Virtual Inertia Emulation and Placement in Power Grids

Séminaire d'Automatique du Plateau de Saclay

Laboratoire de Signaux et Systèmes du Supelec

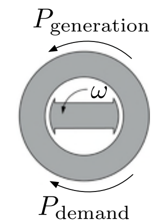
Florian Dörfler

At the beginning of power systems was . . .

At the beginning was the **synchronous machine**:

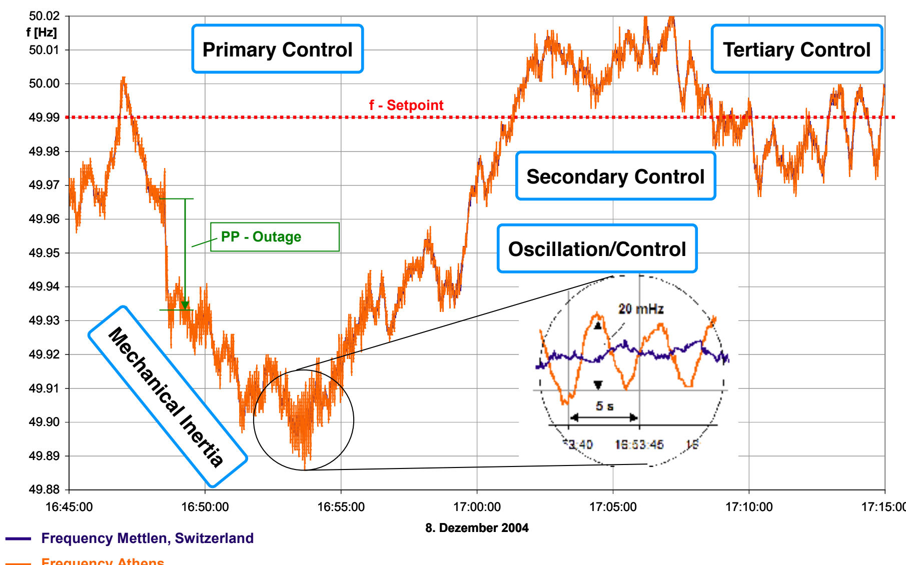
$$M \frac{d}{dt}\omega(t) \;=\; P_{\text{generation}}(t) - P_{\text{demand}}(t)$$

change of kinetic energy $=$ instantaneous power balance



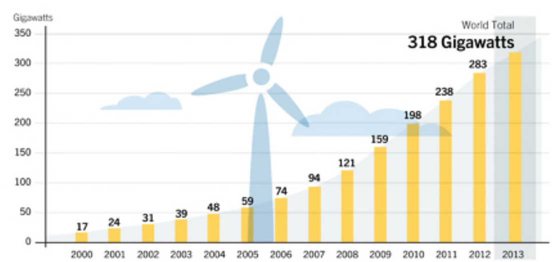
Fact: the AC grid & all of power system operation has been designed around synchronous machines.

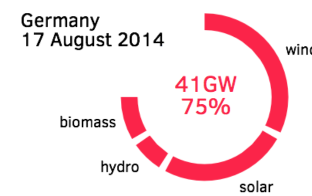
Operation centered around bulk synchronous generation

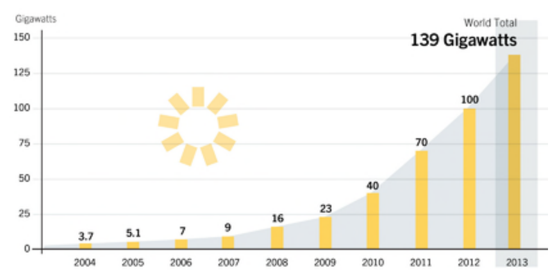


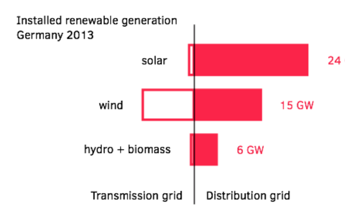
Source: W. Sattinger, Swissgrid

Distributed / non-rotational / renewable generation on the rise









Source: Renewables 2014 Global Status Report

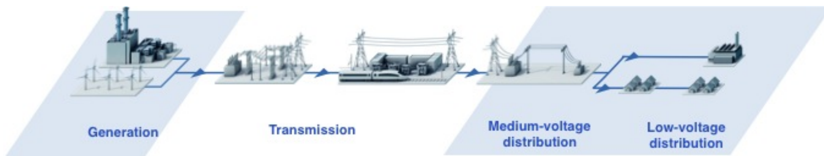
A few (of many) game changers . . .

