TUTORIAL SESSION: Synchronization in Coupled Oscillators: Theory and Applications

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Exploring Synchronization in Complex Oscillator Networks

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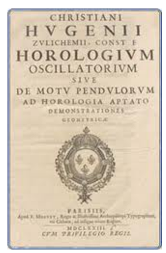
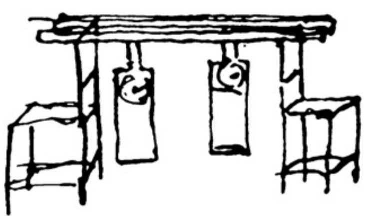
A Brief History of Sync
how it all began

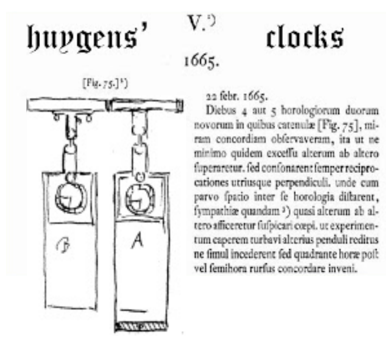
Christiaan Huygens (1629 – 1695)

- physicist & mathematician
- engineer & horologist

observed *"an odd kind of sympathy"* between coupled & heterogeneous clocks
[Letter to Royal Society of London, 1665]







Recent reviews, experiments, & analysis
[M. Bennet et al. '02, M. Kapitaniak et al. '12]

A Brief History of Sync
the odd kind of sympathy is still fascinating

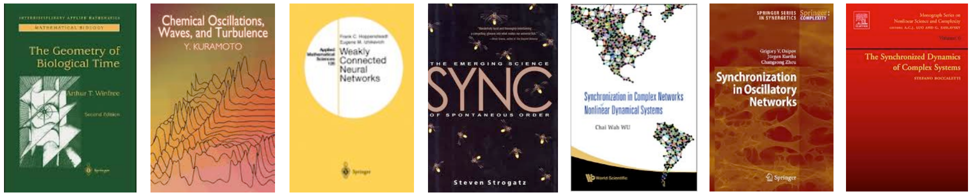
watch movie online here:

http://www.youtube.com/watch?v=JWToUATLGzs&list=UUJIyXclKY8FQQwaKBaawl_A&index=3

Sync of 32 metronomes at Ikeguchi Laboratory, Saitama University, 2012

A Brief History of Sync
a field was born

- Sync in mathematical biology [A. Winfree '80, S.H. Strogatz '03, ...]
- Sync in physics and chemistry [Y. Kuramoto '83, M. Mézard et al. '87...]
- Sync in neural networks [F.C. Hoppensteadt and E.M. Izhikevich '00, ...]
- Sync in complex networks [C.W. Wu '07, S. Bocaletti '08, ...]
- ...and countless technological applications (reviewed later)



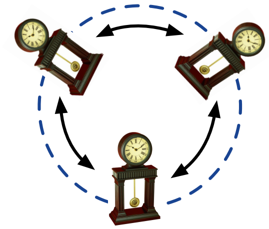
Coupled Phase Oscillators

$\exists$ various models of oscillators & interactions

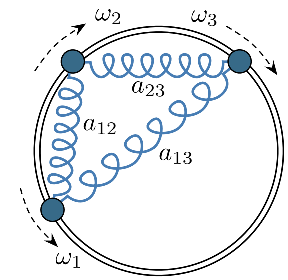
Today: canonical coupled oscillator model
[A. Winfree '67, Y. Kuramoto '75]

Coupled oscillator model:

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$



- n **oscillators** with phase $\theta_i \in \mathbb{S}^1$
- **non-identical** natural frequencies $\omega_i \in \mathbb{R}^1$
- elastic **coupling** with strength $a_{ij} = a_{ji}$
- undirected & connected **graph** $G = (\mathcal{V}, \mathcal{E}, A)$



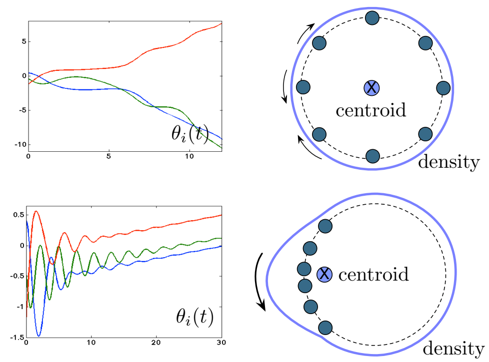
Phenomenology and Challenges in Synchronization

Synchronization is a **trade-off:**
coupling vs. heterogeneity

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

coupling small & $|\omega_i - \omega_j|$ large
$\Rightarrow$ incoherence & no sync

coupling large & $|\omega_i - \omega_j|$ small
$\Rightarrow$ coherence & frequency sync



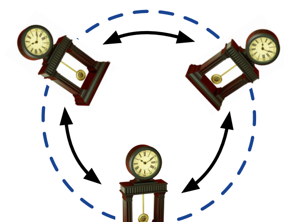
Some central questions:
(still after 45 years of work)

- proper notion of sync & phase transition
- quantify "coupling" vs. "heterogeneity"
- interplay of network & dynamics

Applications of the Coupled Oscillator Model

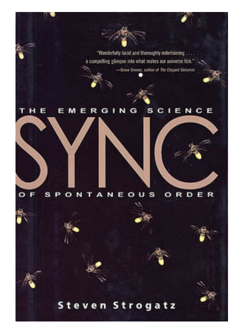
Coupled oscillator model:

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

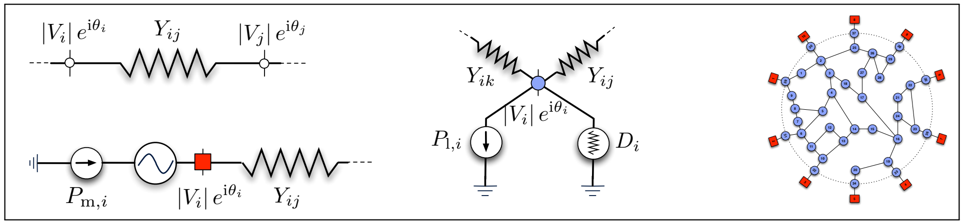


Some related applications:

