Physics of Glaciers

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Gornergletscher, Switzerland, June 2006

The ice-dammed lake (with icebergs) in the confluence drains every year in early summer.

Cold ice of $-13°$ C flows from the highest peaks of Monte Rosa at 4500 m through Grenzgletscher (right of medial moraine) down to elevations of 2500 m, where a central ribbon of ice is at a temperature of $-2.3°$ C, while all other ice is temperate.

Martin Lüthi[1] Fabian Walter[2] Mauro Werder[2]

[1]Geographisches Institut, 3G, Universiät Zürich
[2]Versuchsanstalt für Wasserbau, Hydrologie und Glaziologie VAW, ETH Zürich

Life is like a glacier

Heaven-descended in its origin, it yet takes its mould and conformation from the hidden womb of the mountains which brought it forth. At first soft and ductile, it acquires a character and firmness of its own as an inevitable destiny urges it on its onward career. Fostled and constrained by the crosses and inequalities of its prescribed path, hedged in by impassable barriers which fix limits to its movements, it yielding groaning to its fate, and still travels forward seamed with the scars of many a conflict with opposing obstacles. All this while, although wasting, it is renewed by an unseen power – it evaporates, but is not consumed. On its surface it bears the spoils which, during the progress of existence it has made its own; – often weighty burdens devoid of beauty or value – at times precious masses, sparkling with gems or with ore.

Having at length attained its greatest widths and extension, commanding admiration by its beauty and power, wast predominates over supply, the vital springs begin to fail; it stoops into an attitude of decrepitude; it drops the burdens one by one, which it had borne so proudly aloft; its dissolution is inevitable. But as it is resolved into its elements, it takes all at once a new, livelier, and disembarrassed form; the wreck of its members it arises, "another, yet the same", – a noble, full-bodied, arrowy stream, which leaps, rejoicing over the obstacles which before had stayed its progress, and hastens through fertile valleys towards a freer existence, and a final union in the ocean with the boundless and the infinite.

James D. Forbes (1855)

Chapter

1 Overview

Many people are concerned about the question "what will change on a warmer planet?" Most will know that glaciers react to climate, and that they will melt back dramatically in the future, according to most climate scenarios. Big changes in glaciation will affect earth climate, river hydrology and sea level, among others. In this course we will take a detailed look at the physics of the processes that govern the behavior of glaciers and ice sheets.

We will be mainly concerned about ice sheets and valley glaciers. To get a feeling for the relevant scales, typical sizes of ice sheets and glaciers, their aspect ratios (thickness/length), and reaction times are listed below (after Kuhn, 1995)

Size	Thickness	Length	Thickness/Length	Reaction time
Ice sheets	1000 m	1000 km	0.001	1000 years
Valley glaciers	100 m	10 km	0.01	100 years
Cirque glaciers	10 m	0.1 km	0.1	10 years

The Greenland ice sheet is about 3400 m thick. The maximum ice thickness in East Antarctica is more than 4500 m. Grosser Aletschgletscher, the biggest glacier in the Alps, is 25 km long, typically some 500 m thick, and more than 900 m at Konkordiaplatz.

Distribution of glaciated areas

The distribution of glaciated areas on Earth is shown in Table 1.1. Clearly the huge ice sheets of Antarctica and Greenland dominate by the sheer volume of water they store in form of ice. What if they melt? Is that possible at all? How fast would that happen? What about other areas with smaller ice caps, or mountain glaciers?

Ice age ice sheets

Ice sheets have waxed an waned in the past. Their extent and volumes are collected in Table 1.2. The Laurentide Ice Sheet that covered most of north-eastern

Continent	Region	Area (km^2)	Total (km^2)	
Antarctica	Subantarctic islands	7'000		
	Antarctic continent	13'586'310	13'593'310	85.7 %
Greenland	Greenland ice sheet	1'726'400	1'726'400	10.9 %
North America	Canada	200'806		
	USA	75'283		
	Mexico	11	276'100	1.7 %
Asia	Russia and former SU states	77'223		
	Turkey, Iran and Afghanistan	4'000		
	Pakistan and India	40'000		
	Nepal and Bhutan	7'500		
	China (incl. Tibet)	56'481		
	Indonesia	7	185'211	1.2 %
Africa	whole continent	10	10	
Europe	Iceland	11'260		
	Svalbard	36'610		
	Scandinavia	3'174		
	Alps	2'909		
	Pyrenees	12	53'967	0.3 %
South America	Argentina and Chile	23'328		
	Peru and Ecuador	1'900		
	Bolivia, Colombia, Venezuela	680	25'908	0.2 %
Australasia	New Zealand	860	860	
			15'861'766	

Table 1.1: *Global distribution of glaciated areas (after Knight, 1999)*

North America (Hudson Bay; Big Lakes; New York) had a volume similar to today's Antarctic Ice Sheet, but vanished completely about 10'000 years ago. Also the ice sheets that covered Scandinavia, Britain and Ireland, the North American Cordillera (Rocky Mountains) and the Alps melted completely, with some small mountain glaciers remaining.

Sea level change

The sea level rose by about 130 m at the end of the last Ice Age (197 − 66 m in the last row in Table 1.2). That was enough to flood for example Beringia, a 1600 km wide land bridge between Siberia (Asia) and Alaska (North America).

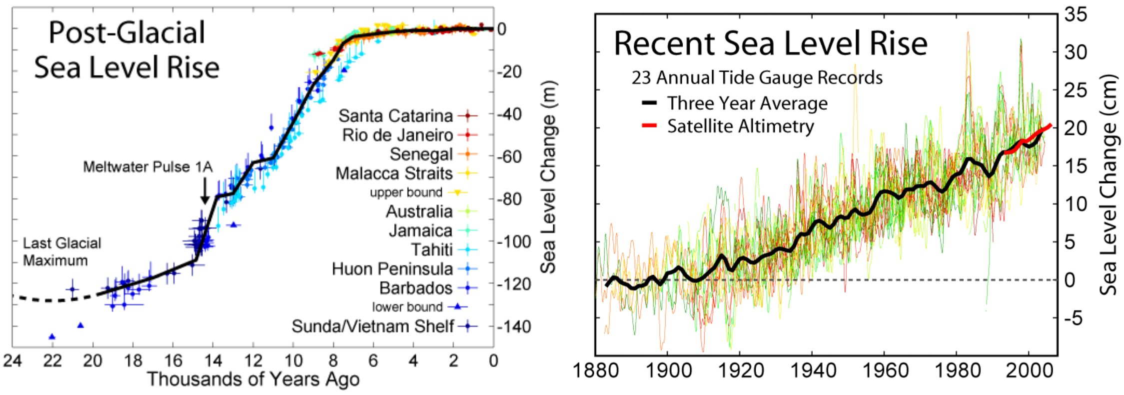

Figure 1.1: *Reconstructed and measured relative sea level. The "Meltwater Pulse 1A" refers to the drainage of late-glacial Lake Agassiz at the margin of the Laurentide Ice Sheet. (From:* `http://www.globalwarmingart.com`*)*

The complete disintegration of the present Greenland Ice Sheet would rise sea level by about 7 m. The Antarctic Ice Sheet has the potential to rise sea level by another 60 m!

Ice sheet	Area ($10^6\,\mathrm{km}^2$)		Volume ($10^6\,\mathrm{km}^3$)		Sea level (m)	
Antarctica	12.5	(13.8)	23.5	(26.0)	59	(66)
Greenland	1.7	(2.3)	2.6	(3.5)	7	(11)
Laurentide	0	(13.4)	0	(29.5)	0	(74)
Cordilleran	0	(2.4)	0	(3.6)	0	(9)
Scandinavian	0	(6.7)	0	(13.3)	0	(34)
Other	0.6	(5.2)	0.2	(1.1)	0.5	(3)
Total	14.9	(43.7)	26.2	(77.0)	66	(197)

Table 1.2: *Approximate sizes of ice sheets at present, and during the glacial maximum (in parentheses). The last column is the equivalent sea level rise represented by the storage of water in those ice sheets (after Knight, 1999).*

Glaciated mountain areas, such as the coastal ranges in Canada/Alaska and the Himalayas contribute considerably to sea level change at present. Measurements of glaciers in Alaska show that they currently contribute $0.27\,\mathrm{mm\,a^{-1}}$ to sea level rise (Arendt et al., 2002). This mass loss is about equal to the current mass loss

Ice sheet	Accumulation	Runoff	Calving	Bottom melting	Net balance	Net Sea level rise
	$\mathrm{Gt\,a^{-1}}$	$\mathrm{Gt\,a^{-1}}$	$\mathrm{Gt\,a^{-1}}$	$\mathrm{Gt\,a^{-1}}$	$\mathrm{Gt\,a^{-1}}$	$\mathrm{mm\,a^{-1}}$
Greenland	520 ± 26	297 ± 32	325 ± 33	32 ± 3	-44 ± 53	0.05 ± 0.05
Antarctica	2246 ± 86	10 ± 10	2072 ± 304	540 ± 26	-376 ± 384	-0.1 ± 0.1
Glaciers and ice caps	688 ± 109	778 ± 114			-91 ± 36	0.3 ± 0.1

Table 1.3: *The mass balance of the Greenland and Antarctic ice sheets and of smaller glaciers and ice caps (after Hooke, 2005; data from the 2001 IPCC report (Houghton J.T. et al., 2001). An ice mass loss of* $360\,\mathrm{Gt}$ *corresponds to a sea level rise of* $1\,\mathrm{mm}$ ($1\,\mathrm{Gt} = 10^{12}\,\mathrm{kg}$).

Period	Mean Specific Mass Balance^a ($\mathrm{kg\,m^{-2}\,a^{-1}}$)	Total Mass Balance^a ($\mathrm{Gt\,a^{-1}}$)	Sea Level Equivalent^a ($\mathrm{mm\,a^{-1}}$)	Mean Specific Mass Balance^b ($\mathrm{kg\,m^{-2}\,a^{-1}}$)	Total Mass Balance^b ($\mathrm{Gt\,a^{-1}}$)	Sea Level Equivalent^b ($\mathrm{mm\,a^{-1}}$)
1960/61 – 2003/04	-283 ± 102	-155 ± 55	0.43 ± 0.1	-231 ± 82	-182 ± 64	0.50 ± 0.18
1960/61 – 1989/90	-219 ± 92	-120 ± 50	0.33 ± 0.14	-173 ± 73	-136 ± 57	0.37 ± 0.16
1990/91 – 2003/04	-420 ± 121	-230 ± 66	0.63 ± 0.18	-356 ± 101	-280 ± 79	0.77 ± 0.22

Table 1.4: *Global average mass balance of glaciers and ice caps for different periods, showing mean specific mass balance (*$\mathrm{kg\,m^{-2}\,a^{-1}}$*); total mass balance (*$\mathrm{Gt\,a^{-1}}$*); and Sea Level Equivalent (SLE;* $\mathrm{mm\,a^{-1}}$*) derived from total mass balance and an ocean surface area of* $362 \cdot 10^6\,\mathrm{km}^2$. *This is Table 4.4 from the 2007 IPCC report (Lemke et al., 2007). Superscripts a and b indicate mass balances excluding and including glaciers and ice caps around ice sheets, respectively.*

of the Greenland Ice Sheet. For High Mountain Asia, mass loss estimates diverge. A recent study considering digital elevation models derived from satellite stereo-imagery of about 92 % of the glaciers finds a total mass change equivalent to $-0.046 \pm 0.009\,\mathrm{mm\,a^{-1}}$ (Brun et al., 2017).

Although glaciers and ice caps are the strongest contributors to current eustatic sea level rise, mass loss from both polar ice sheets accelerates three times faster (Rignot et al., 2011b). Figure 1.5 clearly shows this non-linear increase. If this trend continues, the polar ice sheets will dominate sea level rise within the next decades.

