The type of solutions of the Bessel-differential equation and the carthesian correspondence for the symmetric 3 layer film waveguide

For the graphical representation of cylindrical functions see at summary at the end of the chapter.

We consider the transverse wave number $k_{Ti}^2 = k_i^2 - \beta^2$ corresponding to *medium i.*

For mode confinement as before the eigenvalue β is restricted to the interval $k_2 = k_0 \cdot n_2 < \beta < k_1 = k_0 \cdot n_1$.

General solution for the longitudinal components E_z and H_z for a homogeneous medium section:

$$\left.\begin{matrix}E_z\\H_z\end{matrix}\right\}=\begin{Bmatrix}A_0\\B_0\end{Bmatrix}\cdot Z_0(k_T r)+\sum_{m=1}^{\infty}\begin{Bmatrix}A_1^m\\B_1^m\end{Bmatrix}\cdot Z_m(k_T r)\cdot\cos(m\varphi)+\begin{Bmatrix}A_2^m\\B_2^m\end{Bmatrix}\cdot Z_m(k_T r)\cdot\sin(m\varphi) \qquad (3.153),$$

(depends on azimthal mode number m)

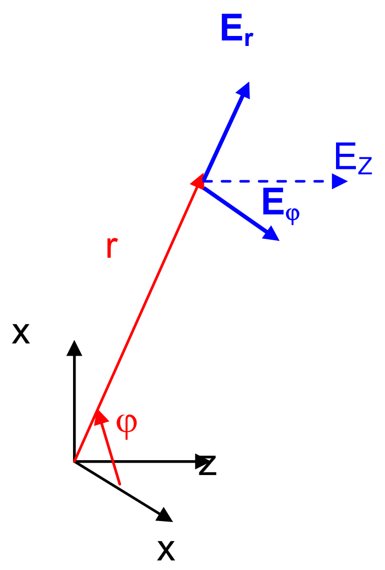
$Z_m(...)$ is a cylinder function from the table for R(r) depending on medium i with n_i (core or cladding) and argument m.

5) Radial continuity conditions of the transversal field at $r = a$ for all $\varphi = 0 \dots 2\pi$:

The formulation of the continuity requires the additional calculation of the transverse field components E_r, E_φ, H_r und H_φ in cylinder coordinates (without proof):

$$\boxed{\begin{aligned}
E_r &= \frac{1}{i k_T^2}\left\{\beta\cdot\frac{\partial}{\partial r}E_z + \omega\mu\cdot\frac{1}{r}\frac{\partial}{\partial\varphi}H_z\right\}\\
E_\varphi &= \frac{1}{i k_T^2}\left\{\beta\cdot\frac{1}{r}\frac{\partial}{\partial\varphi}E_z - \omega\mu\cdot\frac{\partial}{\partial r}H_z\right\}\\
H_r &= \frac{1}{i k_T^2}\left\{\beta\cdot\frac{\partial}{\partial r}H_z - \omega\varepsilon\cdot\frac{1}{r}\frac{\partial}{\partial\varphi}E_z\right\}\\
H_\varphi &= \frac{1}{i k_T^2}\left\{\beta\cdot\frac{1}{r}\frac{\partial}{\partial\varphi}H_z + \omega\varepsilon\cdot\frac{\partial}{\partial r}E_z\right\}
\end{aligned}}
\quad \Rightarrow \quad
\begin{aligned}
E_{z1} - E_{z2} &= 0 \qquad & H_{z1} - H_{z2} &= 0\\
E_{\varphi 1} - E_{\varphi 2} &= 0 \qquad & H_{\varphi 1} - H_{\varphi 2} &= 0
\end{aligned}$$

Boundary conditions



The boundary conditions lead to an infinite set of equations for A_0, B_0, A_1^m, A_2^m, B_1^m und B_2^m, m=1,2, 3 … ∞, but symmetry and rotation invariance properties of the solutions reduce the solution space to:

- ***core ①:*** $n = n_1, \quad k_1 > \beta, \quad r < a$

$$E_z(r,\varphi) = A_1 \cdot J_m(k_T r) \cdot \cos(m\varphi + \varphi_0)$$
$$H_z(r,\varphi) = B_1 \cdot J_m(k_T r) \cdot \sin(m\varphi + \varphi_0)$$

oscillatory transverse wave solution;

- ***cladding ②:*** $n = n_2, \quad k_2 < \beta, \quad r > a$

4 unknown A_1, A_2, B_1, B_2

$$E_z(r,\varphi) = A_2 \cdot K_m(k_T' r) \cdot \cos(m\varphi + \varphi_0)$$
$$H_z(r,\varphi) = B_2 \cdot K_m(k_T' r) \cdot \sin(m\varphi + \varphi_0)$$

decaying, transverse confined , evanescent wave solution

➡ Reduction to 4 terms: A_1, A_2, B_1, B_2 are the desired solutions for a particular m and a particular wave excitation as boundary condition.

The solutions are inserted into the tangential boundary conditions and define a set of equations for A_1, A_2, B_1, B_2 and the unknown propagation constant $\beta(\omega)=k_Z(\omega)$ as eigenvalue:

Using similar substitutions as in the case of the slab waveguide

$$\xi(\beta) = a \cdot k_T = a \cdot \sqrt{k_1^2 - \beta^2} \qquad \eta(\beta) = a \cdot k_T' = a \cdot \sqrt{\beta^2 - k_2^2}$$

we obtain (without proof) following system of eq.

$$\Rightarrow \begin{bmatrix} J_m(\xi) & 0 & -K_m(\eta) & 0 \\ 0 & J_m(\xi) & 0 & -K_m(\eta) \\ \frac{\pm\beta\cdot m}{\xi^2} J_m(\xi) & \frac{\omega\cdot\mu_1}{\xi} J'_m(\xi) & \frac{\pm\beta\cdot m}{\eta^2} K_m(\eta) & \frac{\omega\cdot\mu_2}{\eta} K'_m(\eta) \\ \frac{\omega\cdot\varepsilon_1}{\xi} J'_m(\xi) & \frac{\pm\beta\cdot m}{\xi^2} J_m(\xi) & \frac{\omega\cdot\varepsilon_2}{\eta} K'_m(\eta) & \frac{\pm\beta\cdot m}{\eta^2} K_m(\eta) \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

for m=1,2,3, … ∞ and $Z'(x) = \frac{\partial Z}{\partial x}$

The sign ± corresponds to the *cos*- ($\varphi_0 = 0$) resp. *sin*-solution ($\varphi_0 = \pm\pi/2$) for E_z.

Nontrivial solutions for A_1, A_2, B_1, B_2 only exist if the determinant of the homogeneous system vanishes, leading to the eigenvalue equation for ξ, η:

$$\det \begin{vmatrix} J_m(\xi) & 0 & -K_m(\eta) & 0 \\ 0 & J_m(\xi) & 0 & -K_m(\eta) \\ \frac{\pm\beta\cdot m}{\xi^2} J_m(\xi) & \frac{\omega\cdot\mu_1}{\xi} J'_m(\xi) & \frac{\pm\beta\cdot m}{\eta^2} K_m(\eta) & \frac{\omega\cdot\mu_2}{\eta} K'_m(\eta) \\ \frac{\omega\cdot\varepsilon_1}{\xi} J'_m(\xi) & \frac{\pm\beta\cdot m}{\xi^2} J_m(\xi) & \frac{\omega\cdot\varepsilon_2}{\eta} K'_m(\eta) & \frac{\pm\beta\cdot m}{\eta^2} K_m(\eta) \end{vmatrix} = 0$$

➡ Eigenvalue equation f(η, ξ, m)=0 ➡ β(ω)

Characteristic equation, resp. Eigenvalue equation:

$$\boxed{\left\{ k_1^2\cdot\tilde{J}_m(\xi) + k_2^2\cdot\tilde{K}_m(\eta) \right\}\cdot\left\{ \tilde{J}_m(\xi) + \tilde{K}_m(\eta) \right\} - m^2\beta^2\cdot\left(\frac{1}{\xi^2} + \frac{1}{\eta^2} \right)^2 = 0}$$

with the definitions: $\tilde{J}_m(\xi) = \frac{J'_m(\xi)}{\xi\cdot J_m(\xi)}$; $\tilde{K}_m(\eta) = \frac{K'_m(\eta)}{\eta\cdot K_m(\eta)}$

In analogy to the symmetric planar WG we make use of a single structure parameter or **fiber parameter V(ω)** to eliminate either ξ or η:

$$V(\omega) = a\cdot\sqrt{k_1^2 - k_2^2} = a\cdot k_0\cdot\sqrt{n_1^2 - n_2^2} = a\cdot k_0\cdot NA = \sqrt{\xi^2 + \eta^2} \quad ; \quad k_0 = 2\pi/\lambda_0 = \omega/c_o$$

6) Formal solution procedure for $\beta_{mp}(\omega)$:

- chose an integer m=0, 1, 2, … (azimuthal mode index m) and ω
- eliminate η with the above eigenvalue equation by using $V(\omega)$ and find the *zero* ξ_p with increasing values. There are p zeros (radial mode index p) of the eigenvalue equation for a given m (radial field nodes !)
- from ξ_p we determine $\beta_{mp}(\omega)$ of the modes characterized by the numbers m,p
- for m and p we associate a particular modal solution of **mode *X* $_{m\,p}$** . with X= *HE*-, *EH*-, *TE*- or *TM-modes*.

Classification of Modes X_{pm}: (m= azimuthal mode number, p= radial mode number)

Goal: calculate the propagation constant $\beta_{m,p}(\omega)$ to determine the dispersion properties of the fiber modes

1. class: **m=0** (azimuthally homogeneous), **and β=0** (no cut-off)

1) for m=0 the 4x4 determinant splits into two independent 2x2 sub-determinants for A_1, A_2 (TE) and B_1, B_2 (TM).