- Sync in a population of fireflies
  [G.B. Ermentrout '90, Y. Zhou et al. '06, ...]
- Deep-brain stimulation and neuroscience
  [N. Kopell et al. '88, P.A. Tass '03, ...]
- Sync in coupled Josephson junctions
  [S. Watanabe et. al '97, K. Wiesenfeld et al. '98, ...]
- Countless other sync phenomena in physics, biology, chemistry, mechanics, social nets etc.
  [A. Winfree '67, S.H. Strogatz '00, J. Acebrón '01, ...]



Example 1: AC Power Transmission Network



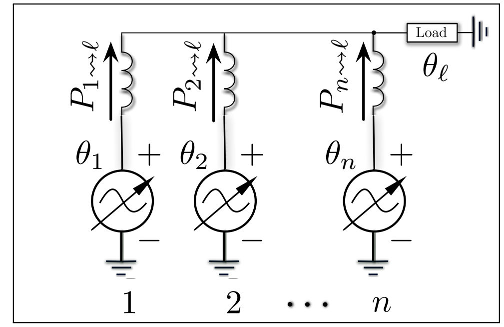
- power transfer on line $i \rightsquigarrow j$: $\underbrace{|V_i||V_j||Y_{ij}|}_{a_{ij}=\text{max power transfer}} \cdot \sin(\theta_i - \theta_j)$
- power balance at node i: $\underbrace{P_i}_{\text{power injection}} = \sum_j a_{ij} \sin(\theta_i - \theta_j)$
- Structure-Preserving Model [A. Bergen & D. Hill '81]:

■: swing eq with $P_{m,i} > 0$ $\quad M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_{m,i} - \sum_j a_{ij} \sin(\theta_i - \theta_j)$

●: $P_{l,i} < 0$ and $D_i \geq 0$ $\quad D_i \dot{\theta}_i = P_{l,i} - \sum_j a_{ij} \sin(\theta_i - \theta_j)$

Example 2: DC/AC Inverters in Microgrids

1. (islanded) microgrid = autonomously managed low-voltage network
2. inverter in microgrid = controllable AC source
3. physics: $P_{i \rightsquigarrow \ell} = a_{i\ell} \sin(\theta_i - \theta_\ell)$



Droop-control [M.C. Chandorkar et. al., '93]: $D_i \dot{\theta}_i = P_i^* - P_{i \rightsquigarrow \ell}$

Closed-loop for inverters & load ℓ: [J.W. Simpson-Porco et. al., '12]

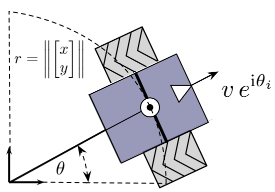
$$D_i \dot{\theta}_i = P_i^* - a_{i\ell} \sin(\theta_i - \theta_\ell)$$
$$0 = P_\ell - \sum_j a_{\ell j} \sin(\theta_\ell - \theta_i)$$

Example 3: Flocking, Schooling, & Vehicle Coordination

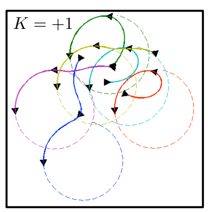
- Network of **Dubins' vehicles**

$$\dot{r}_i = v\, e^{i\theta_i}$$
$$\dot{\theta}_i = u_i(r, \theta)$$

with speed v and steering control $u_i(r, \theta)$



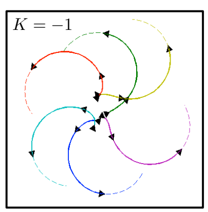
- sensing/comm. graph $G = (\mathcal{V}, \mathcal{E}, A)$ for coordination of autonomous vehicles



"phase sync"

- **relative sensing control** $u_i = f_i(\theta_i - \theta_j)$ for neighbors $\{i, j\} \in \mathcal{E}$ yields closed-loop

$$\dot{\theta}_i = \omega_0(t) - K \cdot \sum_j a_{ij} \sin(\theta_i - \theta_j)$$



"phase balance"

[R. Sepulchre et al. '07, D. Klein et al. '09, L Consolini et al '10]

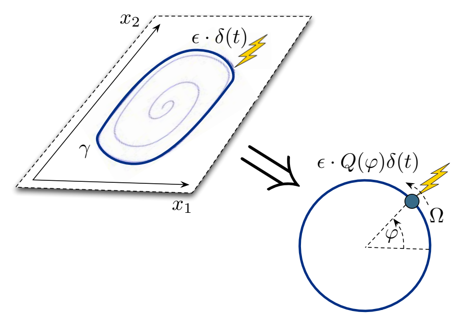
Example 4: Canonical Coupled Oscillator Model

- dynamical system with stable limit cycle γ and weak perturb.

$$\dot{x} = f(x) + \epsilon \cdot \delta(t)$$

- local phase dynamics near γ with phase response curve $Q(\varphi)$

$$\dot{\varphi} = \Omega + \epsilon \cdot Q(\varphi)\delta(t) + \mathcal{O}(\epsilon^2)$$



⇒ same phase reduction applied to interacting oscillators

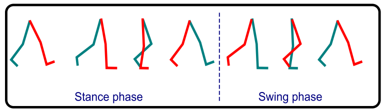
⇒ coord. & time transf. + averaging $\Rightarrow \dot{\theta}_i = \sum_j h_{ij}(\theta_i - \theta_j)$

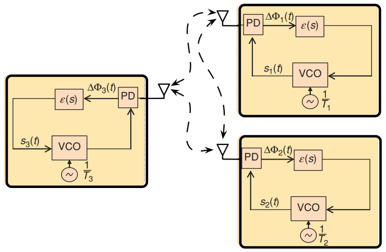
⇒ 0th and 1st (odd) Fourier mode: $\dot{\theta}_i = \omega_i + \sum_j a_{ij} \sin(\theta_i - \theta_j)$

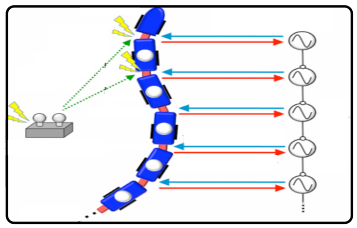
[F.C. Hoppensteadt & E.M. Izhikevich '00, Y. Kuramoto '83, E. Brown et al. '04, ...]