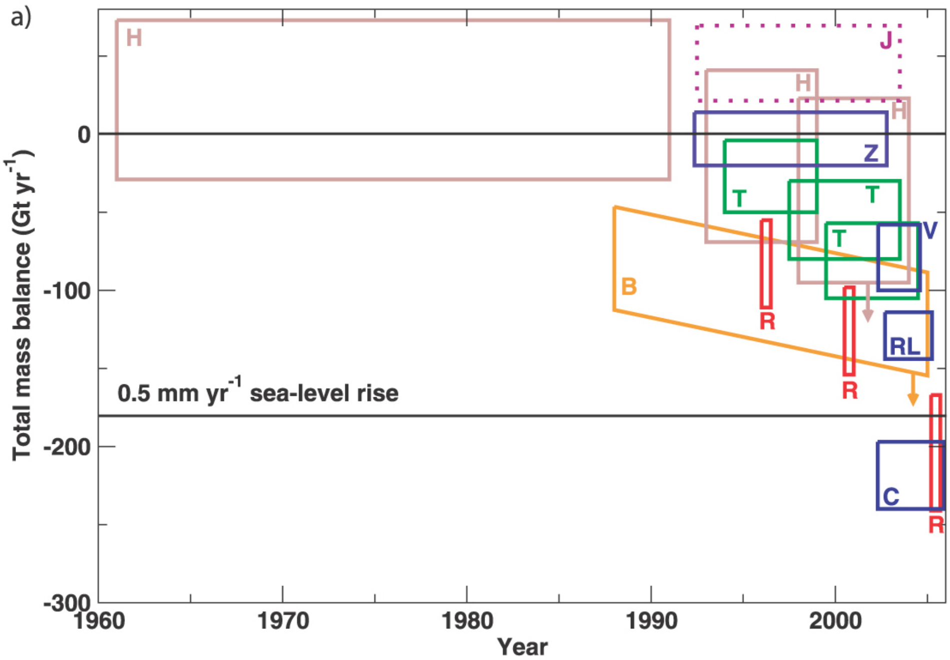

Figure 1.2: *Mass balance estimates for Greenland. The coloured rectangles, following Thomas et al. (2006), indicate the time span over which the measure- ments apply and the estimated range, given as (mean + uncertainty) and (mean - uncertainty) as reported in the original papers. This is Figure 4.18 from the 2007 IPCC report (Lemke et al., 2007), where the meaning of colors and sources are given.*

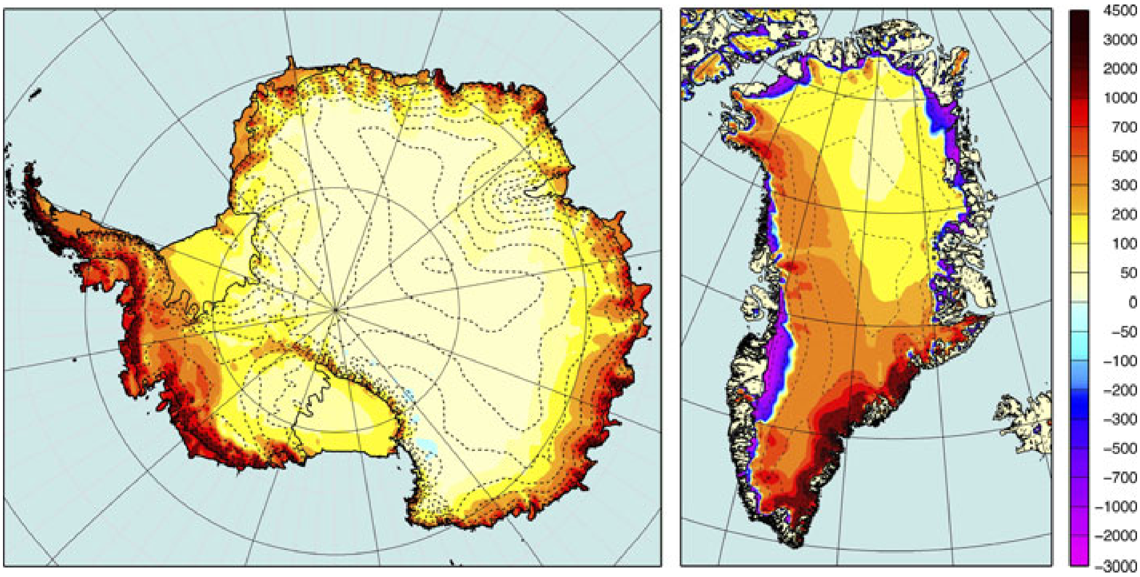

Figure 1.3: *Mass balance estimates for Antarctica and Greenland. (From Van den Broeke et al., 2011)*

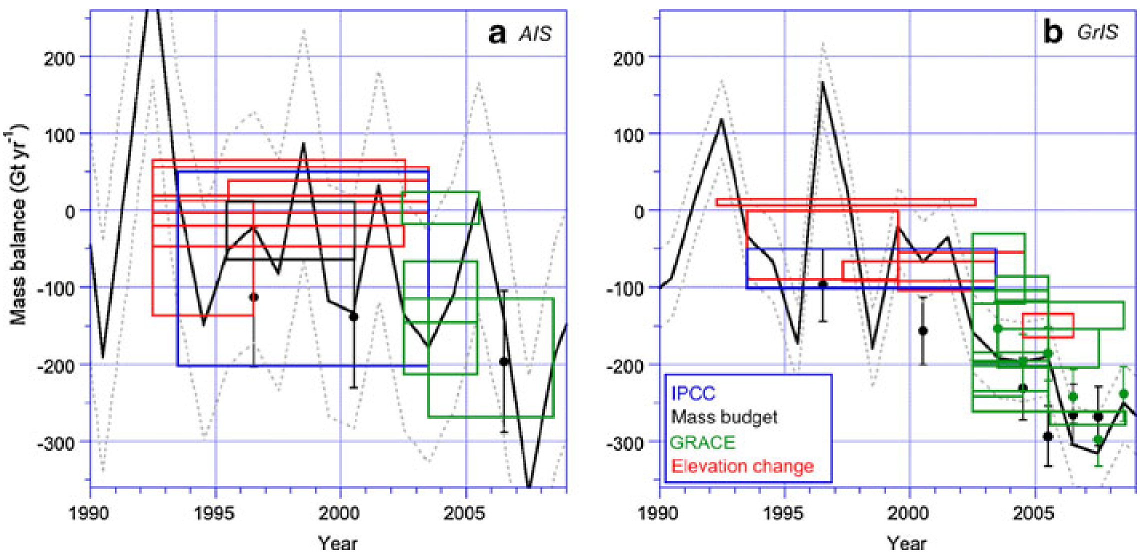

Figure 1.4: *Comparison of different mass balance estimates for the Antarctic Ice Sheet (AIS) and Greenland Ice Sheet (GrIS). (From Van den Broeke et al., 2011)*

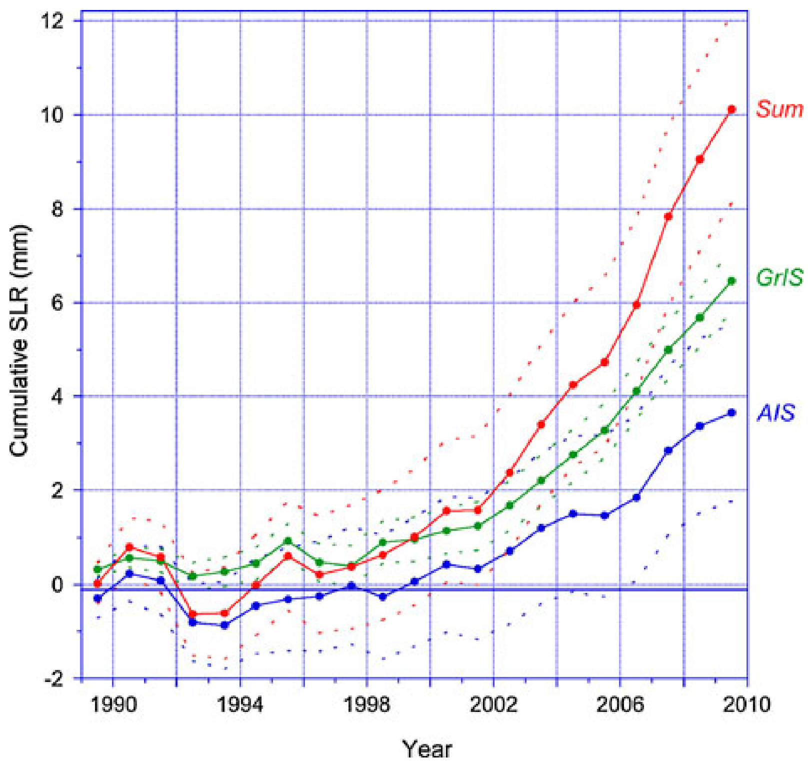

Figure 1.5: *Sea level rise caused by ice loss of the the Antarctic Ice Sheet (AIS) and Greenland Ice Sheet (GrIS). (From Van den Broeke et al., 2011)*

Mass Loss Partitioning for Ice Sheets and Tidewater Glaciers

The mass balance of tidewater glaciers (glaciers terminating in the ocean) and ice sheets is affected by mass gain and mass loss at the surface, at the bed and at the ice-ocean margin. Surface mass balance consists of precipitation, runoff and sublimation, whereas the basal mass loss of grounded glaciers can usually be neglected. At the ice-ocean margin, ice mass is lost in form of iceberg calving or submarine melt. The relative magnitude of these mechanism is not fully understood and varies depending on ocean temperature, subglacial discharge and the presence or absence of a floating ice tongue.

The mass loss of tidewater glaciers and ice sheets is often partitioned into surface mass loss (mostly melt) and dynamic discharge. Dynamic discharge is the ice mass, which tidewater glaciers or outlet glaciers of ice sheets transport to the ocean via ice flow. It does not distinguish between the mass loss process to the ocean (submarine melt or iceberg calving), but quantifies how much mass passes through a pre-defined "flux gate" near the ice-ocean margin. The specification of a flux gate is subjective, often a fjord narrowing is used.

In their current states, the polar ice sheets undergo dynamic thinning, which means that ice flow acceleration near the margins is not compensated by mass replenishment from the ice sheet interior. Consequently, there exist large longitudinal stretching and thinning rates in outlet glaciers. Figure 1.6 confirms the dynamic effect of ice thinning at the polar ice sheets: thinning concentrates at the ice sheet margins and on fast flowing outlet glaciers. Dynamic thinning can be dramatic, for example, during some years of its retreat phase, Columbia Glacier in Alaska lost 10 m per year (Rasmussen et al., 2011). Jakobshavn Isbræ even lost 200 m between 1880 and 1980 (Khan et al., 2015). Dynamic effects do not only lead to thinning. For example, the Antarctic Kamb Ice Stream (formerly Ice Stream C), which flows into the Ross Ice Shelf slowed down about 150 years ago most likely due to hydraulic processes at its bed. The resulting slow flow inhibits mass transport from the ice sheet interior to the ocean, which is why ice is "piling up" at the ice stream, i.e. the ice stream is thickening (1.6).

The dynamic behavior of tidewater glaciers and ice sheet outlet glaciers is complex and not fully understood. There exists a feedback between ice flow and changes at grounding and floating ice tongues. This causes hysteresis in the adjustments to external climatic changes and intrinsic instabilities, which depends on geometrical characteristics of the glacier. Understanding the link between ice dynamics and climate forcing is therefore not straightforward and constitutes perhaps the largest uncertainty for sea level rise predictions.

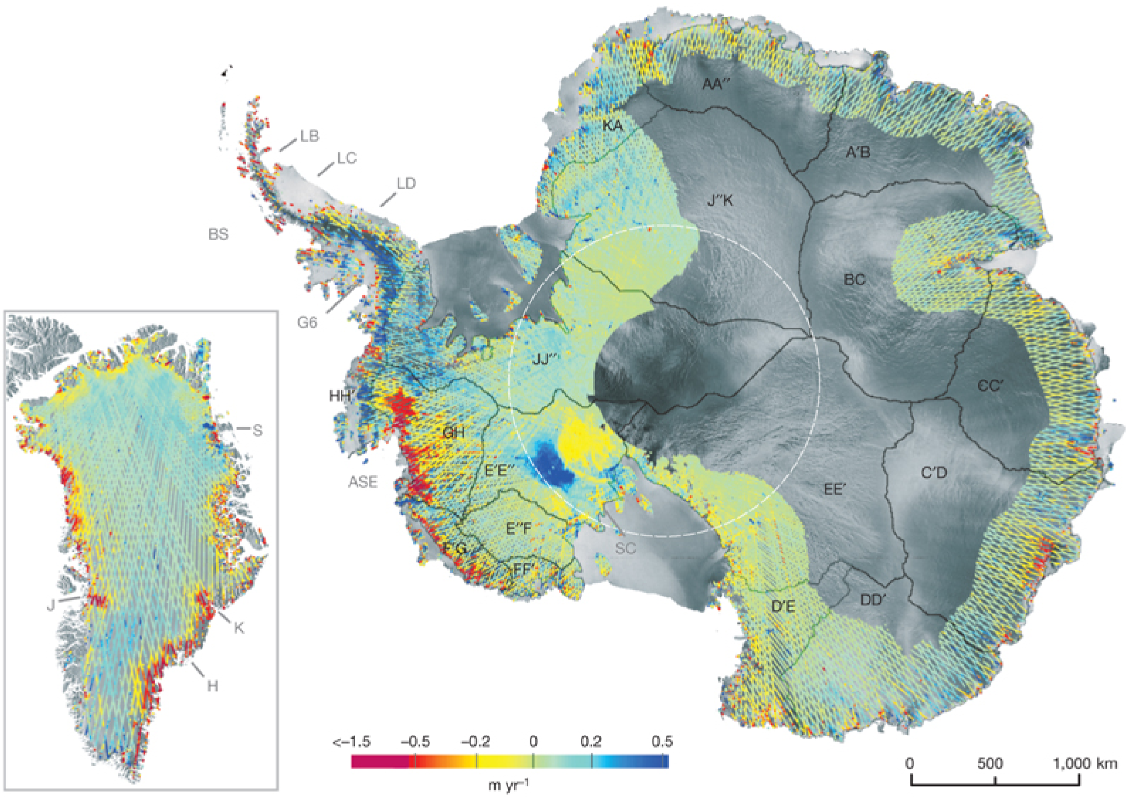

Figure 1.6: *Ice thickness change for Antarctica and Greenland (inset) from satellite laser altimeter data. Note dynamic thinning around Southeast and Northwest Greenland, which concentrates on major fast-flowing outlet glaciers such as Jakobshavn Isbræ (J), Helheim (H) and Kangerdlugssuaq (K) glaciers. In Antarctica, dynamic thinning occurs mainly in West Antarctica. Some dynamic thickening can be noticed in Greenland (Storstrømmen Glacier, S) and on the Kamb Ice Stream in Drainage Sector E'E" in Antarctica. (From Pritchard et al., 2009)*