➡ **TE-** resp. **TM-modes**

$$m=0 \rightarrow$$

$$\begin{bmatrix} J_m(\xi) & 0 & -K_m(\eta) & 0 \\ 0 & J_m(\xi) & 0 & -K_m(\eta) \\ 0 & \frac{\omega\cdot\mu_1}{\xi}J'_m(\xi) & 0 & \frac{\omega\cdot\mu_2}{\eta}K'_m(\eta) \\ \frac{\omega\cdot\varepsilon_1}{\xi}J'_m(\xi) & 0 & \frac{\omega\cdot\varepsilon_2}{\eta}K'_m(\eta) & 0 \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow$$

$$\begin{bmatrix} J_m(\xi) & -K_m(\eta) \\ \frac{\omega\cdot\varepsilon_1}{\xi}J'_m(\xi) & \frac{\omega\cdot\varepsilon_2}{\eta}K'_m(\eta) \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} J_m(\xi) & -K_m(\eta) \\ \frac{\omega\cdot\mu_1}{\xi}J'_m(\xi) & \frac{\omega\cdot\mu_2}{\eta}K'_m(\eta) \end{bmatrix} \cdot \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

2) m=0 TE- and TM-modes have radial symmetry

3) β=0 represents a mode which has its cutoff at ω=0 ➡ HE11-mode

Dispersion relation $\beta(\omega)$ for TE_{op}-modes $(E_z = 0,\ E_r = 0,\ H_\varphi = 0,)$:

$$\left\{ \tilde{J}_m(\xi) + \tilde{K}_m(\eta) \right\} = 0 \text{ using } J_0'(\xi) = -J_1(\xi) \text{ and } K_0'(\eta) = -K_1(\eta) \Rightarrow \boxed{\frac{J_1(\xi)}{\xi\cdot J_0(\xi)} + \frac{K_1(\eta)}{\eta\cdot K_0(\eta)} = 0} \Rightarrow \beta_{TEop}(\omega)$$

Dispersion relation $\beta(\omega)$ for TM_{op}-modes $(H_z = 0,\ H_r = 0,\ E_\varphi = 0,)$:

$$\left\{ k_1^2\cdot\tilde{J}_m(\xi) + k_2^2\cdot\tilde{K}_m(\eta) \right\} = 0 \Rightarrow \boxed{\frac{k_1^2\cdot J_1(\xi)}{\xi\cdot J_0(\xi)} + \frac{k_2^2\cdot K_1(\eta)}{\eta\cdot K_0(\eta)} = 0} \Rightarrow \beta_{TMop}(\omega)$$

2. class: **m≠0, and β≠0** (with mode-cutoff)

general case ➡ **hybrid modes ($E_z \neq 0$, $H_z \neq 0$)**

classification of modes by dominating z-field component:

a) inspection

$$\lim_{V\to\infty}\left\{\frac{E_z}{H_z}\right\} = \begin{cases} 0 & EH \\ \infty & HE \end{cases}$$

TE-like because H_z is dominant

TM-like because E_z is dominant

b) approximation of **weak guiding** $n_1 \approx n_2 \approx n_{eff}$

The general eigenvalue equation simplifies to

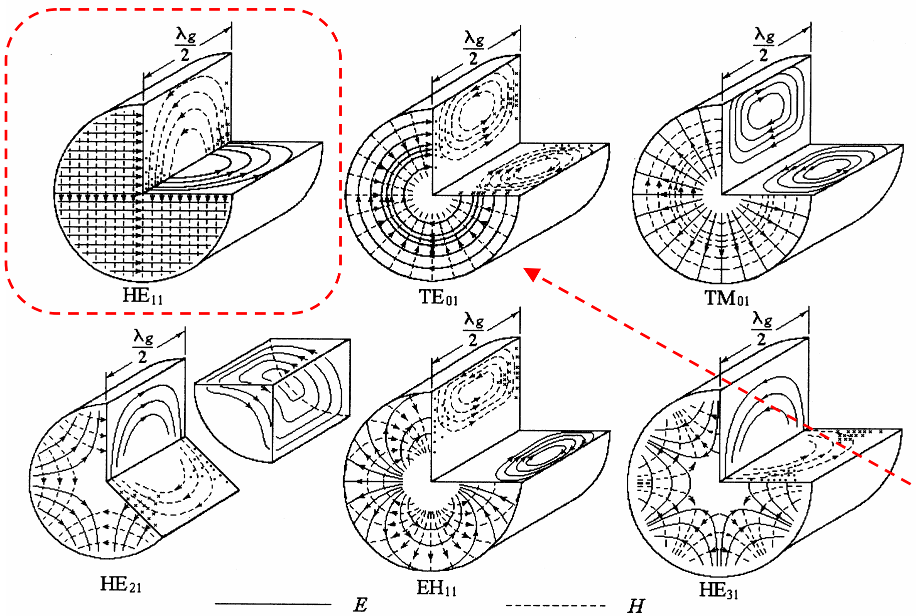
$$\left\{\tilde{J}_m(\xi)+\tilde{K}_m(\eta)\right\} \mp m\cdot\left(\frac{1}{\xi^2}+\frac{1}{\eta^2}\right) = 0$$

sign convention: + for EH_{mp}- and - for HE_{mp}-modes

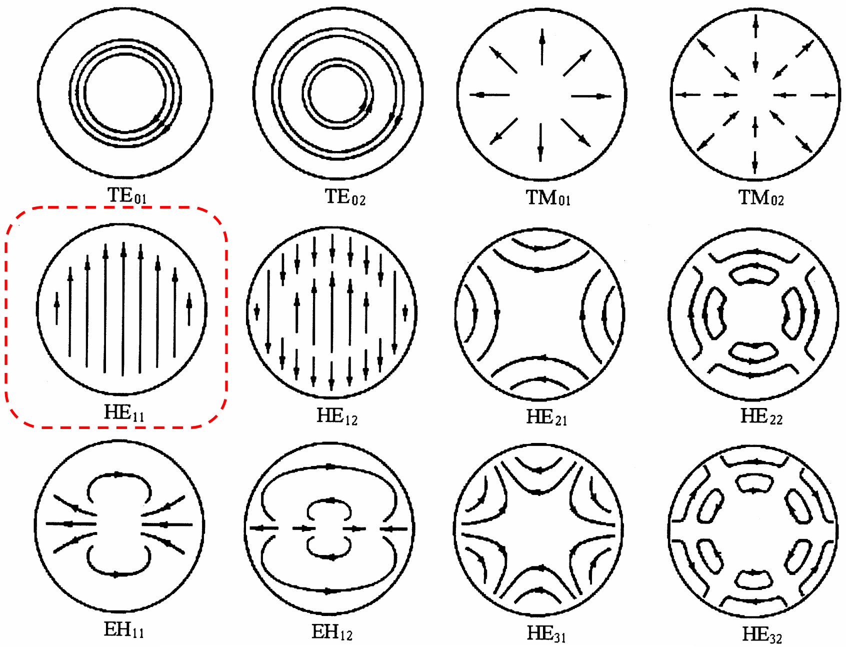
and further to

$$\boxed{\begin{aligned} &\frac{J_{m-1}(\xi)}{\xi\cdot J_m(\xi)}-\frac{K_{m-1}(\eta)}{\eta\cdot K_m(\eta)} \simeq 0 \;\rightarrow\; HE_{mp} \\ &\frac{J_{m+1}(\xi)}{\xi\cdot J_m(\xi)}+\frac{K_{m+1}(\eta)}{\eta\cdot K_m(\eta)} \simeq 0 \;\rightarrow\; EH_{mp} \end{aligned}}$$

➡ Approximate dispersion relation $\beta(\omega)$ for hybrid modes ➡ **$\beta_{HEmp}(\omega)$, $\beta_{EHmp}(\omega)$**



(a) 3-D field patterns for some lower modes

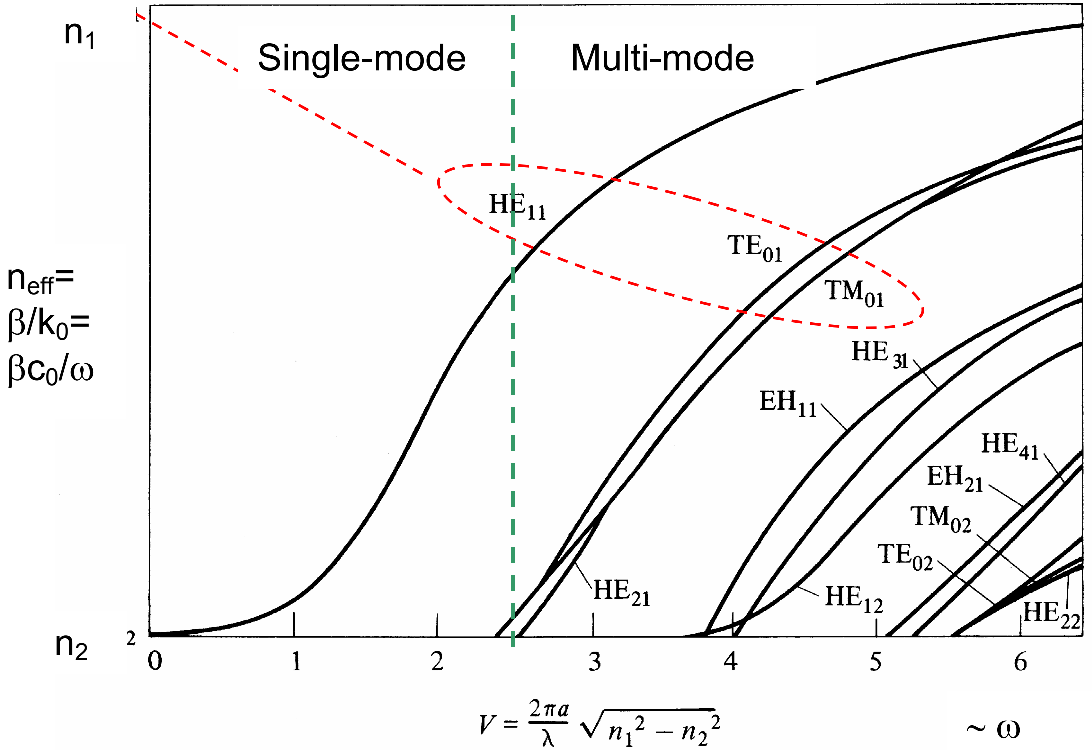


(observe m,p-nodes in the fields!)

Field cross-sections of guided modes in step-index glass fibers:

The most relevant mode is the HE_{11} (m=1, p=1) with a zero-frequency cut-off and single-mode operation.