Example 5: Other technological applications

- Particle filtering to estimate limit cycles [A. Tilton & P. Mehta et al. '12]
- Clock synchronization over networks [Y. Hong & A. Scaglione '05, O. Simeone et al. '08, Y. Wang & F. Doyle et al. '12]
- Central pattern generators and robotic locomotion [J. Nakanishi et al. '04, S. Aoi et al. '05, L. Righetti et al. '06]
- Decentralized maximum likelihood estimation [S. Barbarossa et al. '07]
- Carrier sync without phase-locked loops [M. Rahman et al. '11]







Outline

1. Introduction and motivation
2. Synchronization notions, metrics, & basic insights
3. Phase synchronization and more basic insights
4. Synchronization in complete networks
5. Synchronization in sparse networks
6. Open problems and research directions

Order Parameter

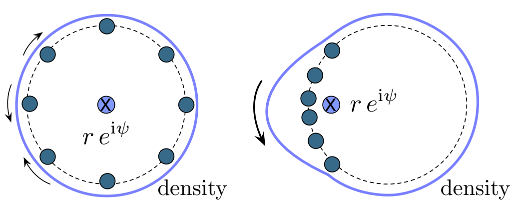
(for homogenous coupling $a_{ij} = K/n$)

Define the order parameter (centroid) by $\boxed{r e^{\mathrm{i}\psi} = \frac{1}{n}\sum_{j=1}^{n} e^{\mathrm{i}\theta_j}}$, then

$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum_{j=1}^{n}\sin(\theta_i - \theta_j) \quad \Leftrightarrow \quad \dot{\theta}_i = \omega_i - Kr\sin(\theta_i - \psi)$$

Intuition: synchronization = entrainment by mean field $re^{\mathrm{i}\psi}$

K small & $|\omega_i - \omega_j|$ large



K large & $|\omega_i - \omega_j|$ small

$\Rightarrow$ analysis based on concepts from statistical mechanics & cont. limit: [Y. Kuramoto '75, G.B. Ermentrout '85, J.D. Crawford '94, S.H. Strogatz '00, J.A. Acebrón et al. '05, E.A. Martens et al. '09, H. Yin et al. '12, ...]

Synchronization Notions & Metrics

1) **frequency sync**: $\dot{\theta}_i(t) = \dot{\theta}_j(t)\ \forall\, i,j$
   $\Leftrightarrow\ \dot{\theta}_i(t) = \omega_{\text{sync}}\ \forall\, i \in \{1,\dots,n\}$

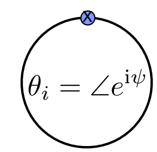
2) **phase sync**: $\theta_i(t) = \theta_j(t)\ \forall\, i,j$
   $\Leftrightarrow\ r = 1$

3) **phase balancing**: $r = 0$
   (e.g., splay state = uniform spacing on $\mathbb{S}^1$)

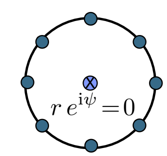
4) **arc invariance**: all angles in $\overline{\mathrm{Arc}}_n(\gamma)$
   (closed arc of length γ) for $\gamma \in [0, 2\pi]$

5) **phase cohesiveness**: all angles in
   $\bar{\Delta}_G(\gamma) = \{\theta \in \mathbb{T}^n : \max_{\{i,j\}\in\mathcal{E}} |\theta_i - \theta_j| \le \gamma\}$
   for some $\gamma \in [0, \pi/2[$

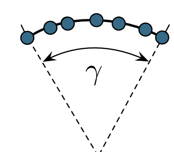
2)



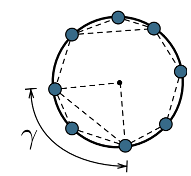
3)



4)



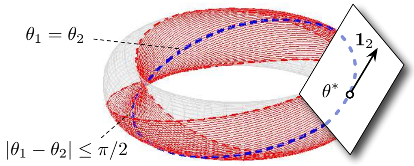
5)



Geometric & Algebraic Insights I
symmetries

Coupled oscillator model:

$$\dot\theta_i = f(\theta) = \omega_i - \sum\nolimits_{j=1}^{n} a_{ij}\sin(\theta_i - \theta_j)$$



- vector field $f(\theta)$ possesses rotational symmetry: $f(\theta^*) = f(\theta^* + \varphi\mathbf{1}_n)$
- $\sum_{i=1}^{n} \dot\theta_i(t) = \sum_{i=1}^{n} \omega_i \overset{!}{=} \sum_{i=1}^{n} \omega_{\text{sync}} \;\Rightarrow\;$ sync frequ. $\omega_{\text{sync}} = \omega_{\text{avg}} = \frac{1}{n}\sum_{i=1}^{n}\omega_i$

$\Rightarrow$ transf. to rot. frame with freq. $\omega_{\text{avg}} \;\Leftrightarrow\; \omega_{\text{sync}} = 0 \;\Leftrightarrow\; \omega_i \mapsto \omega_i - \omega_{\text{avg}}$