Observed mass loss partitioning

Both the Antarctic and Greenland Ice Sheets have positive surface mass balances (Figure 1.7). Consequently, in the absence of dynamic mass loss, both ice sheets would be gaining mass. For both ice sheets, there are large inter-annual variations. In Antarctica, surface melt is negligible and variations in surface mass balance are due to variations in precipitation (see Table 1.3). There exists no overall temporal trend in surface mass balance variations. In contrast, in Greenland, mass loss is approximately equally divided between mass balance at the surface and dynamic discharge to the ocean (van den Broeke et al., 2009). Surface melt is responsible for part of the variations in surface mass balance as well as the negative trend since about 2000 (Figure 1.7b).

The dynamic mass loss of both ice sheets has increased since the 1990's. This has led to persistently negative total mass balances from the polar ice sheets. For Antarctica, the dynamic mass loss is mostly a result of glacier acceleration in West Antarctica and the Antarctic Peninsula prior to 2005. Equivalently, glacier acceleration since around 1996 in southeast, west and northwest Greenland is mainly responsible for increase dynamic mass loss (Van den Broeke et al., 2011).

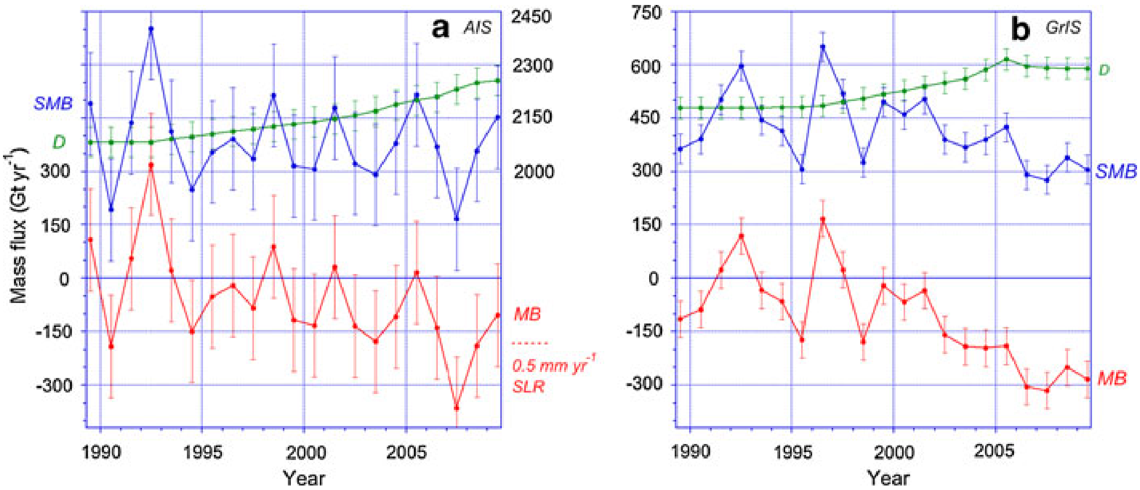

Figure 1.7: *Mass balance estimates for the Antarctic Ice Sheet (AIS) and Greenland Ice Sheet (GrIS). (From Van den Broeke et al., 2011)*

CHAPTER

2 Ice sheet mass balance

2.1 Balance velocity

In the long term, a glacier or ice sheet has to flow fast enough to transport the ice accumulated upstream to the ablation area. At each location on the surface the rate of mass gain or mass loss is called the *specific balance rate* $\dot{b}$ (in units of $\mathrm{kg\,m^{-2}\,a^{-1}}$). For our purposes it is often convenient to use the volumetric specific balance rate $\dot{b}_i = \dot{b}/\rho_i$ (in units of $\mathrm{m^3\,m^{-2}\,a^{-1}} = \mathrm{m\,a^{-1}}$; $\rho_i \simeq 900\,\mathrm{kg\,m^{-3}}$ is the ice density). The ice volume flux Q_{bal} along the flow line at position x, and through a section of width W is

$$Q_{\mathrm{bal}}(x) = W\,q_{\mathrm{bal}}(x) = W \int_0^x \dot{b}_i(x)\,dx\,. \qquad (2.1)$$

Q_{bal} is called the *balance flux*. One can now define the balance velocity

$$u_{\mathrm{bal}}(x) := \frac{Q_{bal}}{W(x)H(x)} = \frac{1}{H(x)} \int_0^x \dot{b}_i(x)\,dx\,, \qquad (2.2)$$

where $H(x)$ is the local ice thickness This equation is an expression of the conservation of volume, and since glacier ice is a nearly *incompressible* medium, the *conservation of mass.*

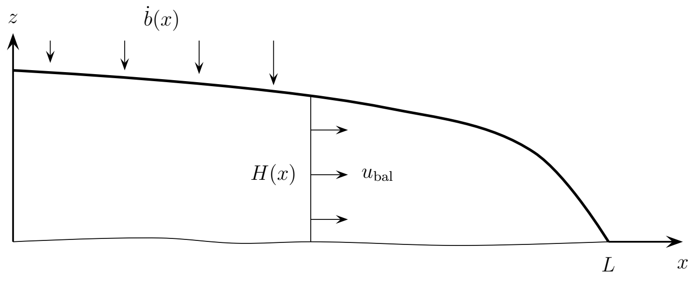

Figure 2.1: *Schematic diagram illustrating the dependence of horizontal balance velocity on accumulation rate.*

Careful: The concept of balance flux and balance velocity is not necessarily useful. It is often used as a diagnostic quantity to delineate areas of an ice sheet where fast flow should occur. The example of Antarctica in Figure 2.2 shows indeed fast flow towards the big ice shelves (such as Ross, Ronne-Filchner and Amery), but the details of many ice streams are quite different in reality.

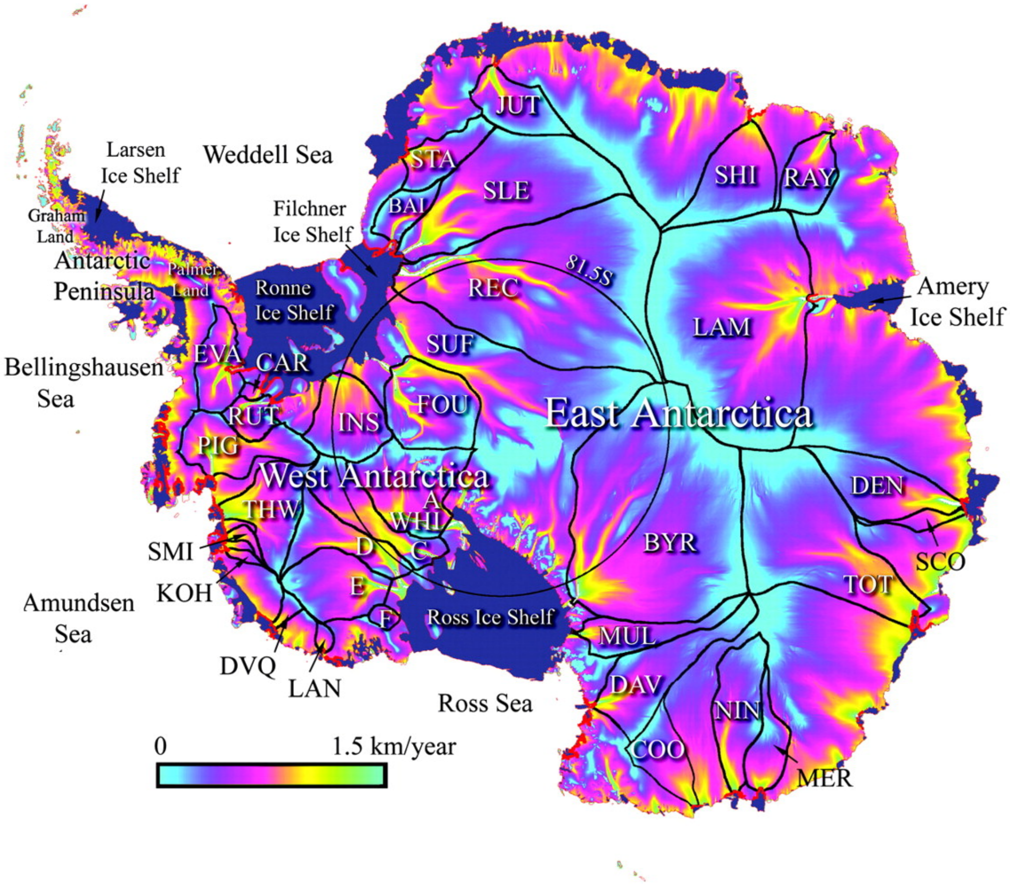

Figure 2.2: *Balance velocity map of the Antarctic Ice Sheet. Clearly visible are the areas of accelerated ice flow towards the coast where ice streams form. Catchment basin boundaries are in black, grounding lines in red, and ice shelves are dark blue (from Rignot and Thomas, 2002).*

Abbreviations for glaciers: Pine Island (PIG), Thwaites (THW), Smith (SMI), Kohler (KOH), DeVicq (DVQ), Land (LAN), Whillans (WHI), A-F (A-F), Byrd (BYR), Mulock (MUL), David (DAV), feeding eastern Cook Ice Shelf (COO), Ninnis (NIN), Mertz (MER), Totten (TOT), Denman (DEN), Scott (SCO), Lambert/Mellor/Fisher (LAM), Rayner (RAY), Shirase (SHI), Jutulstraumen (JUT), Stancomb-Wills (STA), Bailey (BAI), Slessor (SLE), Recovery (REC), Support-Force (SUF), Foundation (FOU), Institute (INS), Rutford (RUT), Carlson (CAR), and Evans (EVA).