Dispersion curves $n_{eff}(V)=\beta(V)/\beta_0$ of guided modes in step-index glass fibers: (compare to film WG p.3-31)

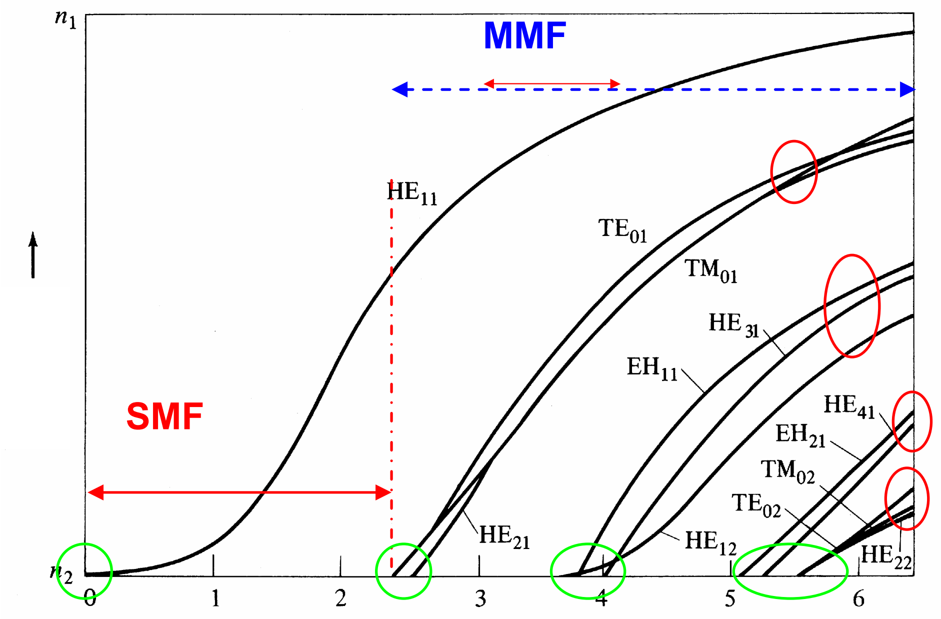


Dispersion in step-index Glassfibers: (similar to the symmetric 3-layer WG)

β(ω) determines the phase- and group-velocities and the modal groupe-velocitiy dispesion D (pulse broadening)

A <u>nonlinear</u> dispersion β(ω) leads to frequency dependent group velocities $v_{gr}(\omega)$ and modal dispersion D(ω)

n_{eff}= β(ω)/k_0



$$V(\omega) = 2\pi a/\lambda\sqrt{n_1^2 - n_2^2} = a\sqrt{\beta^2 - k_2^2}$$

Cutoff-condition V_{mp} of mode (m,p):

- for V< V_{pm}=ξ @ η=0 no pm-mode can exist (cutoff)
- for V> V_{pm} the pm-mode exist and is described by the dispersion relation $\beta_{pm}(\omega)$

At *cutoff* the modes do not decay anymore in the cladding.
Some modes are <u>degenerate</u> at cut-off.

Observe that modes tend to build groups of similar dispersion curves

Cutoff-condition $\eta = a\sqrt{\beta^2 - k_2^2} \rightarrow 0$ **for different modes:** (without proof)

m=0: TE_{0p}, TM_{0p}

$$J_0(\xi) = 0$$

m=1: HE_{1p}, EH_{1p}

$$J_1(\xi) = 0$$

m>1: EH_{mp}

$$J_m(\xi) = 0$$

m>1: HE_{mp}

$$\left(\left(n_1^2/n_2^2\right)+1\right)\cdot J_{m-1}(\xi) = \frac{\xi}{m-1}\cdot J_m(\xi)$$

Conclusions:

1) the **fundamental mode** is the **HE_{11}-mode**, the fiber is fundamental (single) mode for $0<V(\omega)<2.405$ (no intermodal dispersion occurs, but the mode is dispersive)
2) the fundamental mode exists even at $\omega=0$, no cut-off
3) **TE- and TM-modes** are <u>not degenerate</u> (due to rotational symmetry), however they are degenerate at cut-off
4) **Hybrid modes** are 2-times degenerate, because there exist 2 radial solutions

$$cos(n\varphi) \text{ and } cos\left(n\left[\varphi-\frac{\pi}{2n}\right]\right)$$ (90°-rotation. eg. orthogonal polarizations

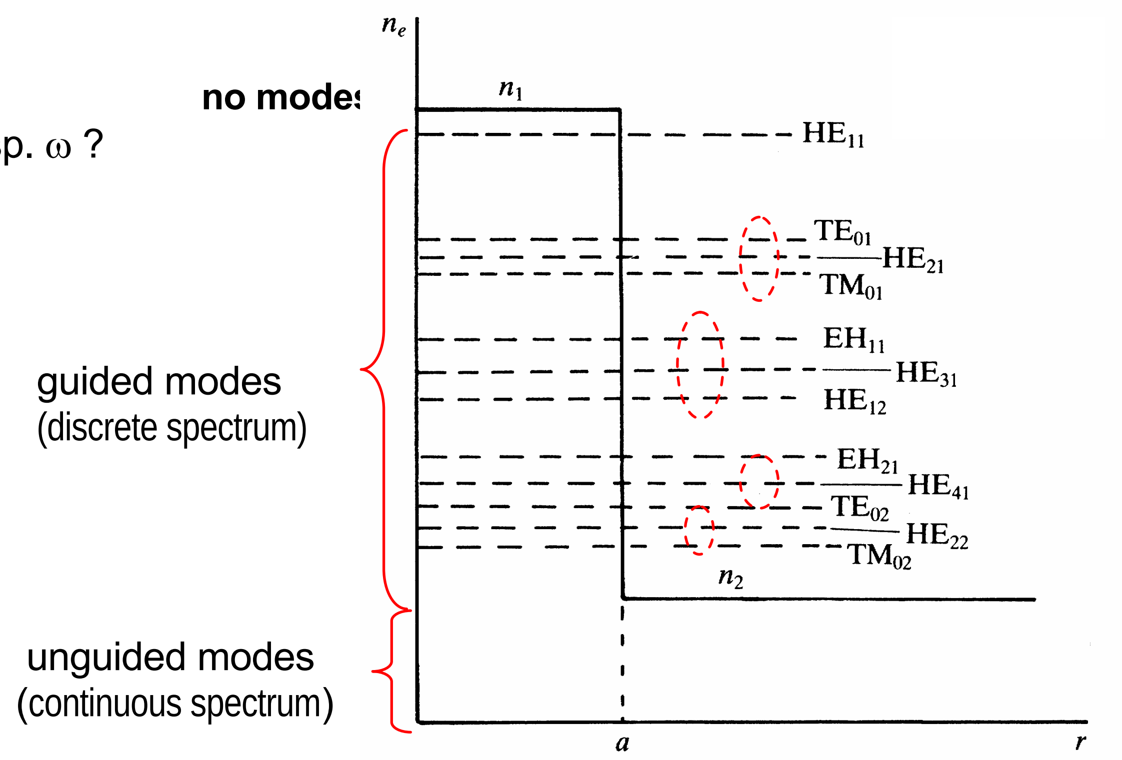
Approximation of Number of modes vers. V:

Question: how many modes N exist for a certain V, resp. ω ?

$$\boxed{N \simeq \frac{V^2}{2} = \frac{a^2 k_o^2 \Delta n}{2}}$$ (without proof)

Step-Index Single Mode Fibers (SMF) must have:

- V<2.45
- small core diameter 2a
- small index difference Δn
- operation at low ω, resp long λ



3.7 Dispersion in weakly guiding optical wave guides

Main transmission limitations of optical waves in optical fibers are

1) ***attenuation*** of the signal by **absorption** (material, scattering) ➨ **amplitude reduction** and

2) frequency dependent propagation, ***dispersion*** (Zerstreuung) ➨ **signal distortion** by the frequency dependence of $n(\omega)$ and $\beta(\omega)$ and spectral width $\Delta\omega$ of the wave .

   ➨ **material or chromatic dispersion** $\beta_{mat}(\omega)$

   ➨ **waveguide dispersion** $\beta_{mode}(\omega)$

3) if several modes X_{pq} can be excited at the signal frequency ω (Multimode Operation in MM-fibers) then the propagation constants $\beta_{pq}(\omega)$ of the modes pq differ, leading to

   ➨ **modal dispersion** (can be avoided by single mode fibers)

Dispersion effects result in pulse broadening (inter-symbol interference) and limit the **data rate B x L – product** of the fiber.

Some dispersion effects can be reversed by **dispersion compensation** introducing dispersion of the opposite sign.

3.7.1 Signal and carrier spectral width $\Delta\omega$:

A quasi-monochromatic ($\Delta\omega_C$) optical carrier wave $A_c(t)e^{i(\omega_o t-\beta_o z)}$ is envelop- or amplitude **modulated** ($\Delta\omega_S$) by a **signal $A_s(t)$** and by intrinsic **fluctuations of the carrier itself $A_C(t)$** in the time-domain (eg. LASER source):

$$\boxed{E(t,z)=A_s(t)A_c(t)e^{i(\omega_o t-\beta_o z)}}$$

time-domain

The total optical spectrum (carrier and signal sidebands) composed of the 2 spectral contributions from signal and carrier:

$$\boxed{E(\omega,z)=A(\omega-\omega_0)=\underbrace{A_s(\omega-\omega_0)}_{\text{signal spectrum}}*\underbrace{A_c(\omega-\omega_0)}_{\text{carrier spectrum}}}$$

frequency-domain ➨ $\Delta\omega$, resp. $\Delta\lambda=?$, *Envelope-spectrum $A(\omega)$* and *time-function $A(t)$* form a Fourier-pair:

$$A(t) \underset{F^{-1}}{\overset{F}{\rightleftarrows}} A(\omega) \quad \textit{with a spectral width: } \Delta\omega$$

The optical spectrum $E(\omega)=A(\omega-\omega_0)$ is obtained by a frequency translation of ω_0.