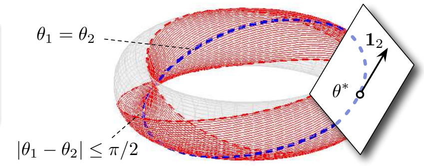
- wlog: assume $\omega_{\text{avg}} = 0 \;\Rightarrow$ **frequency sync = equilibrium manifold**

$$[\theta^*] = \{\theta \in \mathbb{T}^n :\; \theta^* + \varphi\mathbf{1}_n \,, f(\theta^*) = 0 \,, \varphi \in [0, 2\pi]\}$$

Geometric & Algebraic Insights II
Jacobian is Laplacian

Coupled oscillator model:

$$\dot\theta_i = f(\theta) = \omega_i - \sum\nolimits_{j=1}^{n} a_{ij}\sin(\theta_i - \theta_j)$$



- negative Jacobian $-\partial f/\partial\theta$ evaluated at $\theta^* \in \mathbb{T}^n$ is given by

$$L(\theta^*) = \begin{bmatrix} \sum_{k=2}^{n} a_{1k}\cos(\theta_1^* - \theta_k^*) & -a_{12}\cos(\theta_1^* - \theta_2^*) & \dots & -a_{1n}\cos(\theta_1^* - \theta_n^*) \\ \vdots & \ddots & \ddots & \vdots \\ -a_{n1}\cos(\theta_n^* - \theta_1^*) & \dots & -a_{n,n-1}\cos(\theta_n^* - \theta_{n-1}^*) & \sum_{k=1}^{n-1} a_{in}\cos(\theta_n^* - \theta_k^*) \end{bmatrix}$$

$=$ Laplacian matrix of graph $(\mathcal{V}, \mathcal{E}, \tilde{A})$ with weights $\tilde{a}_{ij} = a_{ij}\cos(\theta_i^* - \theta_j^*)$

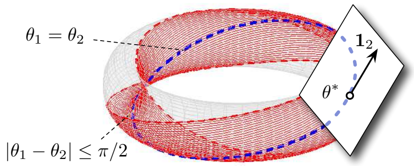
$\Rightarrow$ all weights $\tilde{a}_{ij} > 0$ for $\{i,j\} \in \mathcal{E} \;\Leftrightarrow\;$ $\max_{\{i,j\}\in\mathcal{E}} |\theta_i^* - \theta_j^*| < \pi/2$

$\Rightarrow$ algebraic graph theory: $L(\theta^*)$ is p.s.d. and $\ker(L(\theta^*)) = \text{span}(\mathbf{1}_n)$

Geometric & Algebraic Insights II
Jacobian is Laplacian

Coupled oscillator model:

$$\dot\theta_i = f(\theta) = \omega_i - \sum\nolimits_{j=1}^{n} a_{ij}\sin(\theta_i - \theta_j)$$



Lemma [C. Tavora and O.J.M. Smith '72]

If there exists an equilibrium manifold $[\theta^*]$ in

$$\Delta_G(\pi/2) = \{\theta \in \mathbb{T}^n : \max_{\{i,j\}\in\mathcal{E}} |\theta_i - \theta_j| < \pi/2\} \,,$$

then $[\theta^*]$ is

1. locally exponentially stable (modulo symmetry), and
2. unique in $\bar\Delta_G(\pi/2)$ (modulo symmetry).

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1. Introduction and motivation
2. Synchronization notions, metrics, & basic insights
3. **Phase synchronization and more basic insights**
4. Synchronization in complete networks
5. Synchronization in sparse networks
6. Open problems and research directions

Phase Synchronization
a forced gradient system

$$\dot{\theta}_i = \omega_i - \sum\nolimits_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

$\{$**phase sync**$\} = \{\theta \in \mathbb{T}^n : \theta_i = \theta_j \; \forall \, i, j\}$

Classic intuition [P. Monzon et al. '06, Sepulchre et al. '07]:

- Coupled oscillator model is forced gradient flow $\dot{\theta}_i = \omega_i - \nabla_i U(\theta)$, where $U(\theta) = \sum_{\{i,j\} \in \mathcal{E}} a_{ij} \big(1 - \cos(\theta_i - \theta_j)\big)$ (spring potential)
- assume that $\omega_i = 0 \;\; \forall \, i \in \{1, \dots, n\} \;\Rightarrow\;$ gradient flow $\dot{\theta} = -\nabla U(\theta)$

$\Rightarrow$ global convergence to critical points $\{\nabla U(\theta) = \mathbf{0}\} \supseteq \{$phase sync$\}$

$\Rightarrow$ previous Jacobian arguments: $\{$phase sync$\}$ is local minimum & stable

Phase Synchronization
main result

$$\dot{\theta}_i = \omega_i - \sum\nolimits_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

$\{$**phase sync**$\} = \{\theta \in \mathbb{T}^n : \theta_i = \theta_j \; \forall \, i, j\}$

Theorem: [P. Monzon et al. '06, Sepulchre et al. '07]

The following statements are equivalent:

1. For all $\{i, j\} \in \{1, \dots, n\}$, we have that $\omega_i = \omega_j$; and
2. There exists a locally exp. stable phase synchronization manifold.