synchronous generator

new workhorse

scaling

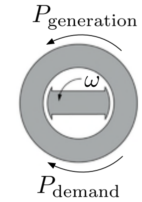
location & distributed implementation



Almost all operational problems can principally be resolved . . . **but one (?)**

Fundamental challenge: operation of low-inertia systems

We slowly loose our giant electromechanical low-pass filter:



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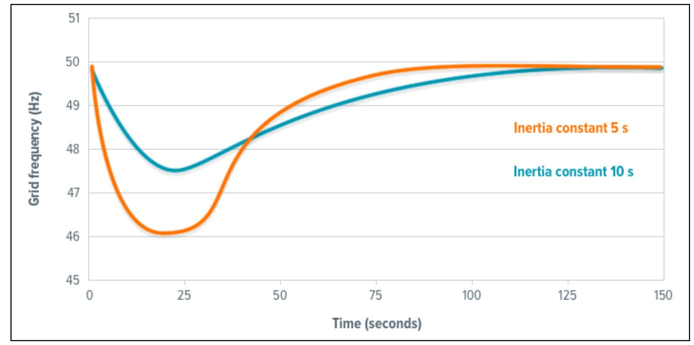
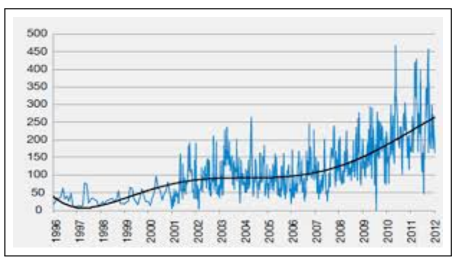
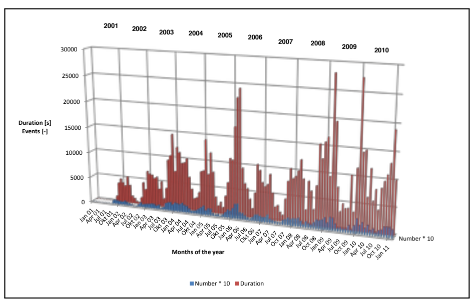


Low-inertia stability: # 1 problem of distributed generation



frequency violations in Nordic grid (source: ENTSO-E)



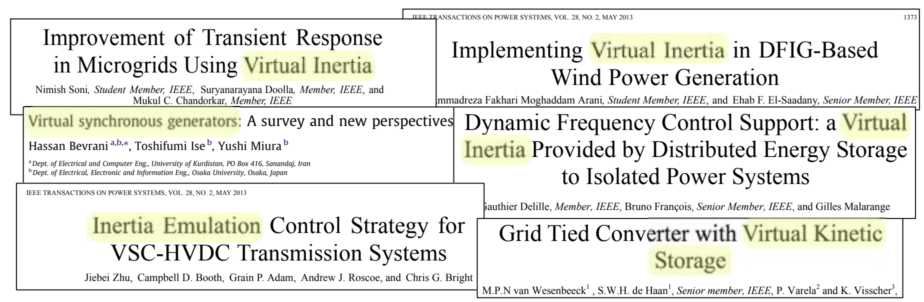
same in Switzerland (source: Swissgrid)

inertia is shrinking, time-varying, & localized, . . . & increasing disturbances

Solutions in sight: none really . . . other than **emulating virtual inertia** through fly-wheels, batteries, super caps, HVDC, demand-response, . . .