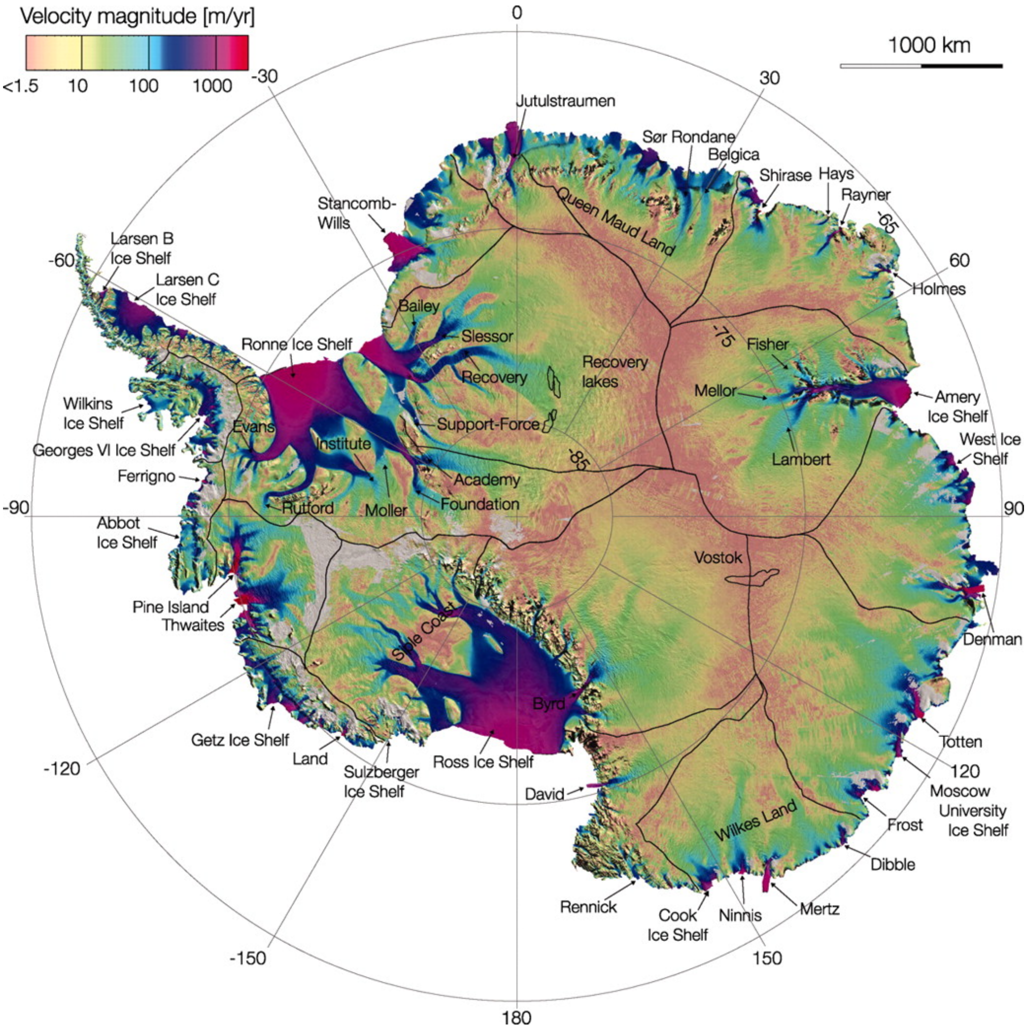

Figure 2.3: *Velocity map of the Antarctic Ice Sheet, derived from radar satellite interferometry. Clearly visible are the areas of accelerated ice flow towards the coast where ice streams form. (from Rignot et al., 2011a).*

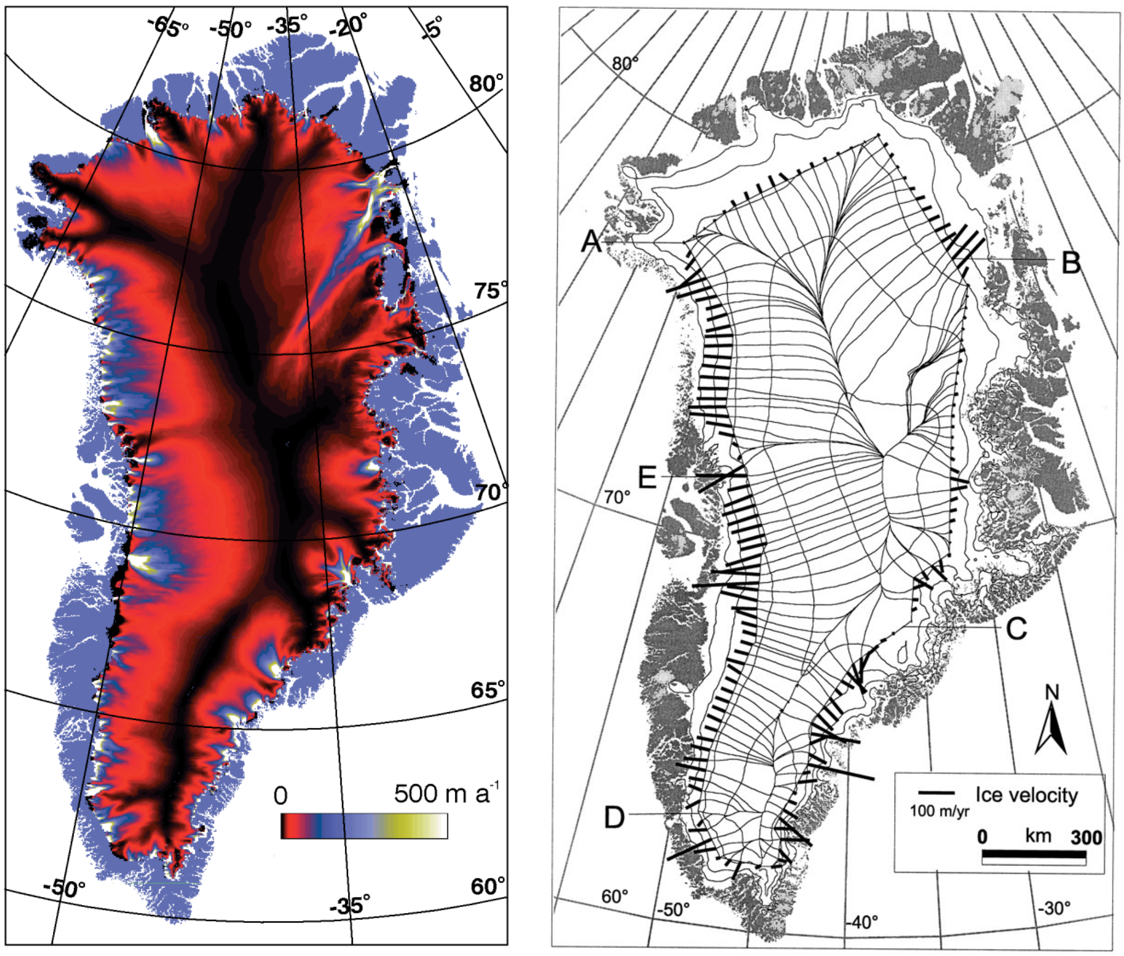

Figure 2.4: *Left: Balance velocity map of the Greenland Ice Sheet (from Bamber et al., 2002). Right: Ice velocities on the Greenland ice sheet at the* 2000 m *contour line (from Thomas et al., 2000).*

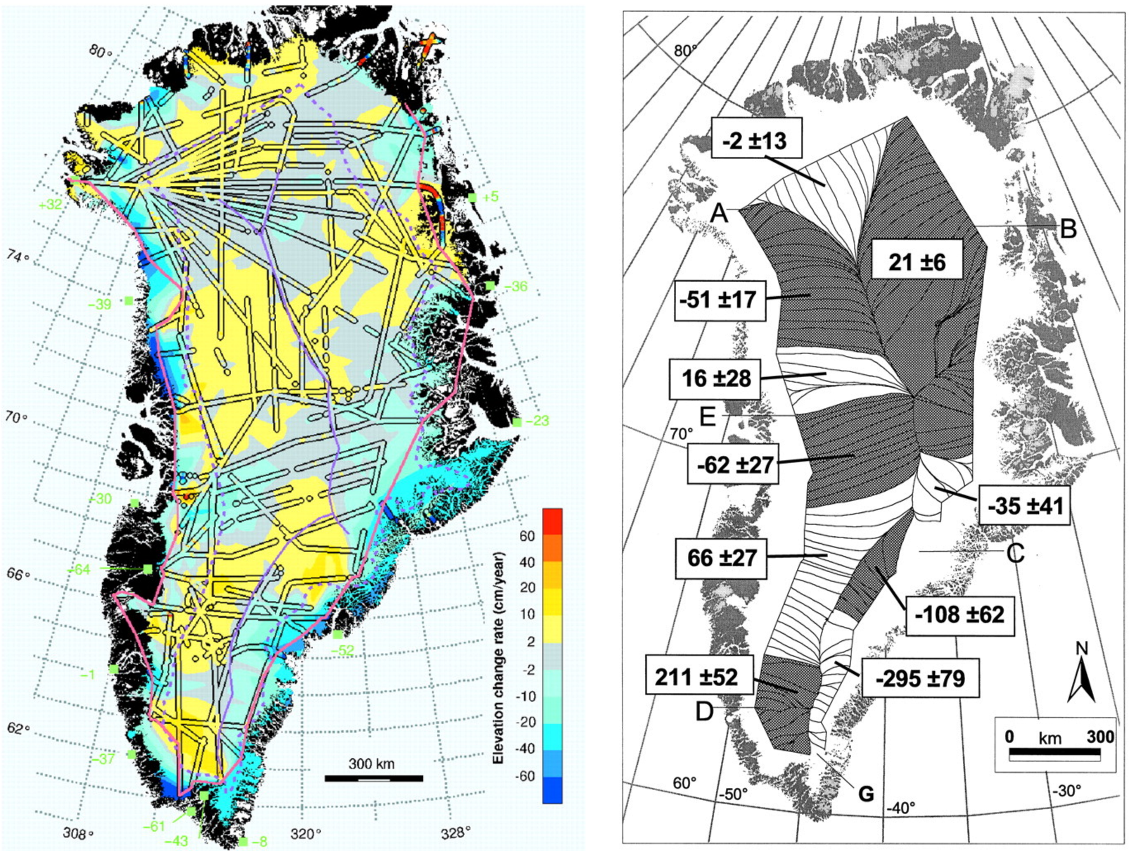

Figure 2.5: *Left: Elevation change rate of the Greenland ice sheet (from Krabill al., 2000). Right: Mass balance of the interior parts of the Greenland ice sheet in millimeters per year (from Thomas et al., 2000).*

2.2 Surface profile of an ice sheet

For the inland parts of ice sheets (but not the ice divide) and some parts of wide glaciers, shearing parallel to the ice surface is the most important contribution to ice flow. In these places the *shallow ice approximation* can be used. The only stress component that contributes to ice deformation is the shear stress parallel to the surface. The overburden pressure cannot cause ice compaction rates since ice is incompressible (of course the overburden pressure leads to elastic deformation).

Shear stress

We derive the magnitude of the shear stress in two different ways. Consider the geometry of Figure 2.6a. The x-axis is parallel to the surface, with an inclination angle α. The z-axis points upwards, perpendicular to the surface. We are interested in the shear stress on a plane at depth h and parallel with the surface. The weight of a column with horizontal extent $S = 1\,\mathrm{m}^2$ is $\rho_i g h S$, where ρ_i is the density of ice and g is the acceleration due to gravity. The component of weight parallel to the (inclined) plane of interest then is

$$\sigma_{xz}^{(a)}\, S = \rho g h \sin\alpha\, S. \tag{2.3}$$

and consequently the stress component

$$\left|\sigma_{xz}^{(a)}\right| = \rho g h \sin\alpha. \tag{2.4}$$

This is called the *driving stress* (Deutsch: Hang-Abtriebskraft). For an equilibrium of forces it has to be balanced by other stresses, such as the *basal drag* τ_b.

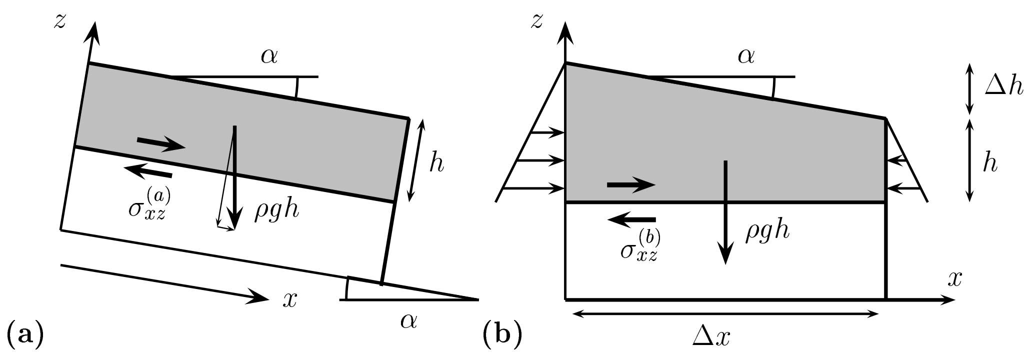

Figure 2.6: *The shear stress σ_{xz} on a plane at depth h below the surface (lower boundary of shaded region) is derived in two different coordinate systems a) surface parallel, inclined coordinate system b) coordinate system is aligned with gravity.*

We now consider the geometry of Figure 2.6b with a coordinate system where the x-axis is horizontal and the z-axis is vertical. The column is again of unit cross-sectional area $S = \Delta x \cdot W$, and the (horizontal) plane of interest is at depths h

to $h + \Delta h$ below the surface. The hydrostatic pressure at this depth (on the right boundary) is approximately $\rho g h$ and varies linearly with depth. Therefore the mean force on the vertical face on the right is $\frac{1}{2}\rho g h \cdot h\, W$ (the second h is due to the area of the face). Similarly, the mean force on the left face is $\frac{1}{2}\rho g (h+\Delta h)^2\, W$. The total shearing force on the horizontal plane of interest is $\sigma_{xz}^{(b)} \Delta x\, W$. Since the block is balanced, the sum of all forces is zero

$$\frac{1}{2}\rho g (h+\Delta h)^2 - \frac{1}{2}\rho g h^2 + \sigma_{zx}^{(b)} \Delta x \overset{!}{=} 0.$$

Expanding the first term, neglecting terms of order Δh^2 and using $\tan\alpha = \Delta h / \Delta x$ leads to

$$\left|\sigma_{xz}^{(b)}\right| = \rho g h \frac{dh}{dx} = \rho g h \tan\alpha. \tag{2.5}$$

This expression is appropriate for a situation in which both the x-axis and the plane of interest are horizontal, and the glacier surface is sloping.