Proof of "$\Rightarrow$": wlog in rot. frame: $\omega_i = \omega_j = 0 \Rightarrow$ follow previous args

Proof of "$\Leftarrow$": phase sync'd solutions satisfy $\theta_i = \theta_j$ & $\dot{\theta}_i = \dot{\theta}_j \Rightarrow \omega_i = \omega_j$

Remark: "almost global phase sync" for certain topologies (trees, cmplt., short cycles) [P. Monzon, E.A. Canale et al. '06-'10, A. Sarlette '09]

Phase Synchronization
further insights when all $\omega_i = 0$

$$\dot{\theta}_i = \omega_i - \sum\nolimits_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

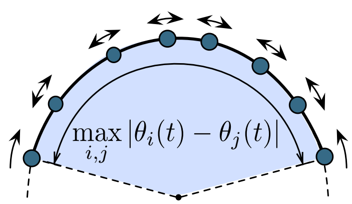
$\{$**phase sync**$\} = \{\theta \in \mathbb{T}^n : \theta_i = \theta_j \; \forall \, i, j\}$

- **Convexity** simplifies life:

  if all oscillators in open semicircle $\mathrm{Arc}_n(\pi)$

  $\Rightarrow$ convex hull $\max_{i,j \in \{1,\dots,n\}} |\theta_i(t) - \theta_j(t)|$ is contracting

  [L. Moreau '04, Z. Lin et al. '08]

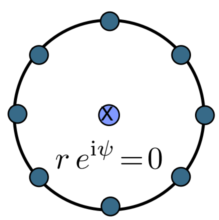


- **Phase balancing**:

  inverse gradient flow (ascent) $\dot{\theta} = +\nabla U(\theta)$

  $\Rightarrow$ phase balancing for circulant graphs

  [L. Scardovi et al. '07, Sepulchre et al. '07]



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4. Synchronization in complete networks
5. Synchronization in sparse networks
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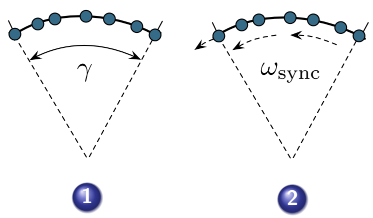
Synchronization in a Complete & Homogeneous Graph
recall definitions

Classic **Kuramoto model** of coupled oscillators:

$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum_{j=1}^{n}\sin(\theta_i - \theta_j)$$

One appropriate **sync notion**:

1. arc invariance: $\theta \in \overline{\text{Arc}}_n(\gamma)$ for small $\gamma \in [0, \pi/2[$
2. frequency sync: $\dot{\theta}_i = \omega_{\text{avg}}$ with $\omega_{\text{avg}} = \frac{1}{n}\sum_{i=1}^{n}\omega_i$



Numerous **results** on sync conditions & bifurcations

[A. Jadbabaie et al. '04, P. Monzon et al. '06, Sepulchre et al. '07, F. de Smet et al. '07, N. Chopra et al. '09, A. Franci et al. '10, S.Y. Ha et al. '10, D. Aeyels et al. '04, , J.L. van Hemmen et al. '93, R.E. Mirollo et al. '05, M. Verwoerd et al. '08, . . .]

Synchronization in a Complete & Homogeneous Graph
brief review

$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum_{j=1}^{n}\sin(\theta_i - \theta_j)$$

frequency sync: $\dot{\theta}_i = \omega_{\text{avg}}\ \forall i$
arc invariance: $\theta \in \overline{\text{Arc}}_n(\gamma)$

- **Implicit equations** for existence of sync'd fixed points
  [D. Aeyels et al. '04, R.E. Mirollo et al. '05, M. Verwoerd et al. '08]
- **Necessary conditions**: $\dot{\theta}_i = \dot{\theta}_j \quad \forall i,j \quad \Rightarrow \quad K > \frac{\omega_{\max}-\omega_{\min}}{2}\cdot\frac{n}{n-1}$
  [N. Chopra et al. '09, A. Jadbabaie et al. '04, J.L. van Hemmen et al. '93]
- **Sufficient conditions**, e.g., $K > \|(\dots,\omega_i-\omega_j,\dots)\|_{2,\infty}\cdot f(n,\gamma)$
  [J.L. van Hemmen et al. '93, A. Jadbabaie et al. '04, F. de Smet et al. '07, N. Chopra et al. '09, G. Schmidt et al. '09, F. Dörfler and F. Bullo '09, S.J. Chung and J.J. Slotine '10, A. Franci et al. '10, S.Y. Ha et al. '10, . . .]

Synchronization in a Complete & Homogeneous Graph
main result

$$\dot{\theta}_i = \omega_i - \frac{K}{n}\sum_{j=1}^{n}\sin(\theta_i - \theta_j)$$

frequency sync: $\dot{\theta}_i = \omega_{\text{avg}}\ \forall i$
arc invariance: $\theta \in \overline{\text{Arc}}_n(\gamma)$

Theorem [F. Dörfler & F. Bullo '11]

The following statements are equivalent:

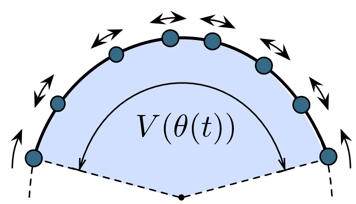
1. Coupling dominates heterogeneity, i.e., $K > K_{\text{critical}} \triangleq \omega_{\max} - \omega_{\min}$;
2. $\exists\ \gamma_{\max} \in\,]\pi/2, \pi]$ s.t. all Kuramoto models with $\omega_i \in [\omega_{\min}, \omega_{\max}]$ and $\theta(0) \in \text{Arc}_n(\gamma_{\max})$ achieve exponential frequency sync; and
3. $\exists\ \gamma_{\min} \in [0, \pi/2[$ s.t. all Kuramoto models with $\omega_i \in [\omega_{\min}, \omega_{\max}]$ feature a locally exp. stable equilibrium manifold in $\overline{\text{Arc}}_n(\gamma_{\min})$.