Dispersion effects depend on the total spectral width $\Delta\omega$ of the modulated wave, therefore we analyze different situations where the **signal-** ($A_s(\omega)$) or the **carrier**-($A_c(\omega)$) spectrum might be dominant.

a) Carrier spectrum $\Delta\lambda_C$, $\Delta\omega\lambda_C$:

ideal coherent light source: $A_c(\omega\text{-}\omega_0)=\delta(\omega\text{-}\omega_0) \rightarrow \Delta\lambda_c \triangleq 0$ *spectral width ; e.g. noise-free Single Frequency Laser*

partial coherent light source: $\rightarrow \Delta\lambda_c \triangleq$ *serveral GHz - 100 GHz (several nm) e.g. Multimode Laser*

incoherent light source (optical noise field): $\rightarrow \Delta\lambda_c \triangleq$ *serveral THz e.g. LED (several 10nm)*

An ideal harmonic optical carrier would have zero spectral $\Delta\lambda_c=0$ width and a Dirac-function spectrum $\delta(\omega-\omega_o)$. A single frequency DFB-Laser can produce such a field approximately with a $\Delta\omega_c$~ 10MHz -10 GHz.

b) Signal spectrum $\Delta\lambda_S$; $\Delta\omega\lambda_S$:

Envelope-spectrum: $A_S(t) \underset{F^{-1}}{\overset{F}{\rightleftarrows}} A_S(\omega)$ *with a spectral width:* $\Delta\omega_s$ $\Delta\omega_s \sim 1/B$ typ. GHz – several 10 GHz

c) The total spectrum $\Delta\lambda$; $\Delta\omega\lambda$ of the modulated carrier wave is dominated

a) by the signal spectrum $E(\omega,0)=A_S(\omega-\omega_0)$; $\Delta\omega \cong \Delta\omega_s$ (ideal coherent light source, dynamic SM-LD)

b) by the carrier source $E(\omega,0)=A_C(\omega-\omega_0)$, $\Delta\omega \cong \Delta\omega_c$ (MM-LD, LED)

c) both carrier and source $\Delta\omega \cong f(\Delta\omega_s, \Delta\omega_c)$ (real quasi-single mode LD)

3.7.2 Signals with finite spectral width $\Delta\omega$, $\Delta\lambda$ in media with non-linear dispersion $\beta(\omega)$:

Chap.2 showed that frequency components traveling in a dispersive medium at different, frequency dependent velocities $v_{ph}(\omega)$, $v_{gr}(\omega)$ need different transit times $\tau(\omega)=L/v_{gr}$ for a fiber length L:

1) the carrier wave travels with the **phase velocity** $v_{ph}(\omega)=\omega/\beta$ and

2) the envelop A travels with the **group velocity** $v_{gr}(\omega)=\partial\omega/\partial\beta=1/(\partial\beta/\partial\omega)$

The resulting dispersion is characterized by the

Group velocity dispersion (GVD): (definition for λ)

$\Delta\tau_g$ is the propagation delay difference over the spectral width $\Delta\omega$

$$\Delta\tau_g = \frac{d\tau_g}{d\lambda}\cdot\Delta\lambda = -\frac{L}{2\pi c_0}\cdot\left\{2\lambda_0\cdot\frac{d\beta}{d\lambda}+\lambda_0^2\cdot\frac{d^2\beta}{d\lambda^2}\right\}\cdot\Delta\lambda = |D|L\Delta\lambda$$

➡ need to know $\beta(\omega)$ for the mode and the material !

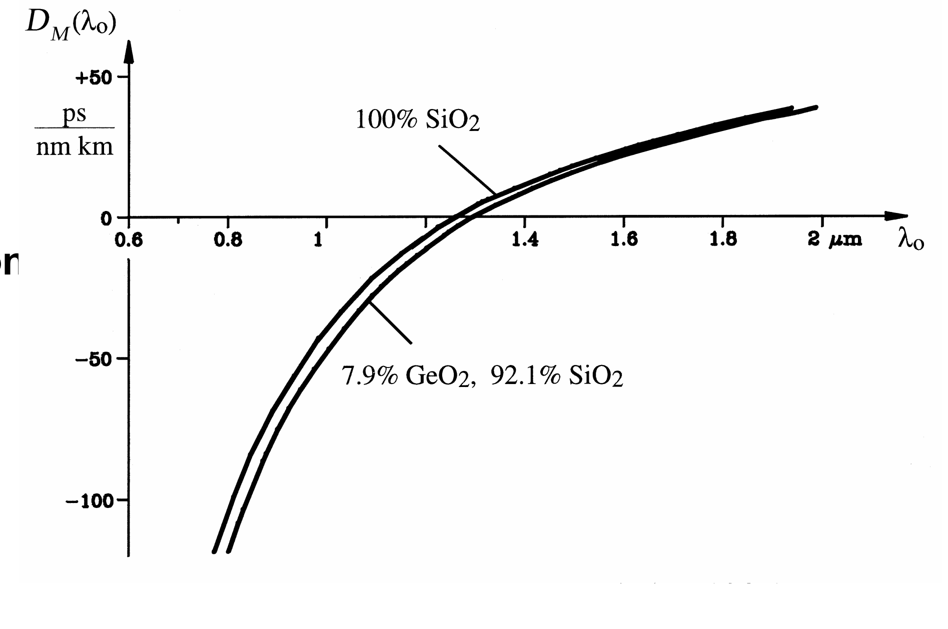
Material and modal Fiberdispersion

a) Material dispersion (without WG)

Dispersion due to the **frequency dependence of the polarization $P(\omega)$** is described by the **frequency dependence of the refractive $n(\omega)$.**

$$\underline{\Delta\tau_g} = \frac{d\tau_g}{d\lambda}\cdot\Delta\lambda = -\frac{L}{2\pi c_0}\cdot\left\{2\lambda_0\cdot\frac{d\beta}{d\lambda}+\lambda_0^2\cdot\frac{d^2\beta}{d\lambda^2}\right\}\cdot\Delta\lambda =$$

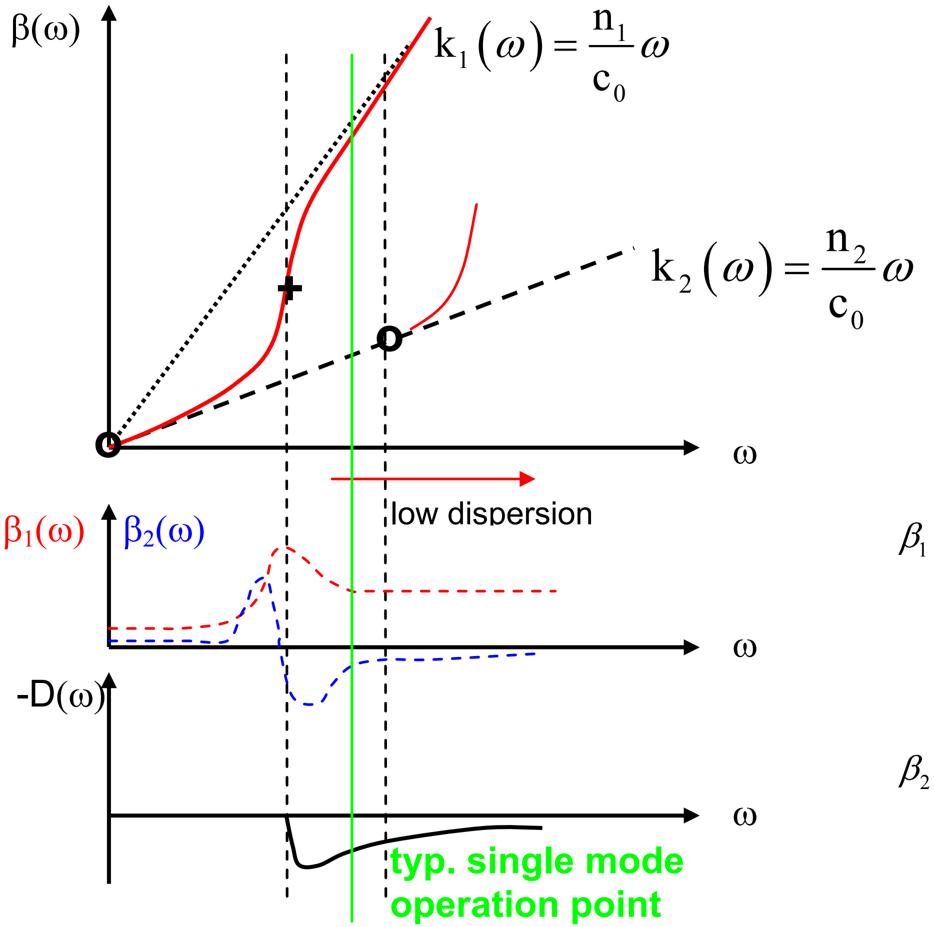
$$= -\frac{L}{c_0}\cdot\left\{\lambda_0\cdot\frac{dn}{d\lambda}+\lambda_0\cdot\frac{d^2n}{d\lambda^2}\right\}\cdot\Delta\lambda = \underline{|D_{mat}|L\Delta\lambda}$$



Material dispersion parameter $D_M(\ _0)$ for SiO_2 and GeO_2–SiO_2 glasses

b) Waveguide Modal Dispersion (without material dispersion n≠f(ω))

Qualitative description of dispersion in the β(ω)-**representation**: $n_1 > n_2$



Interpretation:

The frequency dependency of β(ω) of a WG with n≠f(ω) results from the fact, that the <u>transverse mode-profile is frequency dependent</u>.

➨ the mode "sees" different portions of the "fast" cladding and the "slow" core with changing frequency ω.

$$\beta_1(\omega) = \frac{\partial \beta}{\partial \omega} \rightarrow v_{gr}(\omega) = \frac{1}{\beta_1}$$

$$\beta_2(\omega) = \frac{\partial^2 \beta}{\partial \omega^2} \rightarrow D = \beta_2\left(-\frac{2\pi c_0}{\lambda_0}\right)$$

➨ $D_{tot} \cong D_{material} + D_{modal}$

Normalized representation representation of dispersion B(V) instead of β(ω):

Formal definitions for weakly guiding fibers ($n_{clad} \sim n_{core}$ ➡ $\beta \sim k_1 \sim k_2$):

1) Normalized refractive index difference Δ(ω):

$$Definition\ \Delta: \quad n_2 = n_1 \cdot (1-\Delta) \xrightarrow[\Delta<<1]{} \Delta = \frac{n_1 - n_2}{n_1} \simeq \frac{n_1^2 - n_2^2}{2n_1^2}$$ or

2) Normalized Frequency by using V(ω): ω-transformation

$$V(\omega) = k_0 a \cdot NA = k_0 a / n_2 \cdot \sqrt{n_1^2 - n_2^2} \simeq k_1 a \cdot \sqrt{2\Delta} \approx k_2 a \cdot \sqrt{2\Delta} = \frac{n_2}{c_0} \omega\, a \cdot \sqrt{2\Delta} \quad \sim \omega$$

using: $n_1^2 - n_2^2 \approx n_1^2 \cdot 2\Delta \approx n_2^2 \cdot 2\Delta$

3) Normalized Phase B(ω):