Virtual inertia emulation

devices commercially available, required by grid-codes or incentivized through markets



Improvement of Transient Response in Microgrids Using Virtual Inertia

Nimish Soni, Student Member, IEEE, Suryanarayana Doolla, Member, IEEE, and Mukul C. Chandorkar, Member, IEEE

IEEE TRANSACTIONS ON POWER SYSTEMS, VOL. 28, NO. 2, MAY 2013

Implementing Virtual Inertia in DFIG-Based Wind Power Generation

...mmadreza Fakhari Moghaddam Arani, Student Member, IEEE, and Ehab F. El-Saadany, Senior Member, IEEE

Virtual synchronous generators: A survey and new perspectives

Hassan Bevrani[a,b,*], Toshifumi Ise[b], Yushi Miura[b]

Dynamic Frequency Control Support: a Virtual Inertia Provided by Distributed Energy Storage to Isolated Power Systems

...authier Delille, Member, IEEE, Bruno François, Senior Member, IEEE, and Gilles Malarange

IEEE TRANSACTIONS ON POWER SYSTEMS, VOL. 28, NO. 2, MAY 2013

Inertia Emulation Control Strategy for VSC-HVDC Transmission Systems

Jiebei Zhu, Campbell D. Booth, Grain P. Adam, Andrew J. Roscoe, and Chris G. Bright

Grid Tied Converter with Virtual Kinetic Storage

M.P.N van Wesenbeeck[1], S.W.H. de Haan[1], Senior member, IEEE, P. Varela[2] and K. Visscher[3]

$$\mathbf{M}\,\frac{d}{dt}\,\omega(t) \;=\; P_{\text{generation}}(t) - P_{\text{demand}}(t) \qquad \text{. . . essentially } \textbf{D-control}$$

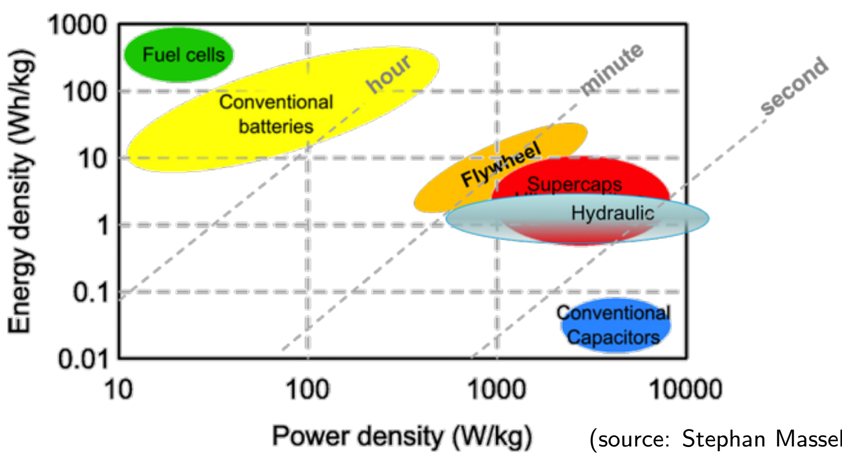
⇒ plug-&-play (decentralized & passive), grid-friendly, user-friendly, . . .

⇒ **today:** where to do it? how to do it properly?

Outline

inertia emulation

Classification & choice of actuators

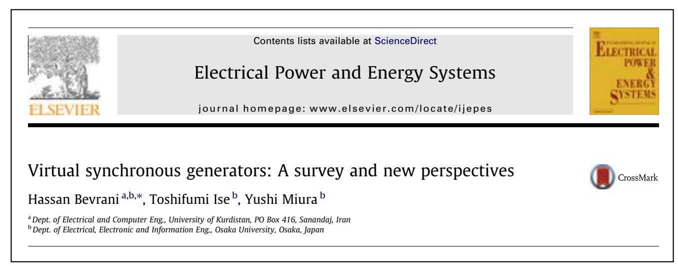


(source: Stephan Masselis)

each of these (& far more) have been proposed for virtual inertia emulation

Inertia emulation & virtual synchronous machines

1. naive **D-control** on $\omega(t)$: $\mathbf{M}\,\frac{d}{dt}\,\omega(t) \;=\; P_{\text{generation}}(t) - P_{\text{demand}}(t)$
2. more sophisticated **emulation of virtual synchronous machine**