Notice that the expressions (2.4) and (2.5) are different because of different coordinate systems, and therefore different meaning of σ_{xz}. While $\sigma_{xz}^{(a)}$ is the shear stress on an inclined plane, $\sigma_{xz}^{(b)}$ is the shear stress on a horizontal plane below an inclined surface. For small angles α they are almost equal since $\sin\alpha \sim \tan\alpha \sim \alpha$.

Surface profile of an ice sheet: plastic ice

We consider a steady-state ice sheet on a flat horizontal bed (Fig. 2.1). The stress at the ice sheet base – a horizontal plane of interest – is (Eq. 2.5)

$$\tau_b = \sigma_{xz}^{(b)} = \rho g H \frac{dH}{dx}\,. \tag{2.6}$$

First we assume the simplest case, namely that the shear stress at the base cannot exceed a threshold value τ_0. This is the case if the ice or substrate at the glacier base behaves as a *perfect-plastic* material with yield stress τ_0. If the whole base is at the *yield stress* τ_0, as will be the case if ice accumulates at the surface, Equation (2.6) can be integrated between x and L

$$H^2 = \frac{2\tau_0}{\rho g}(L - x)\,, \tag{2.7}$$

which is a parabola (Nye, 1952). The thickness at the center is $H_0 = (2\tau_0 L/\rho g)^{1/2}$. If we use $\tau_0 = 100\,\mathrm{kPa} (= 0.1\,\mathrm{MPa} = 1\,\mathrm{bar})$ and the horizontal extent of the Greenland ice sheet $L = 450\,\mathrm{km}$ we obtain $3160\,\mathrm{m}$ which is about the elevation of Summit. Notice that no assumption about mass balance has entered this calculation.

We see that typical (shear) stresses in glaciers are of the order $100\,\mathrm{kPa}$. Applying the concept to a mountain glacier we can get a feeling for typical ice thicknesses: for $\alpha = 5^\circ$ the ice thickness would be $127\,\mathrm{m}$, for $\alpha = 10^\circ$ it decreases to $63\,\mathrm{m}$. The next step is to replace the assumption of perfect plasticity with a flow law for glacier ice.

Flow velocity

The flow velocity of an ice sheet depends on the ice thickness H and the surface slope $\frac{dz_s}{dx}$, which for a flat base is $\frac{dH}{dx}$. If the ice is moving over the base with the *basal velocity* u_b, the velocity at the glacier surface is

$$u_s = \underbrace{\frac{2A}{n+1}\left(\rho g \frac{dH}{dx}\right)^n H^{n+1}}_{\text{ice deformation}} + \underbrace{u_b}_{\text{basal motion}} \tag{2.8}$$

This is a vertically integrated form of the force equilibrium equation, complemented with the famous flow law that Glen published in 1952. Glen's (and Steinemann's) flow law – which will be explained in detail later – is

$$\dot{\varepsilon} = A\tau^n. \tag{2.9}$$

This equation states that the horizontal shear strain rate $\dot{\varepsilon} = \dot{\varepsilon}_{xz}$ depends on the n-th power of the horizontal shear stress $\tau = \sigma_{xz} = \rho g H \frac{dH}{dx}$. The power-law exponent is an material property and is close to $n = 3$. The quantity A is a softness parameter that depends on temperature (and also grain size, water and impurity content, etc.) and is $A = 75.7\,\mathrm{MPa}^{-3}\,\mathrm{a}^{-1} = 2.4 \times 10^{-24}\,\mathrm{s}^{-1}\,\mathrm{Pa}^{-3}$ for $0°$ C. See Table B1 for values at colder temperatures.

The horizontal flow velocity at a depth $h := H - z$ below the surface (where z is the distance above bedrock) is

$$u(h) = u_s - \frac{2A}{n+1}\left(\rho g \frac{dH}{dx}\right)^n h^{n+1} \qquad \text{or} \tag{2.10}$$

$$u(h) = \frac{2A}{n+1}\left(\rho g \frac{dH}{dx}\right)^n \left(H^{n+1} - h^{n+1}\right) + u_b\,. \tag{2.11}$$

These equations are also known as the *shallow ice approximation.* To obtain the ice flux Q through a vertical section of unit width $W = 1\,\mathrm{m}$ (so that we can suppress W), we integrate Equation (2.10) over the ice thickness

$$\begin{aligned}
Q &= \int_0^H u(h)\,dh \\
&= u_s H - \frac{2A}{n+1}\left(\rho g \frac{dH}{dx}\right)^n \int_0^H h^{n+1}\,dh \\
&= u_s H - \frac{2A}{(n+1)(n+2)}\left(\rho g \frac{dH}{dx}\right)^n H^{n+2} \qquad \text{(with Eq. 2.8)} \\
&= \frac{2A}{n+2}\left(\rho g \frac{dH}{dx}\right)^n H^{n+2} + u_b\,H\,.
\end{aligned} \tag{2.12}$$

The flux q through a vertical section of unit width W is given in units of $\mathrm{m^2\,a^{-1}}$. The ice flow velocity averaged over depth is

$$\bar{u} = \frac{Q}{H} = \frac{2A}{n+2}\left(\rho g \frac{dH}{dx}\right)^n H^{n+1} + u_b \tag{2.13}$$

In the absence of basal motion, and with the usual assumption $n = 3$ the depth-averaged flow velocity is $\frac{n+1}{n+2} = 0.8 = 80\,\%$ of the surface flow velocity $u_s = u(H)$.

If the local ice flux q and the balance velocity q_{bal} agree, the ice sheet is in a *steady state*: the ice sheet is fully adjusted to the climate and no changes of geometry over time occur. Notice that ice sheets and some glaciers never reach a steady state due to intrinsic instabilities.

Surface profile of an ice sheet: viscous ice

We are now in a position to calculate the shape of an ice sheet. For simplicity we assume a uniform mass balance rate $\dot{b}$ (in units of $\mathrm{m\,a^{-1}}$), and that mass loss is due to calving at the edge. In a steady state the balance flux through a position x, $q_{\mathrm{bal}}(x) = \dot{b}x$ has to be equal to the volume flux q_{flow} through this cross section. Setting Equations (2.1) and (2.12) equal, and ignoring basal motion, leads to

$$\dot{b}x \overset{!}{=} \frac{2A}{n+2}\left(\rho g \left|\frac{dH}{dx}\right|\right)^n H^{n+2}\,. \tag{2.14}$$

The solution of this differential equation gives the profile

$$H^{2+2/n} = K\left(L^{1+1/n} - x^{1+1/n}\right), \tag{2.15}$$

with

$$K = \frac{2(n+2)^{1/n}}{\rho g}\left(\frac{\dot{b}}{2A}\right)^{1/n}. \tag{2.16}$$

At the highest point the variables are $x = 0$ and $H(0) = H_0$. Therefore we can write the equation as

$$\left(\frac{H}{H_0}\right)^{2+2/n} + \left(\frac{x}{L}\right)^{1+1/n} = 1. \tag{2.17}$$

Perfect plasticity corresponds to $n \to \infty$, which reduces this equation to the parabola of Equation (2.7). This solution was first described by Vialov (1958).

Chapter

3 Glacier mass balance

3.1 Mass balance terminology

Climate changes lead to changes in the mass budget of a glacier or ice sheet, the mass balance. The measure for this change is the *specific balance rate*, which is defined as the rate at which mass is added to, or removed from a glacier [1]. We designate[2] by $\dot{b}$ the rate of ice accumulation or melting at location $\mathbf{x}$ on the glacier surface. The *net balance* is defined as the sum of mass gain (accumulation) and mass loss (ablation) during a certain time span. In practice, the hydrological year – from 1. October to 30. September – is often used. By definition, the net balance b is evaluated as the integral of $\dot{b}(\mathbf{x}, t)$ over a time interval t_1 to t_2

$$b(\mathbf{x}) = \int_{t_1}^{t_2} \dot{b}(\mathbf{x}, t)\, dt\,. \tag{3.1}$$

The net balance is the mass gain or loss at a location on the glacier surface. The function $b(\mathbf{x})$ describes the spatial distribution of mass balance over the glacier surface. Integrating this function over the glacier surface S leads to the *glacier net balance* B

$$B = \int_S b(\mathbf{x})\, dS\,. \tag{3.2}$$

The glacier net balance is also called the *total net balance*. The glacier net balance is the sum of accumulation and ablation over the whole glacier surface, and therefore the volume change of the glacier (almost: some processes within the glacier and at the base may also contribute to the volume change). Dividing the total mass balance by the glacier surface gives the *average net balance* or *specific net balance*

$$\bar{B} = \frac{B}{S}\,. \tag{3.3}$$

[1] Different designations and notations are in use. Here we adopt the notation of Cuffey and Paterson (2010)

[2] The dot is used to mark a rate. It is part of the symbol, not a time derivative operator.

The following table gives an overview of the mass balance terms (according to Cuffey and Paterson, 2010)

English term	Symbol	Unit	Deutscher Ausdruck
(specific) balance rate	$\dot{b}$	$\mathrm{kg\,m^{-2}\,a^{-1}}$	Massenbilanzrate
volumetric balance rate	$\dot{b}_i$	$\mathrm{m\,a^{-1}}$	Massenbilanzrate
(specific) net balance	b	$\mathrm{kg\,m^{-2}}$	Netto-Massenbilanz
volumetric net balance	b_i	m	Netto-Massenbilanz
glacier net balance	B	kg	gesamte Massenbilanz
average net balance	$\bar{B}$	$\mathrm{kg\,m^{-2}}$	spezifische Nettobilanz
glacier net balance	B_i	$\mathrm{m^3}$	gesamte Massenbilanz
average net balance	$\bar{B}_i$	m	spezifische Nettobilanz

Unfortunately not everybody is using these terms and symbols consistently, but the concept is simple enough that no confusion should arise.

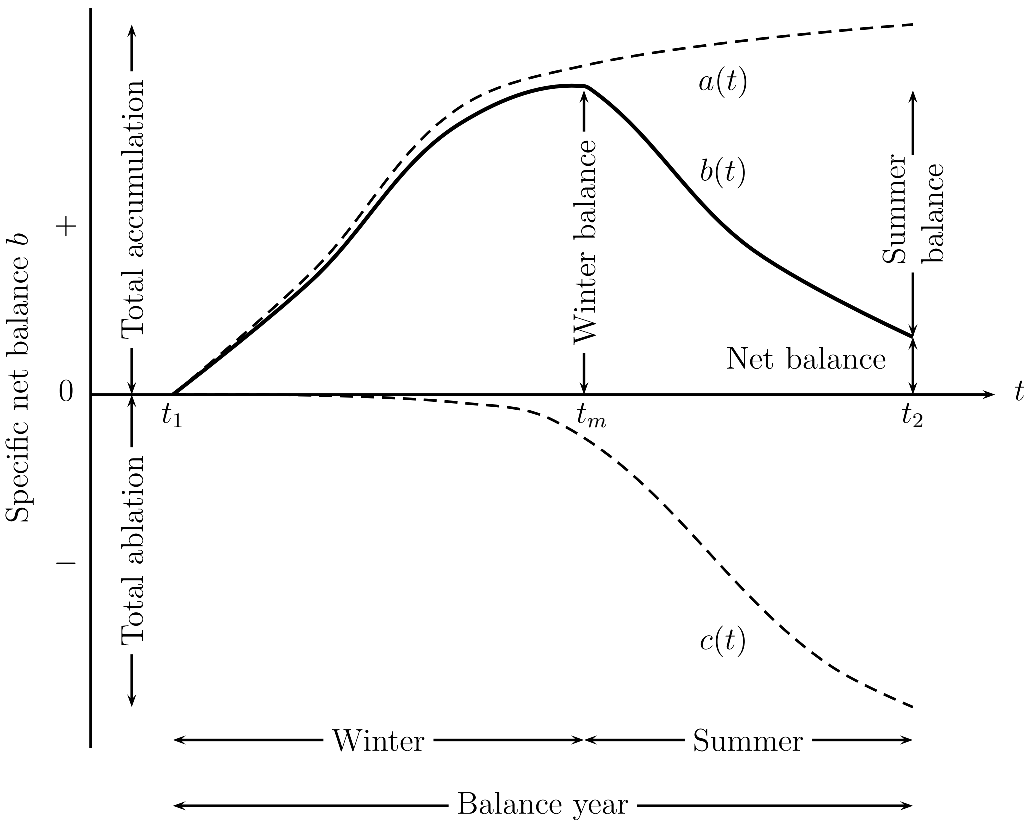

Figure 3.1: *Variation of the net balance over the course of a balance year. Shown are accumulation rate $\dot{a}$, ablation rate $\dot{c}$ and specific balance rate $\dot{b} = \dot{a} + \dot{c}$. A graph similar to this applies for any location on the glacier.*