➡ **β-transformation to the [0,1]-interval**

The eigenvalue $\beta(\omega)$ in the interval

$\beta \in [k_2, k_1]$

of the characteristic eigenvalue equation

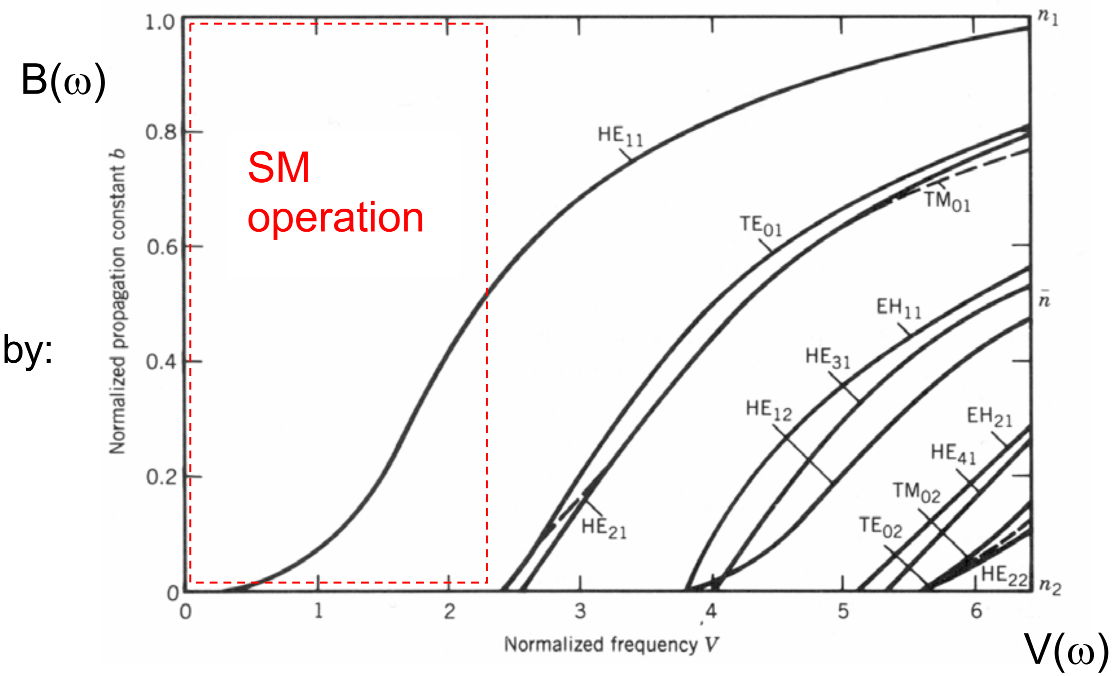
$C(\beta, \omega) = 0$ is transformed into the

normalized Phase B(ω) in the unit-interval $B \in [0, 1]$ by:

$$B(\omega) \underbrace{=}_{Definition} \frac{\beta^2(\omega) - k_2^2}{k_1^2 - k_2^2} = 1 - \frac{\xi^2}{V^2} = \frac{\eta^2}{V^2} \approx \frac{\beta - k_2}{k_1 - k_2}$$

$$\underset{[0,1]}{B(V(\omega))} \longleftarrow \underset{[0,\infty]}{\beta(\omega)}$$

Typical dimensionless model dispersion diagram B(V) of fiber-modes: (from Agrawal)



Total (material and waveguide) dispersion expressed from B(V) and n(ω):

Core and cladding indices $n_1(\omega)$ and $n_2(\omega)$ are now also frequency-dependent. In addition the frequency dependence of the solution of the modal eigenvalue problem for $\beta(\omega)$ and $B(\omega)$ describes the **structural dispersion**.

➡ both frequency dependencies define the **total dispersion**.

For weakly ($n_1(\omega)$~$n_2(\omega)$ ➞ β~k_1~k_2) guiding structures:

$$B(\omega) \approx \frac{\beta(\omega)-k_2(\omega)}{k_1(\omega)-k_2(\omega)} \quad \rightarrow \quad \beta(\omega) \approx k_2(\omega)+B(\omega)\cdot\left(k_1(\omega)-k_2(\omega)\right) \approx k_2(\omega)\cdot\left(1+B(\omega)\cdot\Delta(\omega)\right) \quad with \quad \Delta(\omega)=\left(k_1(\omega)-k_2(\omega)\right)/k_2(\omega)$$

(3.204).

The general definition of the **group delay time τ_g using β and B** is:

$$\tau_g = \frac{L}{v_g} = L\cdot\frac{d\beta}{d\omega} \overset{defing\ \phi=L\beta}{\overbrace{=}} \frac{d\phi}{d\omega} = \frac{L}{c_0}\cdot\frac{d\beta}{dk_0} \underbrace{=}_{k=2\pi/\lambda} -\frac{L}{c_0}\cdot\frac{\lambda_0^2}{2\pi}\cdot\frac{d\beta}{d\lambda}$$

leads with the substitution of β by B to:

$$\tau_g = \frac{L}{c_0}\cdot\frac{d\beta}{dk_0} \approx \frac{L}{c_0}\cdot\frac{d}{dk_0}\left\{k_2+B\cdot\left(k_1-k_2\right)\right\}$$

For the calculation of $\frac{d}{dk_0}\left\{k_2+B\cdot\left(k_1-k_2\right)\right\}$; *resp.* $\frac{dB}{dk_0}$ we use of the weak guiding approximations ($n_1(\omega)$~$n_2(\omega)$):

$$\frac{dk_i}{dk_0} = \frac{d\left(k_0\cdot n_i\right)}{dk_0} = n_i + k_0\cdot\frac{dn_i}{dk_0} = n_{gr,i} \quad \text{group index (material contribution)}$$

$$\frac{dB}{dk_0} = \frac{dB}{dV}\cdot\left(\frac{dV}{dk_0}\right) \underbrace{\approx}_{\text{assuming: } n_1(\omega)\sim n_2(\omega)} \frac{dB}{dV}\cdot\left(\frac{V}{k_0}\right) \quad \text{waveguide structure contribution} \quad ; \quad \text{having used:} \quad \frac{dV}{dk_0}\approx\frac{V}{k_0}$$

$$\underline{\frac{d}{dk_0}\{B\cdot(k_1-k_2)\}} = \frac{dB}{dV}\cdot\frac{V}{k_0}\cdot(k_1-k_2)+B\cdot\left(n_{gr,1}-n_{gr,2}\right) \approx \underline{\left(n_{gr,1}-n_{gr,2}\right)\cdot\frac{d(VB)}{dV}} \qquad \text{using} \quad k_1 - k_2 = k_0\left(n_1 - n_2\right) \simeq k_0\left(n_{gr,1}-n_{gr,2}\right) \quad (3.206).$$

The simplification $dV/dk_0 \approx V/k_0$ meaning that we neglect the frequency dependence of the $\Delta\omega$, resp. frequency dependence of core and cladding is the same:

$$V = a\sqrt{k_1^2 - k_2^2} = ak_0\sqrt{n_1^2 - n_2^1} \xrightarrow[\Delta=(n_1^2-n_2^1)/2n_1^2]{\text{neglecting the frequency dependence of}} \frac{dV}{dk_0} = \left(\frac{V}{k_0}\right)$$

The third equation assumes that $n_{gr1} - n_{gr2} \approx n_1 - n_2$ meaning that the material dispersion of core and cladding is similar.

$$\boxed{\tau_g(VB) \simeq \frac{L}{c_0}\cdot\left\{n_{gr,2}+\left(n_{gr,1}-n_{gr,2}\right)\cdot\underbrace{\frac{d(VB)}{dV}}_{structure}\right\} \approx \frac{L\cdot n_2}{c_0}\cdot\left\{1+\Delta\cdot\frac{d(VB)}{dV}\right\}} \qquad \frac{d(VB)}{dV} \text{ mode delay time factor, } \quad \Delta = \frac{n_1-n_2}{n_1}$$

The group delay dispersion $\Delta\tau_g$ can be obtained by a derivation of the group delay time τ_g *with respect to* λ:

$$\Delta\tau_g = \frac{d\tau_g}{d\lambda}\cdot\Delta\lambda = \frac{d}{d\lambda}\left(\frac{L}{c_0}\cdot\left\{n_{gr,2}+\left(n_{gr,2}-n_{gr,1}\right)\cdot\frac{d(VB)}{dV}\right\}\right)\cdot\Delta\lambda$$

The derivation of $\frac{d}{d\lambda}(\{\ldots\ldots\})$ is simplified by assuming

1) $dV/d\lambda \approx - V/\lambda$ because V~ω

2) the material dispersion in core and cladding are assumed to be equal $d\{n_{gr1} - n_{gr2}\}/d\lambda \rightarrow 0$ (equal group indices of refraction)

$$\frac{d}{d\lambda}\left\{\left(n_{gr,2}-n_{gr,1}\right)\cdot\frac{d(VB)}{dV}\right\} = \cancel{\frac{d}{d\lambda}\left(n_{gr,2}-n_{gr,1}\right)}\cdot\frac{d(VB)}{dV}+\left(n_{gr,2}-n_{gr,1}\right)\cdot\frac{d^2(VB)}{dV^2}\cdot\frac{dV}{d\lambda}\approx$$