Moreover, we have $K_{\text{critical}}/K = \sin(\gamma_{\min}) = \sin(\gamma_{\max})$

and **practical phase synchronization**: from $\gamma_{\max}$ arc $\rightarrow$ $\gamma_{\min}$ arc

Synchronization in a Complete & Homogeneous Graph
main proof ideas

1. **Arc invariance**: $\theta(t)$ in γ arc $\Leftrightarrow$ arc-length $V(\theta(t))$ is non-increasing



$$\Leftrightarrow \begin{cases} V(\theta(t)) = \max_{i,j\in\{1,\dots,n\}} |\theta_i(t) - \theta_j(t)| \\ D^+V(\theta(t)) \overset{!}{\leq} 0 \end{cases}$$

true if $\boxed{K\sin(\gamma) \geq K_{\text{critical}}}$

2. **Frequency synchronization** $\Leftrightarrow$ consensus protocol in $\mathbb{R}^n$

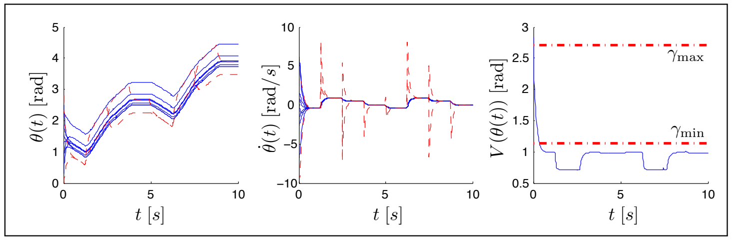
$$\frac{d}{dt}\dot{\theta}_i = -\sum_{j=1}^{n} a_{ij}(t)(\dot{\theta}_i - \dot{\theta}_j)\,,$$

where $a_{ij}(t) = \frac{K}{n}\cos(\theta_i(t) - \theta_j(t)) > 0$ for all $t \geq T$

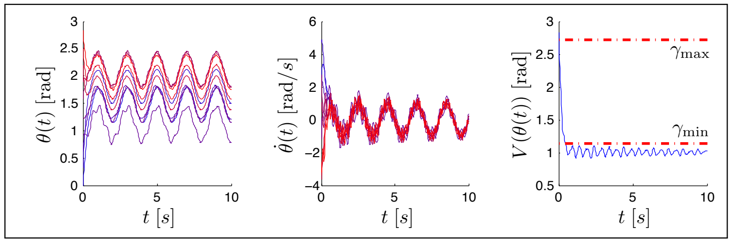
3. **Necessity**: all results exact for bipolar distribution $\omega_i \in \{\omega_{\min}, \omega_{\max}\}$

Synchronization in a Complete & Homogeneous Graph
robustness and extensions

1. **Switching natural frequencies:** dwell-time assumption ✓



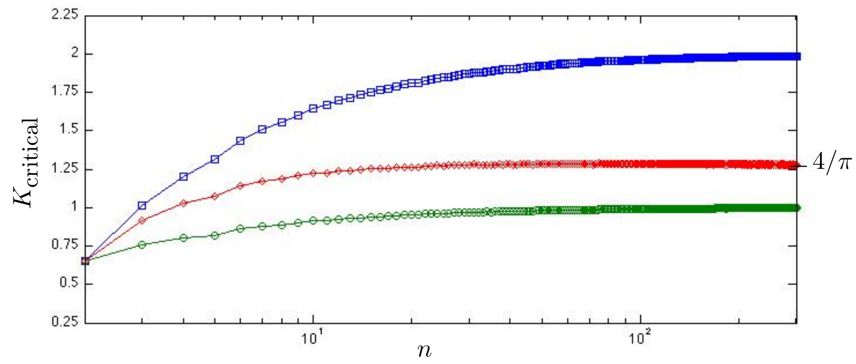
2. **Slowly time-varying:** $\|\ddot{\omega}(t) - \ddot{\omega}_{\text{avg}}(t)\|_\infty$ sufficiently small ✓



Synchronization in a Complete & Homogeneous Graph
scaling & statistical analysis

Kuramoto model with $\omega_i \in [-1, 1]$: $\dot{\theta}_i = \omega_i - \frac{K}{n}\sum_{j=1}^{n} \sin(\theta_i - \theta_j)$

Cont. limit predicts largest $K_{\text{critical}} = 2$ for *bipolar distribution* & smallest $K_{\text{critical}} = 4/\pi$ for *uniform distribution* [Y. Kuramoto '75, G.B. Ermentrout '85]



necessary bound (○), sufficient & tight bound (□), & exact & implicit bound (◊)

Outline

1. Introduction and motivation
2. Synchronization notions, metrics, & basic insights
3. Phase synchronization and more basic insights
4. Synchronization in complete networks
5. **Synchronization in sparse networks**
6. Open problems and research directions

Primer on Algebraic Graph Theory

Undirected graph $G = (\mathcal{V}, \mathcal{E}, A)$ with weight $a_{ij} > 0$ on edge $\{i, j\}$

- adjacency matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ (induces the graph)
- degree matrix $D \in \mathbb{R}^{n\times n}$ is diagonal with $d_{ii} = \sum_{j=1}^{n} a_{ij}$
- Laplacian matrix $L = D - A \in \mathbb{R}^{n\times n}$, $L = L^T \geq 0$

Notions of connectivity

- topological: connectivity, path lengths, degree, etc.
- spectral: 2nd smallest eigenvalue of L is "algebraic connectivity" $\lambda_2(L)$

Notions of heterogeneity

$\|\omega\|_{\mathcal{E},\infty} = \max_{\{i,j\}\in\mathcal{E}} |\omega_i - \omega_j|$, $\|\omega\|_{\mathcal{E},2} = \left(\sum_{\{i,j\}\in\mathcal{E}} |\omega_i - \omega_j|^2\right)^{1/2}$

Synchronization in Sparse Networks
a brief review I

$$\dot{\theta}_i = \omega_i - \sum\nolimits_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

Assume connectivity & $\omega_{\text{avg}} = \frac{1}{n}\sum_{i=1}^{n} \omega_i = 0$

1. **necessary condition**: $\sum_{j=1}^{n} a_{ij} \geq |\omega_i| \quad \Leftarrow \quad$ sync

   [C. Tavora and O.J.M. Smith '72]

   Proof idea: $\dot{\theta}_i = 0$ has no solution if condition is not true

2. **sufficient condition I**: $\lambda_2(L) > \lambda_{\text{critical}} \triangleq \|\omega\|_{\mathcal{E}_{\text{cmplt}},2} \quad \Rightarrow \quad$ sync

   [F. Dörfler and F. Bullo '09]

   Proof idea: analogous Lyapunov proof with $V(\theta) = \sum_{i<j} |\theta_i - \theta_j|^2$;
   condition also implies $\theta^* \in \text{Arc}_n(\lambda_{\text{critical}}/\lambda_2(L)) \Rightarrow$ evtl. too strong!