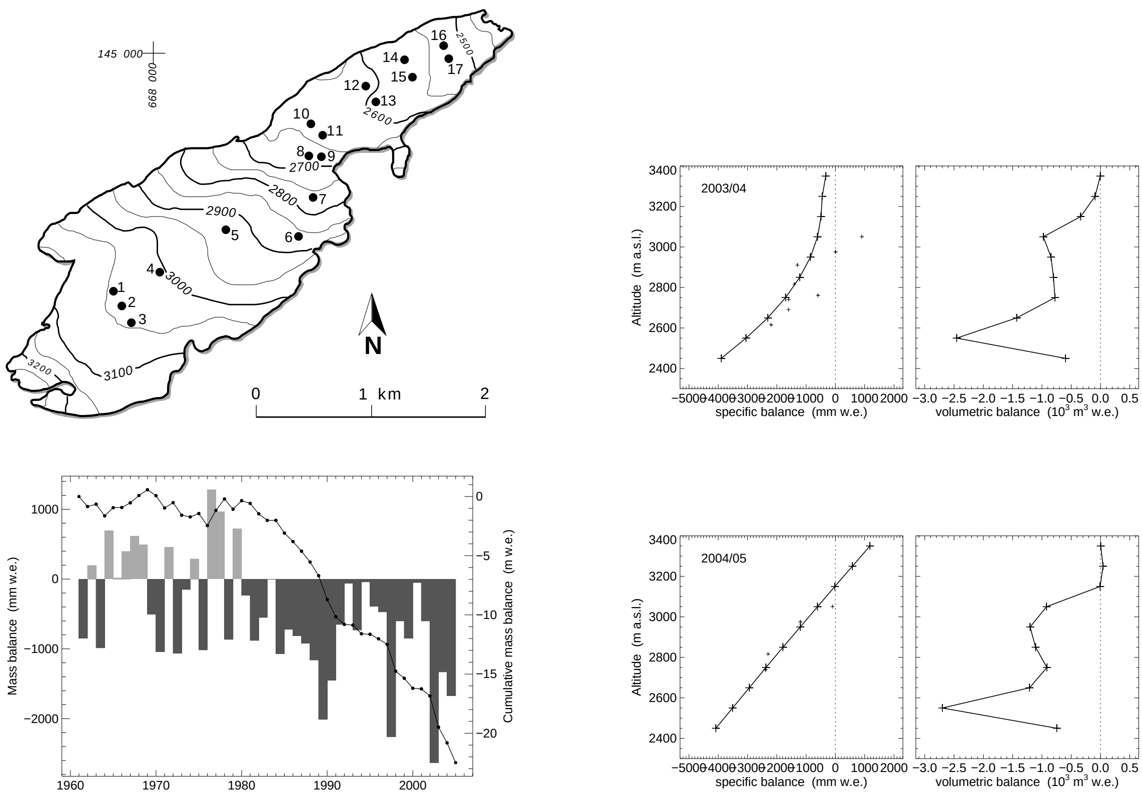

Figure 3.2: *Mass balance data from Griesgletscher (Wallis, Switzerland). Top left: Map with mass balance stakes indicated, Bottom left: net mass balance 1961-2008, Right column: balance with respect to elevation (from Glaciological Reports, 2008).*

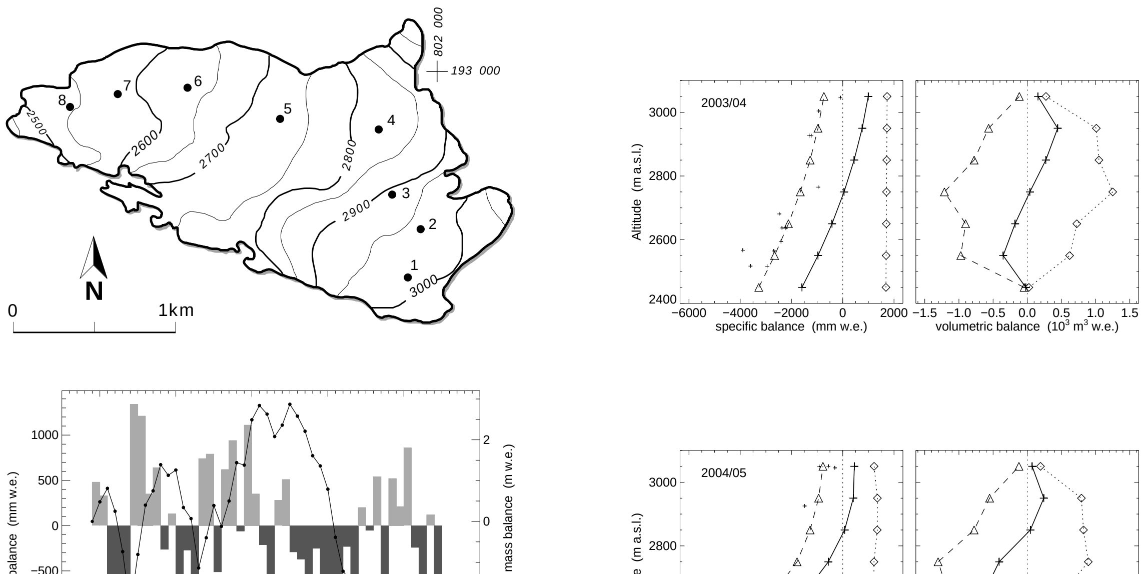

Figure 3.3: *Mass balance data from Silvrettagletscher (Graubünden, Switzerland). Right column: Winter (dotted), summer (dashed) and total balances (solid) are shown (from Glaciological Reports, 2008).*

3.2 Spatial mass balance variations

Net balance varies widely on a glacier or ice sheet, mainly due to temperature, precipitation and radiation, but also due to local effects like aspect (shading), wind redistribution of snow, and avalanches. In the Alps, net balance generally varies with elevation because both temperature and precipitation show strong elevation gradients, whereas on ice sheets the distance form the ocean is a dominating factor.

The variation of net mass balance with elevation usually varies from year to year, as exemplified for Griesgletscher (Fig. 3.2, panels c and d) and Silvrettagletscher (Fig. 3.3, panels c and d). A good average value is $\dot{g} := d\dot{b}(z)/dz = 0.007 - -0.008\,\mathrm{a}^{-1}$. The elevation of the equilibrium line is currently around 3000−3100 m a.s.l., with an important variation between climate regions and due to exposition (Fig. 3.8c).

3.3 Mass balance changes due to climate change

We define climate change as the long term changes of one or more of the climate variables precipitation, radiation and temperature. Figure 3.4 shows nicely that net balance is mainly influenced by the summer balance, i.e. the intensity of melting is more important for the mass balance of a glacier than the amount of snow falls.

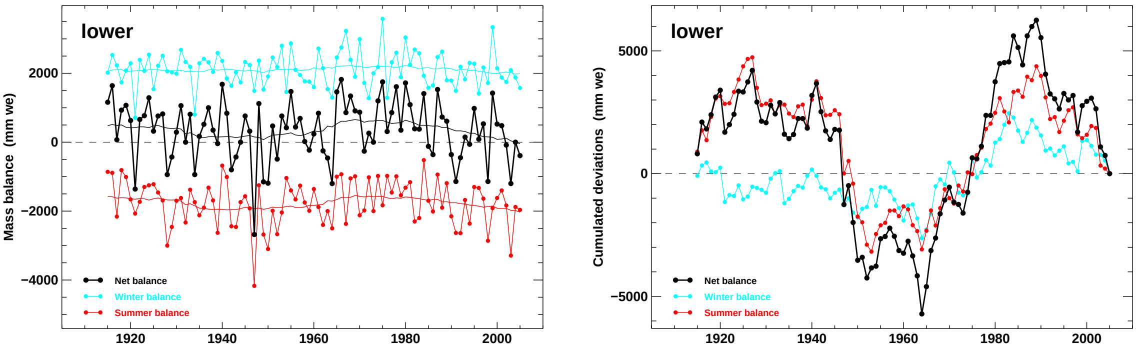

Figure 3.4: *Left: Winter, summer and net balance of a stake on Claridenfirn. Right: Cumulative winter, summer and net balance (from Glaciological Reports, 2008).*

The net balance and the average net balance react very differently to a climate change, as illustrated in Figure 3.5. Initially the glacier is in a steady state, i.e. in equilibrium with the climate. We assume that the climate changes instantly at time t_0 in Figure 3.5a. The melt rate and the accumulation rate change immediately everywhere on the glacier. This leads to a corresponding change in balance rate and net balance. Therefore the net balance rate $\dot{b}$ reacts immediately and without delay to the climate shift (Figure 3.5b).

The average net balance is $\bar{B} = 0$ before the climate change, when the same amount of ice is melted as is accumulated over the course of a year. Immediately after t_0

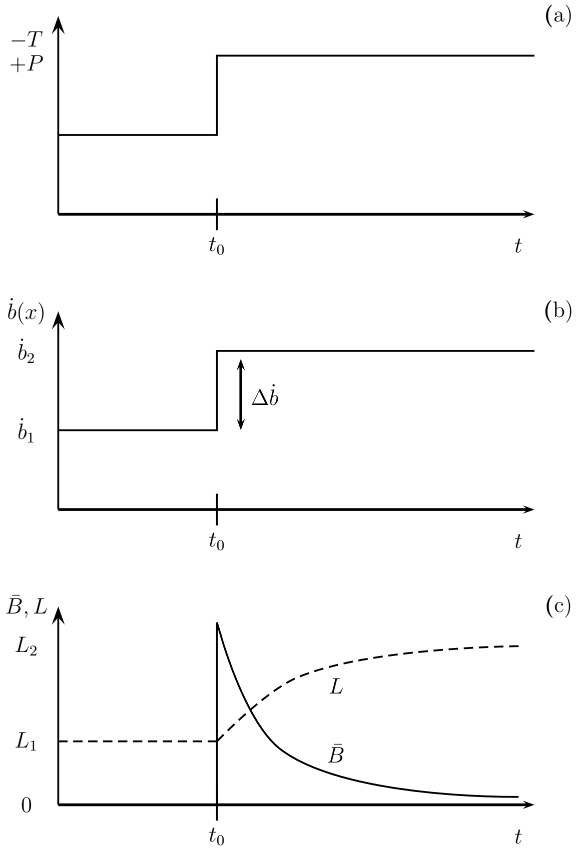

Figure 3.5: *After a sudden climate change (e.g. increase in precipitation P and/or decrease in air temperature T) the glacier reacts with changes of the net balance rate $\dot{b}$, the glacier length L, and of the average net balance $\bar{B}$. Plots a) and b) are valid for any location on the glacier, since we assume the same change of climate at the time $t = t_0$.*

the average net balance is positive or negative until the glacier has reached a new equilibrium (Figure 3.5c). A climate change only leads to a temporary change of the average net balance. Before, and long after the climate change the average net balance is zero.

The glacier length reacts immediately to a climate change, but not jump-like (Figure 3.5c). The length varies until a new equilibrium has been reached. The time needed for the glacier to reach a new equilibrium after a change in the net balance is called the *reaction time*.

3.4 Length change

A glacier in equilibrium with a given climate has a vanishing total net balance

$$B = \int_S b \, dS = 0\,. \tag{3.4}$$

A change in climate usually leads to a change in the distribution of the net balance $b(x)$ everywhere. Consequently the glacier net balance B is different from zero. If the mass balance change was positive, the glacier will reach a new equilibrium by extending its ablation area, which means an advance of the terminus (the glacier gets longer). The additional melt in the new area will compensate the increased accumulation over the initial area.