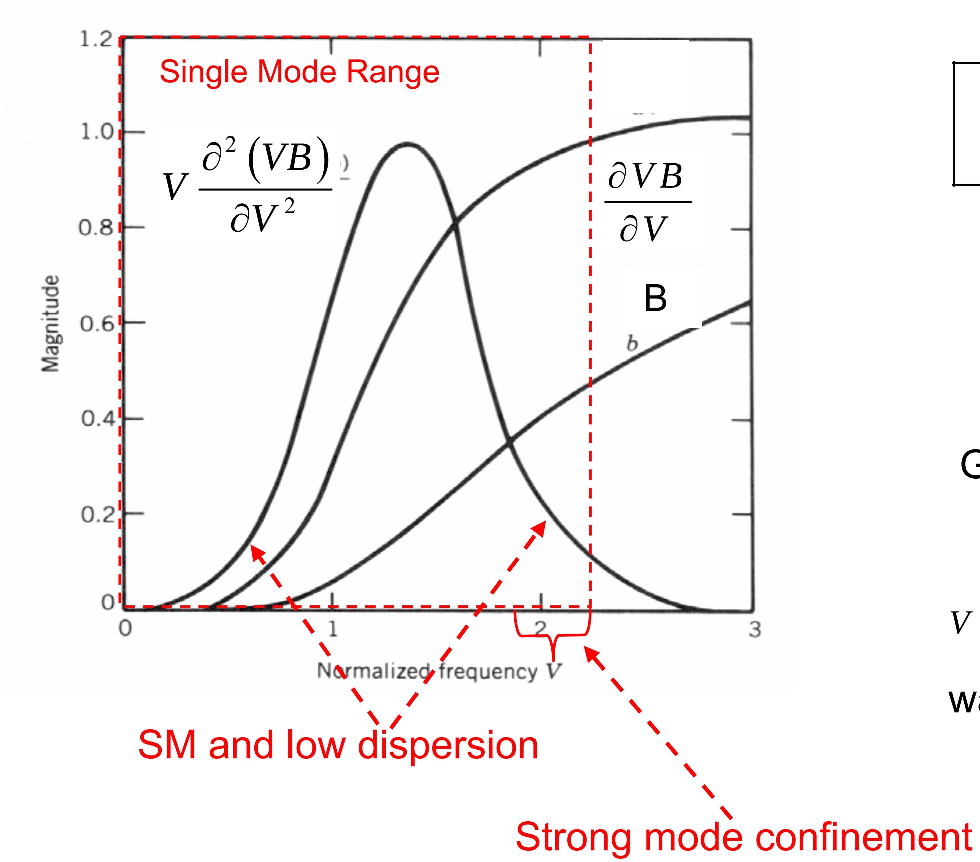
$$\text{using}: \quad \frac{\partial V}{\partial\lambda}=\frac{\partial V}{\partial k}\frac{\partial k}{\partial\lambda}=-\left(\frac{2\pi}{\lambda^2}\right)\frac{\partial V}{\partial k} \quad \rightarrow$$

$$\frac{d}{d\lambda}\left\{\left(n_{gr,2}-n_{gr,1}\right)\cdot\frac{d(VB)}{dV}\right\}\approx-\left(n_{gr,2}-n_{gr,1}\right)\cdot\frac{d^2(VB)}{dV^2}\cdot\frac{V}{\lambda_0} \qquad (3.211)$$

With these simplification the total group-delay $\Delta\tau_g$ including **material (D_m)** and **waveguide (D_W) dispersion** is:

$$\boxed{\Delta\tau_g=\frac{d\tau_g}{d\lambda}\cdot\Delta\lambda=L\cdot\left\{\underbrace{\frac{1}{c_0}\cdot\frac{dn_{gr,2}}{d\lambda}}_{\text{material dispersion (core)}}-\underbrace{\frac{\left(n_{gr,1}-n_{gr,2}\right)}{c_0\cdot\lambda_0}\cdot V\cdot\frac{d^2(VB)}{dV^2}}_{\text{wave guide dispersion}}\right\}\cdot\Delta\lambda=\left(D_{Material}+D_{Waveguide}\right)\cdot L\cdot\Delta\lambda}$$

Dispersion parameters for the HE_{11}-mode (mode with zero frequency cut-off) for the step-index fiber in normalized representation:



$$D_{Waveguide} = -\frac{n_{gr,1} - n_{gr,2}}{c_0 \cdot \lambda_0} \cdot V \cdot \frac{d^2(VB)}{dV^2} \approx -\frac{n_2 \cdot \Delta}{c_0 \cdot \lambda_0} \cdot V \cdot \frac{d^2(VB)}{dV^2}$$

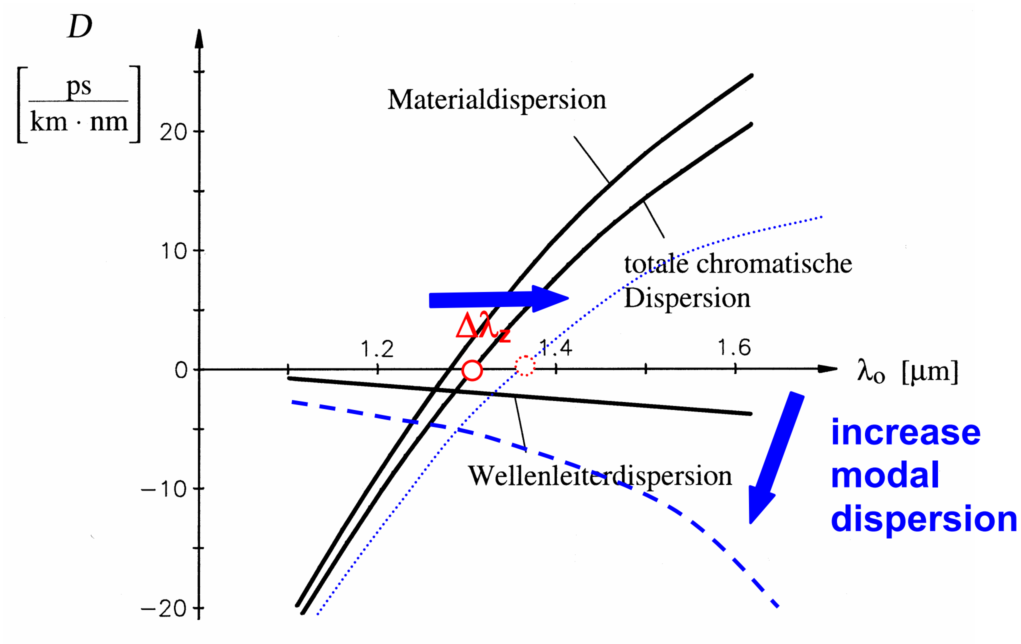
Good approximation when mode is well confined to the core !

$V \cdot \frac{d^2(VB)}{dV^2}$ ~ -D is the **dispersion factor** and determines the waveguide dispersion and goes to 0 for large V (core propagation)

Conclusions:

- Waveguide dispersion for the HE_{11}-mode reaches a maximum between the cut-off frequency and the onset of the next higher order mode. Dispersion is **negative (!)** and **decreases with increasing frequency**.
- For the HE_{11}-mode operation in the V-interval $2 < V < 2.405$ is optimal (most of the field energy is concentrated in the core). The next higher order mode would start at V>2.405.
- The WG has to be operated close to the single frequency operation limit (onset of a new mode)

Total Dispersion for a step-index fiber:



- In general waveguide dispersion is much weaker than the dispersion of the material glass
- Waveguide dispersion has a <u>negative sign</u> and can compensate <u>positive</u> material dispersion

  Enhancing the modal dispersion will shift the zero dispersion to longer wavelength λ_z and flatten the total chromatic dispersion.
- The total waveguide dispersion for a simple glass step-index fibers has a zero at λ_z~1300nm
- The dispersion is however ~15ps/nm km at the loss minimum of ~1500nm

Dispersion compensated fibers: (optional)

Goal: obtain low dispersion over the low loss range from 1300 – 1600nm (telecom range) by compensating effects:

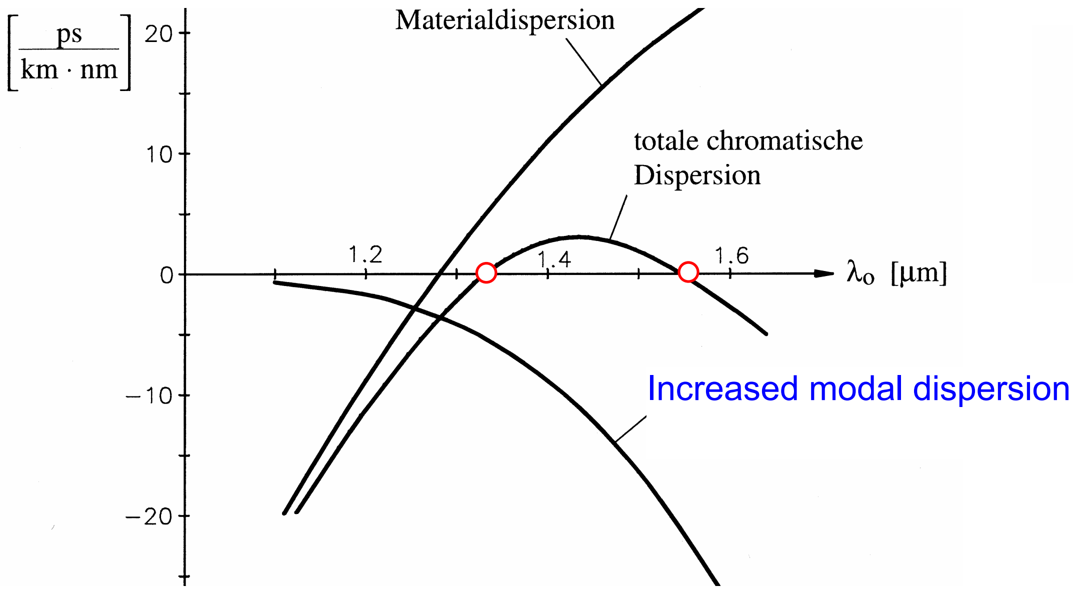
1. solution: **shifting the dispersion zero** ➡ dispersion shifted fibers

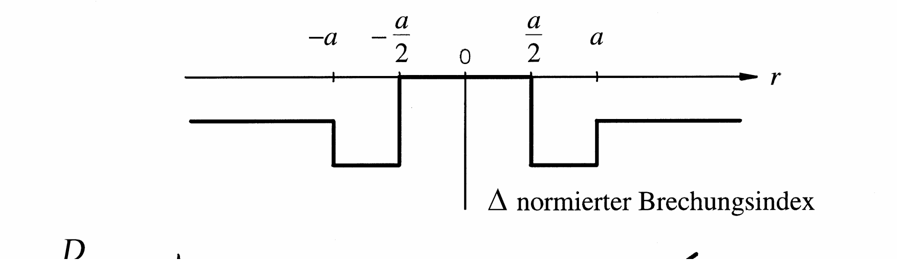
   change of B(V) by reducing the core radius a and increasing the normalized refractive index difference Δ results in a dispersion zero at 1550nm

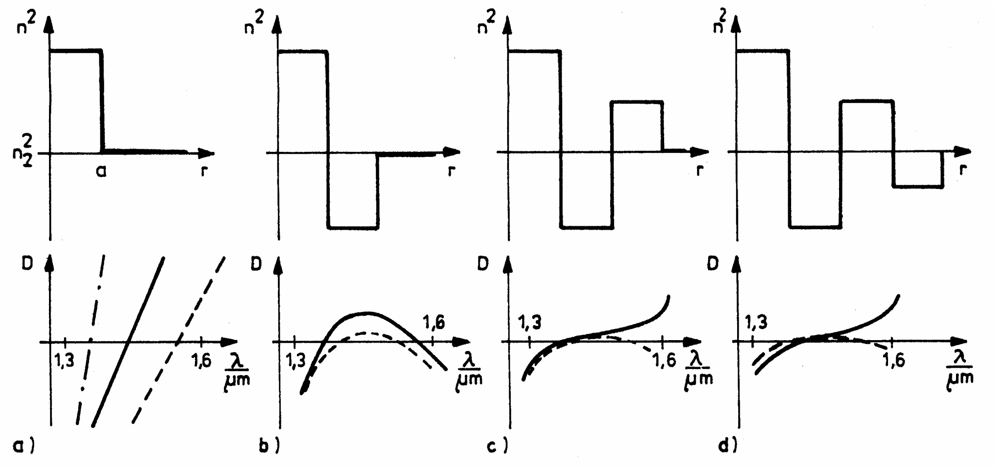
2. solution: **multiple cladding layers** (index-profile modification) ➡ dispersion flattened fibers

   Increase in mode dispersion leads to a 2. dispersion zero and flattening of D between the 1. and 2. zero

Example of a broad band dispersion reduction in a dispersion flattend fiber







shifted flattened

dot-dash line is material dispersion,

full line is the achievable, practical total dispersion

dash line is an idealized total dispersion achievable within material limits.