Synchronization in Sparse Networks
a brief review II

$$\dot{\theta}_i = \omega_i - \sum\nolimits_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

Assume connectivity & $\omega_{\text{avg}} = \frac{1}{n}\sum_{i=1}^{n} \omega_i = 0$

3. **sufficient condition II**: $\lambda_2(L) > \lambda_{\text{critical}} \triangleq \|\omega\|_{\mathcal{E},2} \quad \Rightarrow \quad$ sync

   [F. Dörfler and F. Bullo '11]

   Proof idea inspired by [A. Jadbabaie et al. '04]: fixed point theorem with incremental 2-norms; condition implies $\|\theta^*\|_{\mathcal{E},2} \leq \lambda_{\text{critical}}/\lambda_2(L)$

$\Rightarrow$ $\exists$ similar conditions with diff. metrics on coupling & heterogeneity

Synchronization in Sparse Networks
problems . . .

Problems: the sharpest general nec. & suff. conditions known to date

$$\sum\nolimits_{j=1}^{n} a_{ij} < |\omega_i| \quad , \quad \lambda_2(L) > \|\omega\|_{\mathcal{E}_{\text{cmplt}},2} \quad , \text{ and } \quad \lambda_2(L) > \|\omega\|_{\mathcal{E},2}$$

have a large gap and are conservative **!**

Why?

1. conservative bounding of trigs & network interactions
2. conditions $\theta^* \in \text{Arc}_n\left(\frac{\lambda_{\text{critical}}}{\lambda_2(L)}\right)$ or $\|\theta^*\|_{\mathcal{E},2} \leq \frac{\lambda_{\text{critical}}}{\lambda_2(L)}$ are too strong
3. analysis with 2-norm is conservative

Open problem: quantify "coupling/connectivity" vs. "heterogeneity"

[S. Strogatz '00 & '01, J. Acebrón et al. '00, A. Arenas et al. '08, S. Boccaletti et al. '06]

A Nearly Exact Synchronization Condition
a "back of the envelope calculation"

- Recall: if $\exists$ equilibrium $[\theta^*] \in \bar{\Delta}_G(\gamma)$, then it is unique and stable

$$\omega_i = \sum\nolimits_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \qquad (\star)$$

- Consider linear "small-angle" approximation of $(\star)$:

$$\omega_i = \sum\nolimits_{j=1}^{n} a_{ij}(\delta_i - \delta_j) \quad \Leftrightarrow \quad \omega = L\delta \qquad (\star\star)$$

Unique solution (modulo symmetry) of $(\star\star)$ is $\delta^* = L^\dagger \omega$

$\Rightarrow$ Solution ansatz for $(\star)$: $\boxed{\theta_i^* - \theta_j^* = \arcsin(\delta_i^* - \delta_j^*)}$ (<u>for a tree</u>)

$$\omega_i = \sum\nolimits_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) = \sum\nolimits_{j=1}^{n} a_{ij} \sin\left(\arcsin(\delta_i^* - \delta_j^*)\right) = \omega_i \quad \checkmark$$

$\Rightarrow$ **Theorem:** (<u>for a tree</u>) $\exists\, [\theta^*] \in \bar{\Delta}_G(\gamma) \Leftrightarrow \left\|L^\dagger \omega\right\|_{\mathcal{E},\infty} \leq \sin(\gamma)$

A Nearly Exact Synchronization Condition

Theorem [F. Dörfler, M. Chertkov, and F. Bullo '12]

Under one of following assumptions:

1) graph is either tree, homogeneous, or short cycle ($n \in \{3, 4\}$)
2) natural frequencies: $L^\dagger \omega$ is bipolar, small, or symmetric (for cycles)
3) arbitrary one-connected combinations of 1) and 2)

If $\|L^\dagger \omega\|_{\mathcal{E},\infty} \leq \sin(\gamma)$ where $\gamma < \pi/2$

$\Rightarrow$ $\exists$ a unique & locally exponentially stable equilibrium manifold in

$$\bar{\Delta}_G(\gamma) = \{\theta \in \mathbb{T}^n \mid \max_{\{i,j\} \in \mathcal{E}} |\theta_i - \theta_j| \leq \gamma\} .$$

A Nearly Exact Synchronization Condition
comments

- **Statistical correctness** through Monte Carlo simulations: construct nominal randomized graph topologies, weights, & natural frequencies

$\Rightarrow$ sync "for almost all graphs $G(\mathcal{V}, \mathcal{E}, A)$ & ω" with high accuracy

- Possibly thin sets of degenerate **counter-examples** for large cycles

- **Intuition:** the condition $\|L^\dagger \omega\|_{\mathcal{E},\infty} \leq \sin(\gamma)$ is equivalent to

$$\left\| [\text{eigenvectors of L}] \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & \frac{1}{\lambda_2(L)} & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \frac{1}{\lambda_n(L)} \end{bmatrix} [\text{eigenvectors of L}]^T \omega \right\|_{\mathcal{E},\infty} \leq \sin(\gamma)$$

$\Rightarrow$ includes previous conditions on $\lambda_2(L)$ and degree ($\approx \lambda_n(L)$)