To illustrate this principle, we imagine a glacier of constant width, and a net balance that only varies with x. Initially, the glacier is in equilibrium (steady state), and has a length L_0

$$\int_0^{L_0} b(x)\, dx = 0, \qquad \text{for} \quad t < t_0. \tag{3.5}$$

At time $t = t_0$ a step change in climate leads to a change in mass balance $\Delta b(x)$. After a certain time the glacier reaches a new equilibrium at a new length $L_0 + \Delta L$

$$\int_0^{L_0+\Delta L} (b(x) + \Delta b(x))\, dx = 0, \qquad \text{for} \quad t - t_0 > \tau_R. \tag{3.6}$$

The *reaction time* τ_R is the time span the glacier needs to adjust to the new net balance (we will come back to τ_R later). We rewrite the integral in Equation (3.6) to elucidate the meaning of the individual terms (using linearity of the integral operator)

$$\begin{aligned} 0 &= \int_0^{L_0+\Delta L} (b(x) + \Delta b(x))\, dx \\ &= \underbrace{\int_0^{L_0} b(x)\, dx}_{0} + \underbrace{\int_0^{L_0} \Delta b(x)\, dx}_{\Delta\bar{b}L_0} + \underbrace{\int_{L_0}^{L_0+\Delta L} b(x)\, dx}_{\bar{b}_t\Delta L} + \underbrace{\int_{L_0}^{L_0+\Delta L} \Delta b(x)\, dx}_{\Delta\bar{b}\Delta L \simeq 0}\,. \end{aligned} \tag{3.7}$$

The first term is zero because of Equation (3.5). The second term is the integral of the mass balance change $\Delta b(x)$ over the original length of the glacier. It can also be written as

$$\Delta\bar{b} := \frac{1}{L_0} \int_0^{L_0} \Delta b(x)\, dx \tag{3.8}$$

The third term is the integral of the original mass balance distribution over the new length of the glacier. Here we used the definition of the mean net balance at the terminus

$$\bar{b}_t := \frac{1}{\Delta L} \int_{L_0}^{L_0+\Delta L} b(x)\, dx \tag{3.9}$$

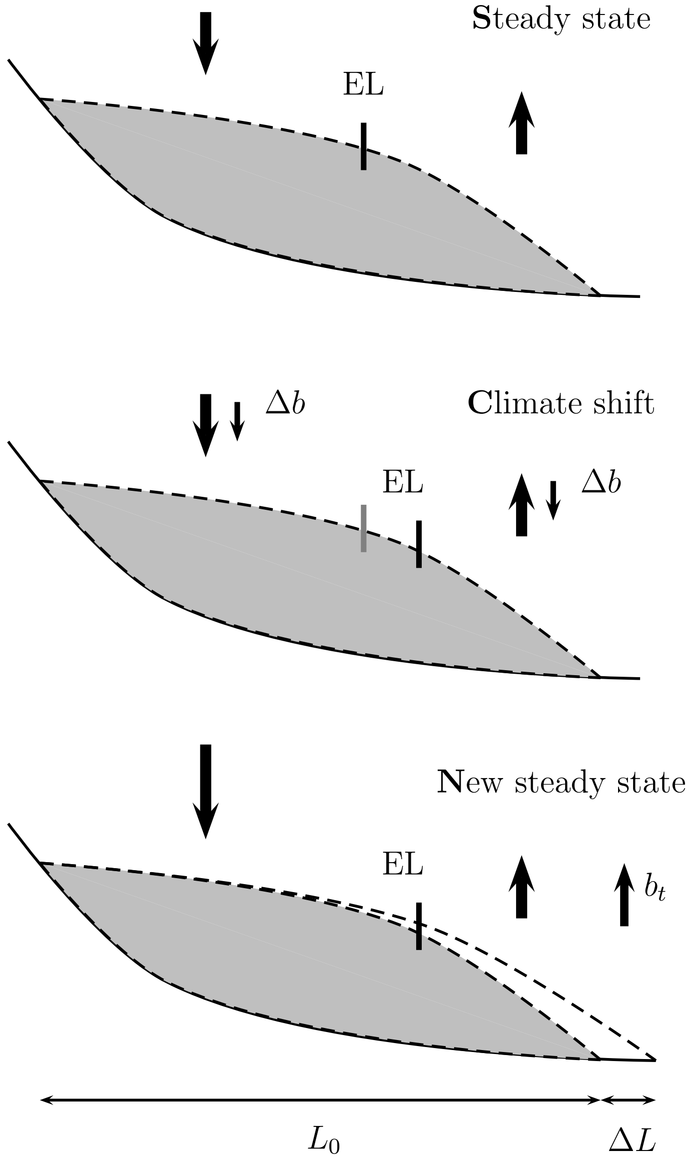

Figure 3.6: *In a steady state the glacier is in equilibrium with the climate. The total mass balance is equal to zero, and the same amount of ice is added in the accumulation area as is removed by melting in the ablation area. Through a climate change that balance is destroyed and has to be re-established by a change in geometry, in this example an extension of the ablation area.*

to write the third term as $\bar{b}_t \Delta L$. The fourth term is a product of the length change and the change of the net balance integrated over ΔL. This term is of second order, and usually negligible compared to the second and third term. With these simplifications we can now write

$$L_0 \Delta \bar{b} + \bar{b}_t \Delta L \simeq 0. \tag{3.10}$$

This equation expresses the fact, that a change of "input" ($L_0 \Delta \bar{b}$) has to be compensated by a change of "output" ($\bar{b}_t \Delta L$). To cope with increased mass balance, the glacier has to extend its ablation area by ΔL.

3.5 Reaction time scales

Jóhannesson volume time scale

How long does it take a glacier to reach a new equilibrium state, after it has been perturbed by a change in net balance $\Delta b(x)$ at time $t = t_0$? Equation (3.10) gives a relation between the change in net balance Δb and the length increase ΔL. Linked to the length change is a change in ice volume ΔV. How long will it take to fill up the newly created volume with the extra mass balance? This time span τ_v is called the *volume time scale*

$$\tau_v := \frac{\text{volume change}}{\text{mass balance change}} = \frac{\Delta V}{\Delta \dot{B}}\,. \tag{3.11}$$

Observation and results from numerical modeling studies show that during a glacier advance the geometry of the accumulation area stays almost the same. The increase of surface elevation in the accumulation area is very small compared to the changes in the ablation area. Also the shape of a glacier tongue stays almost the same. We imagine that the whole tongue is shifted down-slope by the distance ΔL to the new position. The size of the gap that forms at the thickest part of the glacier then gives a good estimate of the volume difference

$$\Delta V \simeq H_{\max} \Delta L. \tag{3.12}$$

For the volume time scale we therefore get the following approximation

$$\begin{aligned} \tau_{v_J} &:= \frac{\Delta V}{\Delta \dot{B}} = \frac{\Delta V}{\int_0^{L_0} \Delta \dot{b}(x)\, dx} \\ &\simeq \frac{H_{\max} \Delta L}{L_0 \Delta \bar{\dot{b}}} \\ &\simeq \frac{H_{\max} \Delta L}{-\bar{\dot{b}}_t \Delta L} && \text{with Eq. (3.10)} \end{aligned} \tag{3.13}$$

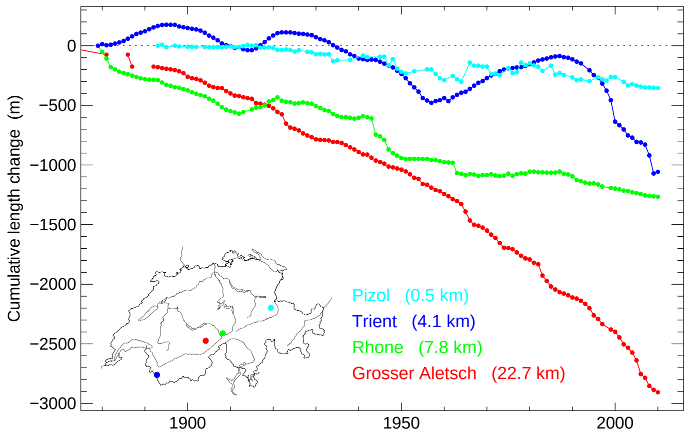

Figure 3.7: *Cumulative length changes of four different-sized glaciers (length is given in parentheses) of the Swiss Glacier Monitoring Network. (Data source: http://glaciology.ethz.ch/swiss-glaciers)*

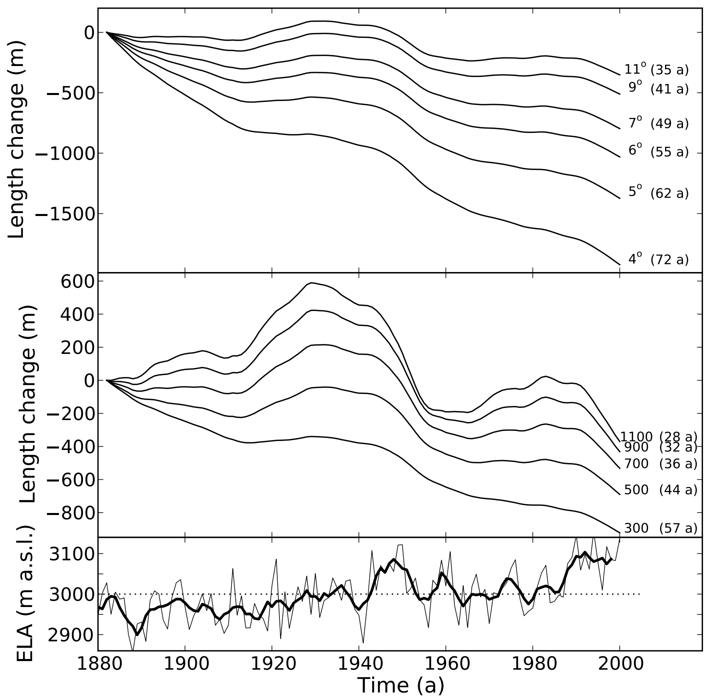

Figure 3.8: *Modeled length changes for different glaciers under the same climate history. (a) Glaciers on different bedrock slopes β (for a vertical extent of the accumulation area of $Z = 400\,\mathrm{m}$). (b) Glaciers with different vertical extents Z of the accumulation area (for $\beta = 7°$). Values of β and Z are indicated next to curves, the volume time scale τ_{v_H} is given in parentheses. (c) The variation of equilibrium line altitude is shown as thin line, and smoothed with a 5 years running average (wide line) (From Lüthi and Bauder, 2010).*

This is the Jóhannesson et al. (1989) volume time scale that depends only on two quantities that are relatively easy to determine

$$\tau_{v_J} \simeq \frac{H_{\max}}{(-\bar{\dot{b}}_t)}\,. \tag{3.14}$$

The mean net balance rate at the glacier tongue $\bar{\dot{b}}_t$ is always negative, so that the volume time scale is positive.

The volume time scale is the minimum time that the glacier needs to adjust to a new climate. It is possible that it takes much longer for the glacier to transport the extra mass to the glacier tongue. However, for most glaciers the relation $\tau_R \simeq \tau_v$ is valid.

Application With Equation (3.14) we can estimate the reaction time of typical Alpine glaciers. Ice thickness is of the order $100 - 200\,\mathrm{m}$ (up to $450\,\mathrm{m}$ for Gornergletscher, $550\,\mathrm{m}$ for Grosser Aletschgletscher). The melt rate at the glacier tongue is typically $4 - 7\,\mathrm{m\,a^{-1}}$. This gives volume time scales of several decades. Since the climate is always changing, most glaciers will have a spatial extent that is not in equilibrium with the current climate.

Harrison volume time scale

The dependence of mass balance upon elevation has not been taken into account in the derivation of the Jóhannesson volume time scale. To achieve this, we start by writing the total balance rate as a function of glacier volume V and area A as $\dot{B}(V, A, t)$. The linear expansion around an arbitrary reference state (A', V') with total balance $\dot{B}' = \dot{B}(V', A', t)$ is

$$\begin{aligned}\dot{B}(V, A, t) &= \dot{B}(V', A', t) + \frac{\partial \dot{B}}{\partial V}(V - V') + \frac{\partial \dot{B}}{\partial A}(A - A') \\ &= \dot{B}' + \underbrace{\frac{\partial \dot{B}}{\partial V}}_{=:\dot{g}_e} \Delta V + \underbrace{\frac{\partial \dot{B}}{\partial A}}_{=:\dot{b}_e} \Delta A\,,\end{aligned} \tag{3.15}$$

where $\Delta A = A - A'$ and $\Delta V = V - V'$. To replace ΔA with ΔV, we define the effective ice thickness $H_e^{-1} := \partial A / \partial V$. Now we can replace $\Delta A = \partial A / \partial V \Delta V = \Delta V / H_e$. Also noting that $\dot{B} = dV/dt = d(\Delta V)/dt$ (in absence of mass changes at the base), we finally write

$$\begin{aligned}\frac{d\Delta V}{dt} &= \dot{g}_e \Delta V + \dot{b}_e \Delta A + \dot{B}' \\ &= \dot{g}_e \Delta V + \frac{\dot{b}_e}{H_e} \Delta V + \dot{B}'\,.\end{aligned}$$

From this an evolution equation for the glacier volume can be written as

$$\frac{d\Delta V}{dt} = \left(\dot{g}_e + \frac{\dot{b}_e}{H_e}\right)\Delta V + \dot{B}' = -\frac{1}{\tau_{v_H}}\Delta V + \dot{B}' \,. \tag{3.16}$$

The term $\dot{B}'$ is a (constant or time-varying) forcing term, and is defined as the total balance rate on the original geometry. It can be shown (Harrison et al., 2001; Lüthi, 2009) that $\dot{g}_e = \dot{g} = d\dot{b}(z)/dz$, and that $\dot{b}_e = \dot{b}_t$ is the balance rate at the terminus. The Harrison volume time scale τ_{v_H} in Equation (3.16) therefore is given by

$$\tau_{v_H} = -\left(\dot{g} + \frac{\dot{b}_t}{H_e}\right)^{-1} = \frac{H_e}{(-\dot{b}_t) - \dot{g}H_e} \,. \tag{3.17}$$

The effective ice thickness H_e is about 25% higher than the maximum ice thickness $H_{\max}$ (a result probably only valid on a simple geometry; Lüthi, 2009).