3. solution: **series connection of a fiber with opposite dispersion** (or grating delay line) ➡ dispersion compensator

3.7.3 Systems aspect of Dispersion, Data Rate – Distance Product

In chap.2 we showed that the dispersion $\beta(\omega)$ leads to pulse envelope deformation and to a reduction of the maximum data rate x length product B^{x}L due to **symbol interference** (digital) or **waveform distortion** (analog).

Practically we have several additive distortion (delay) mechanisms adding to the total delay time dispersion $\delta\tau$

1) **statistically independent** (uncorrelated) dispersion mechanisms $\Delta\tau_l$ add up

statistically: $\langle\Delta\tau\rangle^2 = \sum_i \langle\Delta\tau_i\rangle^2 \quad \Rightarrow \quad D = \sqrt{\sum_i D_i^2}$

2) **correlated** dispersion mechanism (eg. material and waveguide dispersion, acting on the same spectrum)

additive: $D = D_{Material} + D_{Waveguide}$

resulting in the **bit-rate x length product** (BxL) approximation for the chromatic dispersion:

$$\boxed{B\cdot\langle\Delta\tau\rangle = \frac{\langle\Delta\tau\rangle}{T} = BL\cdot|D|\cdot\Delta\lambda < 1}$$

(simple interference approximation with the bit-interval T=1/B !)

$\Delta\lambda$ is the total spectral width of carrier and modulating signal with the typical optical narrow band assumption $\Delta\lambda \ll \lambda_0$

A calculation of dispersion effects on the bit-envelope $A_s(t,z)$ must include the **nonlinearity** of the **dispersion** $\beta(\omega)$.

$\beta(\omega)$ is represent modal and material effects by a Taylor-expansion around the optical carrier ω_0:

$$\beta(\omega) = \sum_{n=0}^{\infty}\frac{1}{n!}\cdot\frac{d^n\beta}{d\omega^n}\cdot(\omega-\omega_0)^n \approx \beta_0 + \underbrace{\beta_1}_{1/v_{gr}}\cdot\Delta\omega + \tfrac{1}{2}\underbrace{\beta_2}_{\substack{GVD\\ parameter}}\cdot\Delta\omega^2 + \tfrac{1}{6}\beta_3\cdot\Delta\omega^3 + \dots \quad with \quad \beta_i = \frac{1}{i!}\cdot\frac{\partial^i}{\partial\omega^i}\bigg|_{\omega 0}\beta_i$$

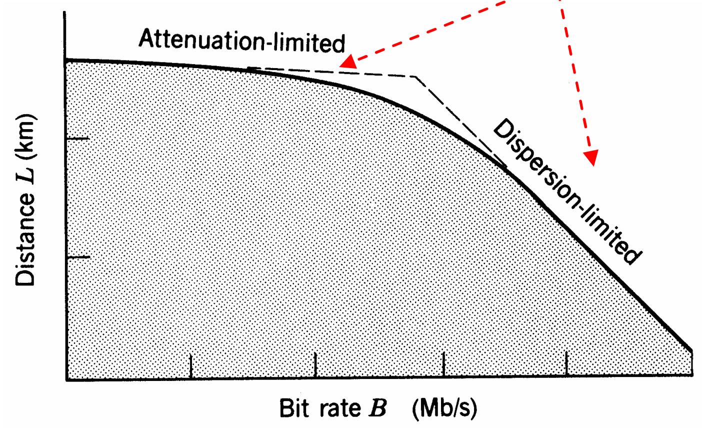
Pulse broadening leads to symbol interference and subsequent bit error rate degradation which we restrict by a simplified statement to obtain the max. bit rate B:

$T_0'(L) < T_B/4$ with $T_B = 1/B$ = bit-time slot and B=bit rate ➡

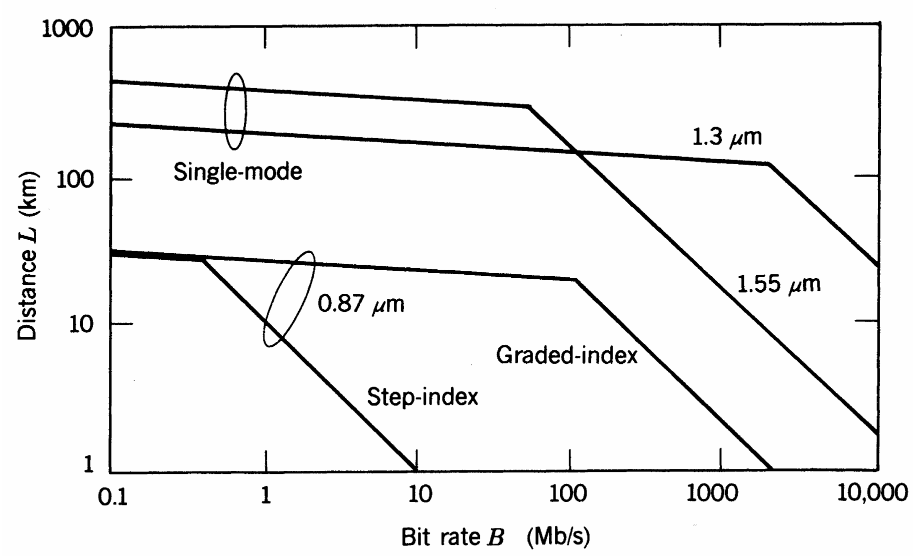
$$BL \cdot |D| \cdot \Delta\lambda_0 \leq \tfrac{1}{4}$$ **Dispersions limit** B ~ 1/L, resp. L_{max} ~ 1/B

If the attenuation dominated dominates: **Attenuation limit**

$$P_{Quelle[dB]} - \alpha \cdot L > P_{Empfänger[dB]} \sim B \rightarrow L_{max} = L_0 - (10/\alpha) \cdot \log(B)$$



Schematic of attenuation and dispersion limit



Attenuation and dispersion limit for different fibers and wavelengths

Conclusions and summary:

- For harmonic guided EM-field the solutions can be obtained from an eigenvalue of an eigenvalue equation of the Helmholtz-equations for E_z, H_z including boundary conditions

  The eigenvalue determines the transverse field profile and the longitudinal propagation constant.

  The two longitudinal field components E_z, H_z are a minimal set of independent field variables.

- Because the longitudinal components E_z, H_z serve as independent field variables, the other components can be derived from E_z and H_z.

- The propagation constant $\beta(\omega)$ is frequency dependent, even if the refractive indices are frequency independent and represents the waveguide dispersion.

- material dispersion $\beta(\omega)$ leads to pulse broadening and limited transmission rates. Waveguide dispersion can be used to compensate material dispersion of opposite sign.

- The waveguide dispersion results from the fact that the transverse mode pattern is also frequency dependent and the transverse mode profile “sees” different portion of the “fast” cladding and the “slow” core. This results in a frequency dependent group velocity of the mode.

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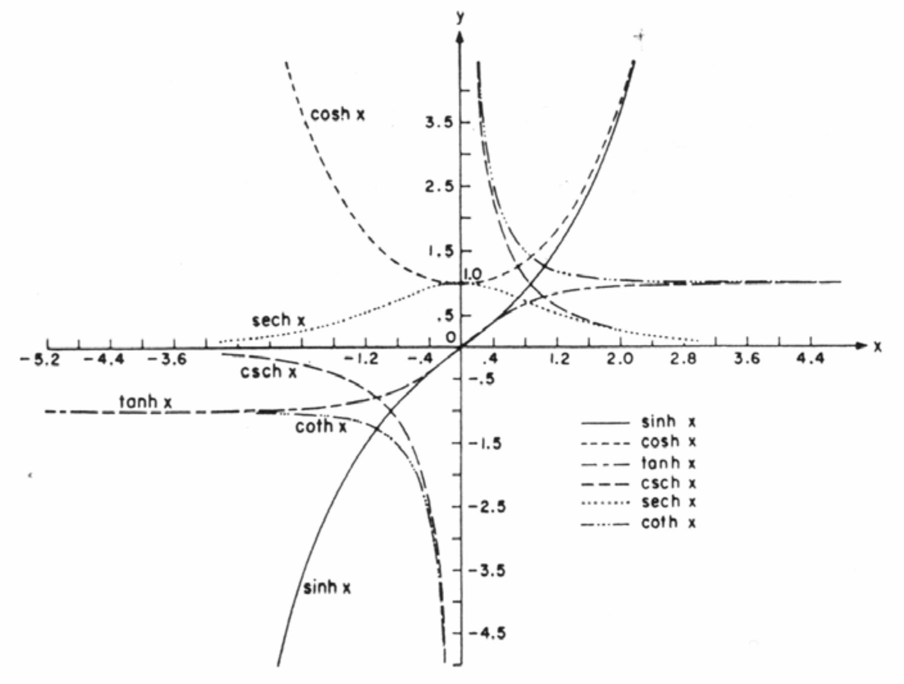
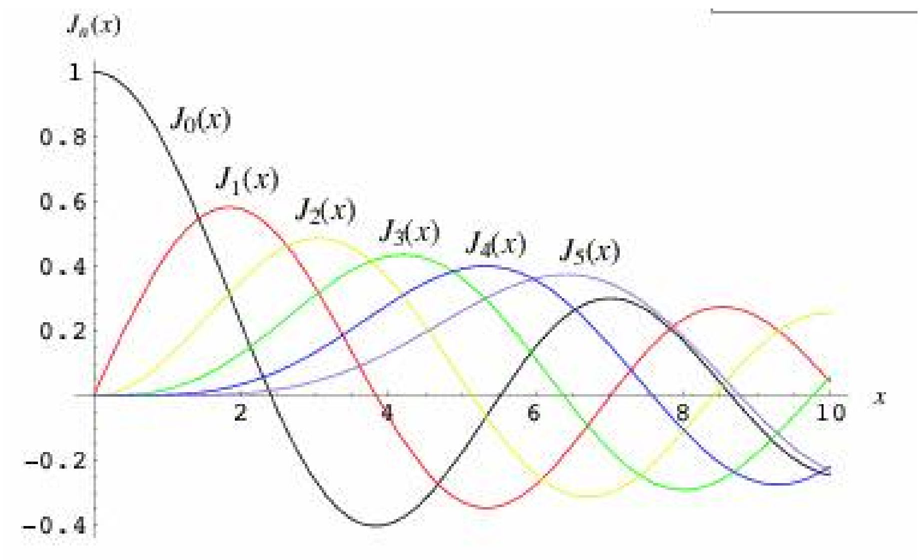
Graphical summary of Cylinder Functions:

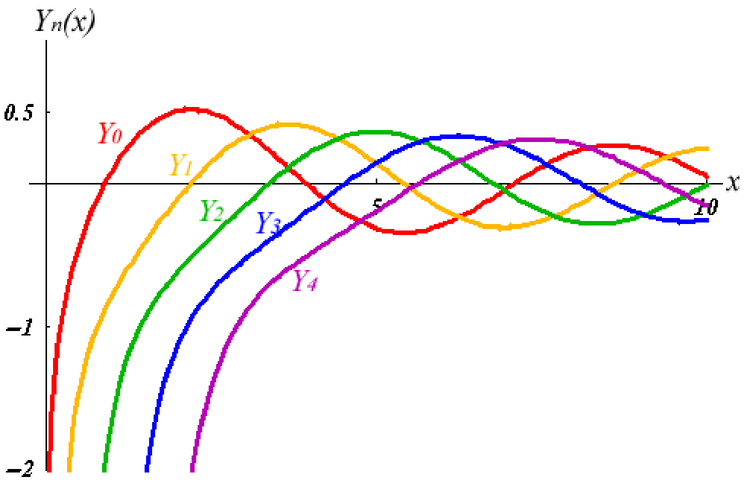
Hyperbolic functions:



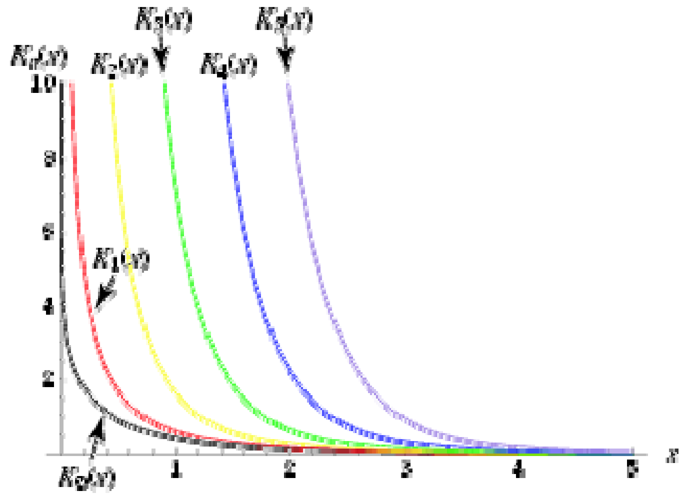
Bessel functions (first kind):



Bessel function (second kind):



Hankel function:



Appendix 1:

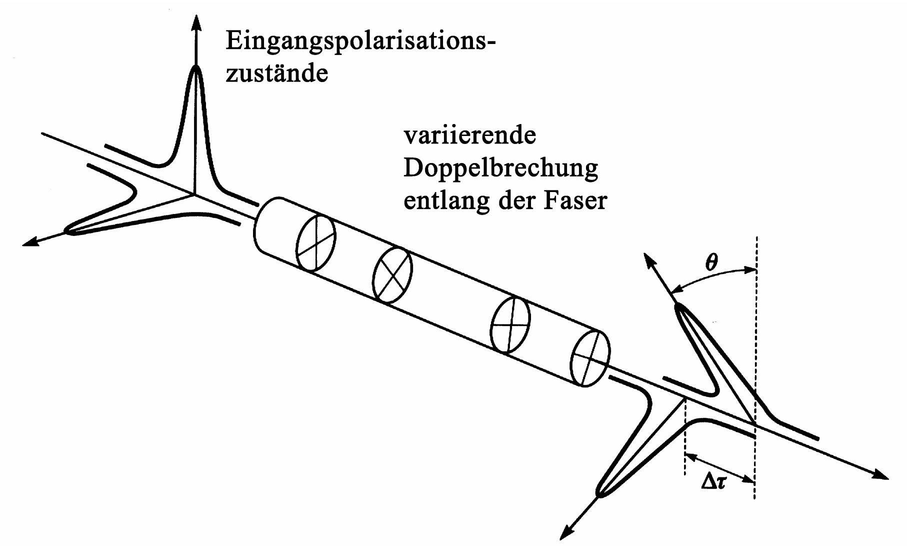
Polarization mode dispersion (optional)

The dispersion of light waves can also depend on the **polarization state** of the optical field in structures where the refractive index depends on the direction of the field vector due to:

- **Form or shape birefringence** (Form-Doppelbrechung) in asymmetric waveguides (eg. rectangular cores)
- asymmetric **stress** (bending, twisting) in the waveguide
- asymmetric **density variations**, etc.

Contrary to material and waveguide dispersion, the last 2 effects can vary stochastically along the fiber and in time.

➡ the 2 possible orthogonal polarizations states at the fiber input are **delayed** differently in time $\Delta\tau=\delta\tau_{pol}$ and **rotate** the polarization directions Θ:



Schematic representation of variable birefringence for

The 2 orthogonal polarized wave are characterized by **2 group velocities v_{gx} and v_{gy}:**

$$\delta\tau_{pol} = L \cdot \left| \frac{1}{v_{gx}} - \frac{1}{v_{gy}} \right|$$

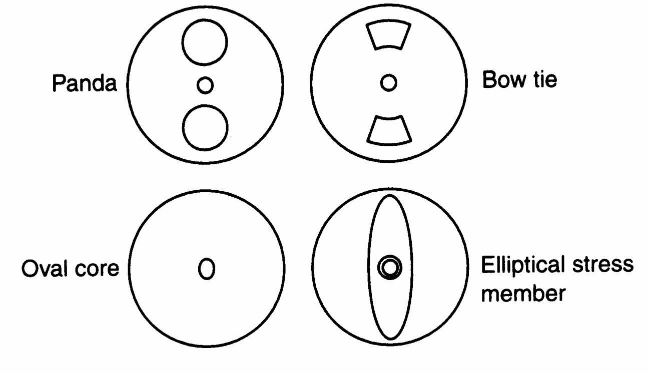
leading by statistical averaging to:

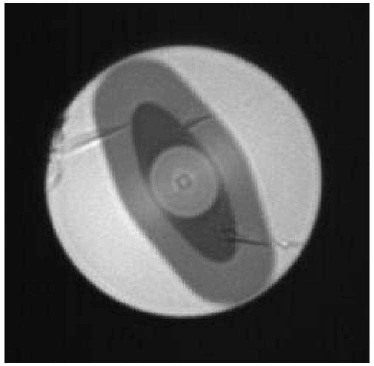
$$\left\langle \delta\tau_{pol} \right\rangle = D_{PMD} \cdot \sqrt{L}$$

D_{PMD}=Polarizationdispersion parameter (typ.0.1-1ps/√km)

Artificially high or low birefringent fibers:

Concept: Polarization filtering of the fiber structure by making one polarization direction relatively lossy and thus filter out the unwanted polarization.





High birefringent fibers with pronounced polarization directions

Appendix 2:

Remark to the Multi-Mode Gradient-Index Fiber (MMF)

Single mode fibers (SMF) require for 1) single mode operation at 1.3 / 1.5µm and 2) low dispersion:

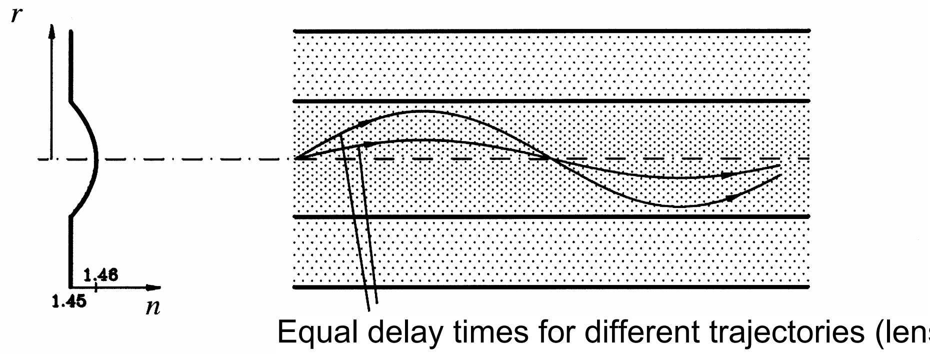
a) a low index differences Δn (~1%) between core and cladding
b) a relative small core diameters d ~5 – 10µm

➨ precise (<0.1µm) and expensive coupling between laser source and fiber, no LEDs are possible because of the large source area, typ ~50µm ∅

Low-cost data-links for moderate data rates (1-10 GB/s) and short distances (<100m) require LEDs and simple fiber coupling:

➨ use **step-index multimode fibers (MMF)** with core diameters of 50-60µm, with simple and efficient coupling, but these fibers are multi-transverse mode (several 100 modes) resulting in a huge intermodal dispersion.

➨ **graded index MMF reduce intermodal dispersion in by a lens-like, graded index profile**:



Equal delay times for different trajectories (lens like)

$$n(r) = \begin{cases} n_1 \cdot \sqrt{1 - 2\Delta \cdot \left(\frac{r}{a}\right)^g} & r < a \\ n_1 \cdot \sqrt{1 - 2\Delta} & r > a \end{cases}$$

Parabolic index profile of the core

Concept in the light ray picture: lens-like dispersion of the core

The beams traveling off-axis "see" on the "average" a lower refractive index n and travel faster than the rays close to the high-index core →
as a result on- and off- axis rays travel at about the same speed in the z-direction resulting in low dispersion.