A Nearly Exact Synchronization Condition
statistical analysis for power networks

Randomized power network test cases

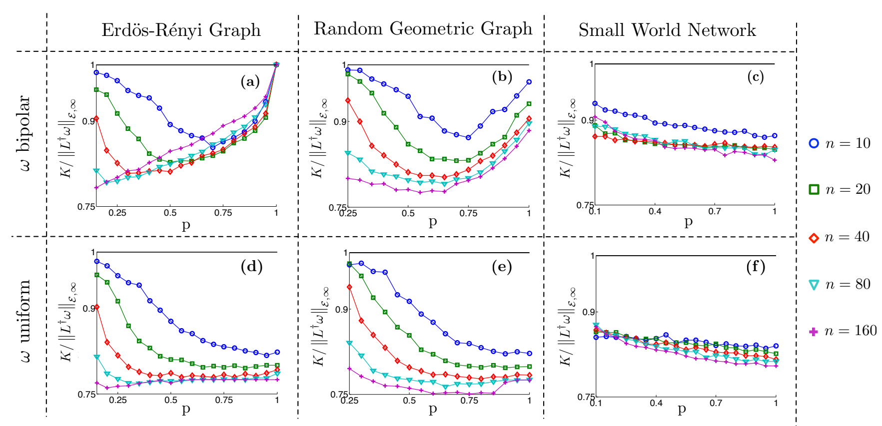
with 50 % randomized loads and 33 % randomized generation

Randomized test case (1000 instances)	Correctness of condition: $\|L^\dagger\omega\|_{\mathcal{E},\infty} \leq \sin(\gamma)$ $\Rightarrow \max_{\{i,j\}\in\mathcal{E}} \|\theta_i^* - \theta_j^*\| \leq \gamma$	Accuracy of condition: $\max_{\{i,j\}\in\mathcal{E}} \|\theta_i^* - \theta_j^*\| - \arcsin(\|L^\dagger\omega\|_{\mathcal{E},\infty})$	Phase cohesiveness: $\max_{\{i,j\}\in\mathcal{E}} \|\theta_i^* - \theta_j^*\|$
9 bus system	always true	$4.1218 \cdot 10^{-5}$ rad	0.12889 rad
IEEE 14 bus system	always true	$2.7995 \cdot 10^{-4}$ rad	0.16622 rad
IEEE RTS 24	always true	$1.7089 \cdot 10^{-3}$ rad	0.22309 rad
IEEE 30 bus system	always true	$2.6140 \cdot 10^{-4}$ rad	0.1643 rad
New England 39	always true	$6.6355 \cdot 10^{-5}$ rad	0.16821 rad
IEEE 57 bus system	always true	$2.0630 \cdot 10^{-2}$ rad	0.20295 rad
IEEE RTS 96	always true	$2.6076 \cdot 10^{-3}$ rad	0.24593 rad
IEEE 118 bus system	always true	$5.9959 \cdot 10^{-4}$ rad	0.23524 rad
IEEE 300 bus system	always true	$5.2618 \cdot 10^{-4}$ rad	0.43204 rad
Polish 2383 bus system (winter peak 1999/2000)	always true	$4.2183 \cdot 10^{-3}$ rad	0.25144 rad

$\Rightarrow$ condition $\|L^\dagger \omega\|_{\mathcal{E},\infty} \leq \sin(\gamma)$ is extremely accurate for $\gamma \leq 25°$

A Nearly Exact Synchronization Condition
statistical analysis for complex networks

Comparison with exact K_{critical} for $\dot{\theta}_i = \omega_i - K \cdot \sum_{j=1}^n a_{ij} \sin(\theta_i - \theta_j)$



$\Rightarrow$ condition $\|L^\dagger \omega\|_{\mathcal{E},\infty} \leq \sin(\gamma)$ is extremely accurate for $\gamma = \pi/2$

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Exciting Open Problems and Research Directions

1. **Q:** What about networks of **second-order oscillators**?

$$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = \omega_i - \sum\nolimits_{j=1}^{n} a_{ij}\sin(\theta_i - \theta_j)$$

Apps: mechanics, synchronous generators, Josephson junctions, ...

Problems: kinetic energy is a mixed blessing for transient dynamics

2. **Q:** What about **asymmetric interactions**?

e.g., directed graphs: $a_{ij} \neq a_{ji}$ or phase shifts: $a_{ij}\sin(\theta_i - \theta_j - \varphi_{ij})$

Apps: sync protocols, lossy circuits, phase/time-delays, flocking, ...

Problems: algebraic & geometric symmetries are broken

3. **Q:** How to derive **sharper results** for heterogeneous networks?

Exciting Open Problems and Research Directions

4. **Q:** What about the **transient dynamics** beyond $\text{Arc}_n(\pi)$, **general equilibria** beyond $\Delta_G(\pi/2)$, or the **basin of attraction**?

Apps: phase balancing, volatile power networks, flocking, ...

Problems: lack of analysis tools (only for simple cases), chaos, ...

5. **Q:** Beyond **continuous, sinusoidal, and diffusive coupling**?

$$\dot{\theta}_i \in \omega_i - \sum\nolimits_{\{i,j\}\in\mathcal{E}} f_{ij}(\theta_i, \theta_j)\,, \quad \theta \in \mathcal{C} \subset \mathbb{T}^n$$

$$\theta_i^+ \in \theta_i + \sum\nolimits_{\{i,j\}\in\mathcal{E}} g_{ij}(\theta_i, \theta_j)\,, \quad \theta \in \mathcal{D} \subset \mathbb{T}^n$$

Apps: impulsive coupling, relaxation oscillators, neuroscience, ...

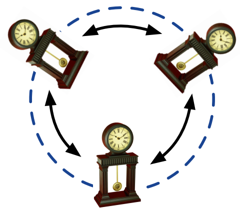
Problems: lack of analysis tools, coping with heterogeneity, ...

6. **Q:** Does anything extend from phase to **state space oscillators**?

Conclusions

- Coupled oscillator model:

$$\dot{\theta}_i = \omega_i - \sum\nolimits_{j=1}^{n} a_{ij}\sin(\theta_i - \theta_j)$$



- history: from Huygens' clocks to power grids
- applications in sciences, biology, & technology
- synchronization phenomenology
- network aspects & heterogeneity
- available analysis tools & results