The second form of Equation (3.17) makes the relation to τ_{v_J} explicit (Eq. 3.14), and shows that the volume time scale can change sign. Since $\dot{b}_t$ is always negative, the sign depends on the relative magnitude of the terms $\left|-\dot{b}_t\right|$ and $\dot{g}H_e$.

$$\begin{aligned} \tau_{v_H} &> 0 \qquad \text{for} \qquad (-\dot{b}_t) > \dot{g}H_e \,, \\ \tau_{v_H} &< 0 \qquad \text{for} \qquad (-\dot{b}_t) < \dot{g}H_e \,. \end{aligned} \tag{3.18}$$

Integration of Equation (3.16) allows us to write down the evolution of glacier volume from one steady state to another under a step change in climate

$$\Delta V(t) = \dot{B}'\,\tau_{v_H}\left(1 - e^{-\frac{t}{\tau_v}}\right) = \Delta V_\infty\left(1 - e^{-\frac{t}{\tau_v}}\right) \,, \tag{3.19}$$

with the final volume change $\Delta V_\infty = \dot{B}'\,\tau_{v_H}$. We see that for positive τ_{v_H} the volume change (response) is always finite. For *negative* volume time scale τ_{v_H}, Equation (3.19) predicts that the response is unstable. For positive $\dot{B}'$ the glacier grows without limit, for negative $\dot{B}'$ it decays. The latter behavior is only qualitatively correct since the assumption of small changes (3.15) breaks down.

From Equation (3.18) we see that the term $\dot{b}_t$ is stabilizing glacier response, whereas the term $\dot{g}H_e$ is destabilizing (the former leads to positive, the latter to negative τ_{v_H}). This is also intuitively clear: glacier growth is limited through melting at the terminus. As a glacier extends to lower elevation, the melt area and the rate of melting at the terminus increase, thus stabilizing the glacier geometry. On the other hand, the feedback between ice thickness and accumulation (implied by $\dot{g}$) can lead to a unlimited growth of a glacier. This is indeed the case for an ice sheet, which topography is rising into higher and higher elevations, thus increasing solid precipitation. Since ice sheets rest on roughly horizontal beds, the ablation area cannot increase fast enough to cope with the increasing accumulation. The growth is eventually stopped when the ice sheet reaches the ocean where iceberg calving provides an efficient mechanism of mass loss.

It is noteworthy (as you will show in homework series 2) that long glaciers have a shorter reaction time scale than short glaciers (on the same slope!), and steep glaciers have a shorter reaction time scale than flat glaciers. The reaction time scale is also inversely proportional to the vertical gradient of mass balance rate $\dot{g}$.

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Appendix

A List of Symbols

Latin letters

Symbol	Description	Units
A	softness parameter, a constant in Glen's flow law	$\mathrm{MPa^{-3}\,a^{-1}}$
$\dot{b}$	specific mass balance rate	$\mathrm{kg\,m^{-2}\,a^{-1}}$
$\dot{b}_i$	specific volumetric mass balance rate	$\mathrm{m\,a^{-1}}$
$B(T)$	temperature dependence of viscosity	
C	specific heat capacity	$\mathrm{J\,kg^{-1}\,K^{-1}}$
E	Young's modulus of elasticity	MPa
g	acceleration due to gravity	$\mathrm{m\,s^{-2}}$
$\dot{g}$	vertical gradient of balance rate $\partial\dot{b}_i/\partial z$	$\mathrm{a^{-1}}$
h	vertical coordinate, depth below surface	m
H	ice thickness	m
k	heat conductivity	$\mathrm{W\,m^{-1}\,K^{-1}}$
n	exponent in Glen's flow law	
p	pressure	MPa
P	heat production	W
Q	heat flux	$\mathrm{W\,m^{-2}}$
q	ice flux, water flux	$\mathrm{m^3\,s^{-1}}$
t	time (seconds, years)	s, a
T	temperature	$^\circ$ C
u_{bal}	balance velocity	$\mathrm{m\,a^{-1}}$
u, v, w	components of the velocity vector $\mathbf{v}$	$\mathrm{m\,a^{-1}}$
$\mathbf{v}$	velocity vector, $\mathbf{v} = (u, v, w)$	$\mathrm{m\,a^{-1}}$
w.eq.	water equivalent	
x, y, z	space coordinates	m
$\mathbf{x}$	position vector, $\mathbf{x} = (x, y, z)$	m
z	vertical coordinate, pointing upwards	m
z_b	bedrock elevation	m
z_{ELA}	equilibrium line altitude	m
z_s	surface elevation	m

Greek letters

Symbol	Description		Units
α	surface slope $\tan\alpha = \frac{dz_s}{dx}$		$^\circ$
β	bed slope $\tan\beta = \frac{dz_b}{dx}$		$^\circ$
$\dot{\boldsymbol{\varepsilon}}$	strain rate tensor with components $\dot{\varepsilon}_{ij}$		a^{-1}
η	shear viscosity		$\mathrm{MPa}\cdot\mathrm{a}$
γ	Clausius-Clapeyron constant	$[\sim 0.074\,\mathrm{K\,MPa^{-1}}]$	$\mathrm{K\,MPa^{-1}}$
κ	thermal diffusivity		
ν	elastic Poisson ratio		
ρ_i	density of ice	$[900 - 917\,\mathrm{kg\,m^{-3}}]$	$\mathrm{kg\,m^{-3}}$
ρ_w	density of water		$\mathrm{kg\,m^{-3}}$
σ_e	effective uniaxial stress	$[\sigma_e := (\frac{3}{2}\sigma_{ij}^{(d)}\sigma_{ij}^{(d)})^{\frac{1}{2}} = \sqrt{3}\tau]$	MPa
σ_m	mean stress	$[\sigma_m := \frac{1}{3}\sigma_{kk}]$	MPa
$\boldsymbol{\sigma}$	stress tensor with components σ_{ij}		MPa
$\boldsymbol{\sigma}^{(d)}$	stress deviator tensor $[\sigma_{ij}^{(d)} := \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij} = \sigma_{ij} - \sigma_m\delta_{ij}]$		MPa
τ	effective shear stress	$[\tau := (\frac{1}{2}\sigma_{ij}^{(d)}\sigma_{ij}^{(d)})^{\frac{1}{2}} = \frac{1}{\sqrt{3}}\sigma_e]$	MPa

Appendix

B Useful quantities

Quantity	Symbol	Value	Unit
Mechanical properties			
Density of water (0° C)	ρ_w	999.84	$\mathrm{kg\,m^{-3}}$
Density of bubble free ice (0° C)	ρ_i	917	$\mathrm{kg\,m^{-3}}$
Young modulus of ice	E	$8.7 \cdot 10^9$	Pa
Shear modulus of ice	μ	$3.8 \cdot 10^9$	Pa
Poisson ratio of ice	ν	0.31	
Creep activation energy (<−10° C)	Q	78	$\mathrm{kJ\,mol^{-1}}$
Thermal properties			
Specific heat capacity of water	C_w	4182	$\mathrm{J\,K^{-1}\,kg^{-1}}$
Specific heat capacity of ice	C_i	2093	$\mathrm{J\,K^{-1}\,kg^{-1}}$
Thermal conductivity of ice (at 0° C)	k	2.1	$\mathrm{W\,m^{-1}\,K^{-1}}$
Thermal diffusivity of ice (at −1° C)	κ	$1.09 \cdot 10^{-6}$	$\mathrm{m^2\,s^{-1}}$
Latent heat of fusion (ice/water)	L	333.5	$\mathrm{kJ\,kg^{-1}}$
Depression of melting point (Clausius-Clapeyron constant)			
- pure ice and air-free water	γ_p	0.074	$\mathrm{K\,MPa^{-1}}$
- pure ice and air-saturated water	γ_a	0.098	$\mathrm{K\,MPa^{-1}}$
Constants			
Gravity acceleration	g	9.81	$\mathrm{m\,s^{-2}}$
Triple point temperature	T_{tp}	273.16	K
Triple point pressure	p_{tp}	611.73	Pa
Gas constant	R	8.31	$\mathrm{J\,mol^{-1}\,K^{-1}}$
Avogadro number	N_A	$6.023 \cdot 10^{23}$	
Boltzmann constant	k_b	$1.3807 \cdot 10^{-23}$	$\mathrm{J\,K^{-1}}$
Stefan-Boltzmann constant	σ_{sb}	$5.67 \cdot 10^{-8}$	$\mathrm{W\,m^{-2}\,K^{-4}}$
Solar constant (radiation)	Q_{solar}	1368	$\mathrm{W\,m^{-2}}$

Flow law parameter

T (° C)	A ($s^{-1}\,Pa^{-3}$)	A ($a^{-1}\,MPa^{-3}$)	A_P ($s^{-1}\,kPa^{-3}$)	A_P ($a^{-1}\,MPa^{-3}$)
0	$2.4 \cdot 10^{-24}$	75.7	($6.8 \cdot 10^{-15}$)	(215)
-2	$1.7 \cdot 10^{-24}$	53.6		
-5	$9.3 \cdot 10^{-25}$	29.3	($1.6 \cdot 10^{-15}$)	(50.5)
-10	$3.5 \cdot 10^{-25}$	11.0	($4.9 \cdot 10^{-16}$)	(15.5)
-15	$2.1 \cdot 10^{-25}$	6.62	($2.9 \cdot 10^{-16}$)	(9.2)
-20	$1.2 \cdot 10^{-25}$	3.78	($1.7 \cdot 10^{-16}$)	(5.4)
-30	$3.7 \cdot 10^{-26}$	1.17	($5.1 \cdot 10^{-17}$)	(1.6)
-40	$1.0 \cdot 10^{-26}$	0.315	($1.4 \cdot 10^{-17}$)	(0.44)
-50	$2.6 \cdot 10^{-27}$	0.082	($3.6 \cdot 10^{-18}$)	(0.11)

Table B.1: *Flow law parameter A recommended by Cuffey and Paterson (2010), and the older values A_P recommended by Paterson (1999).*

It is common to assume that the flow law parameter A can be split into a constant rate factor at a reference temperature A_0 and a parameter absorbing the temperature dependence $B(T)$ (e.g. Hutter, 1983; Paterson, 1999). At temperatures below $-10°$ C the rate factor is of Arrhenius type with an activation energy of about 60 kJ mol^{-1} (Paterson, 1994). A double exponential fit derived by Smith and Morland (1981, eq. 21) is often used

$$B(T) = 0.9316\,\exp(0.32769\,T) + 0.0686\,\exp(0.07205\,T)\,, \quad T \geq -7.65°\,\mathrm{C}, \tag{B.1}$$
$$B(T) = 0.7242\,\exp(0.59784\,T) + 0.3438\,\exp(0.14747\,T)\,, \quad T < -7.65°\,\mathrm{C}, \tag{B.2}$$

where T is the Celsius temperature. This parameterization is almost identical to the values given in Table B1 (Paterson, 1999, p. 97).

The rate factor A in Glen's flow law is also affected by the percentage of water μ within the ice (Duval, 1977; Paterson, 1999)

$$\begin{aligned} A(\mu) &= (3.2 + 5.8\mu) \cdot 10^{-15}\,\mathrm{kPa}^{-3}\,\mathrm{s}^{-1} \\ &= (101 + 183\mu)\,\mathrm{MPa}^{-3}\,\mathrm{a}^{-1}\,, \end{aligned} \tag{B.3}$$

B.1.