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Virtual synchronous generators: A survey and new perspectives

Hassan Bevrani [a,b,*], Toshifumi Ise [b], Yushi Miura [b]

[a] Dept. of Electrical and Computer Eng., University of Kurdistan, PO Box 416, Sanandaj, Iran
[b] Dept. of Electrical, Electronic and Information Eng., Osaka University, Osaka, Japan

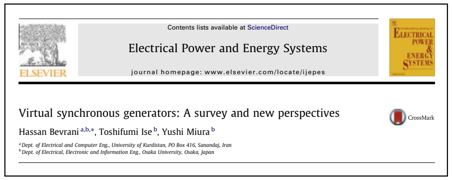
3. everything in between . . . and much more . . .

⇒ by measuring AC current/voltage/power/frequency

⇒ software model of virtual machine provides converter setpoints

⇒ actuation via modulation (switching) or DC injection (batteries etc.)

Challenges in real-world converter implementations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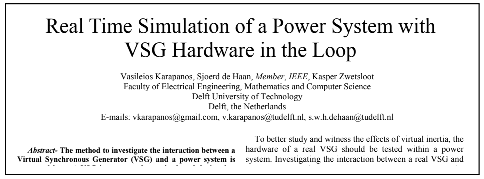

Real Time Simulation of a Power System with VSG Hardware in the Loop

Vasileios Karapanos, Sjoerd de Haan, Member, IEEE, Kasper Zwetsloot

Faculty of Electrical Engineering, Mathematics and Computer Science

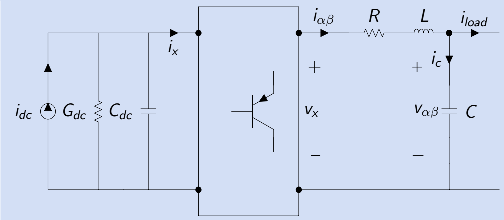
Delft University of Technology

Delft, the Netherlands

1. **delays** in measurement acquisition, signal processing, & actuation
2. **accuracy** in AC measurements (averaged over $\approx$ 5 cycles)
3. **constraints** on currents, voltages, power, etc.
4. **guarantees** on stability and robustness

today: use DC measurement, exploit analog storage, & passive control

Averaged inverter model



DC cap & AC filter equations:

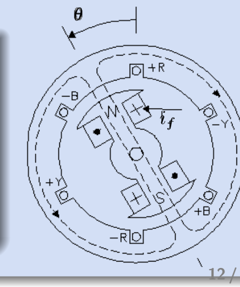
$$C_{dc}\dot{v}_{dc} = -G_{dc}v_{dc} + i_{dc} - \frac{1}{2}m^\top i_{\alpha\beta}$$
$$C\dot{v}_{\alpha\beta} = -i_{load} + i_{\alpha\beta}$$
$$L\dot{i}_{\alpha\beta} = -Ri_{\alpha\beta} + \frac{1}{2}mv_{dc} - v_{\alpha\beta}$$

modulation: $i_x = \frac{1}{2}m^\top i_{\alpha\beta}\,,\; v_x = \frac{1}{2}mv_{dc}$

passive: $(i_{dc}, i_{load}) \to (v_{dc}, v_{\alpha\beta})$

model of a synchronous generator

$$\dot{\theta} = \omega$$
$$M\dot{\omega} = -D\omega + \tau_m + i_{\alpha\beta}^\top L_m i_f \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}$$
$$C\dot{v}_{\alpha\beta} = -G_{load}v_{\alpha\beta} + i_{\alpha\beta}$$
$$L_s\dot{i}_{\alpha\beta} = -Ri_{\alpha\beta} - v_{\alpha\beta} - \omega L_m i_f \begin{bmatrix}-\sin(\theta)\\ \cos(\theta)\end{bmatrix}$$

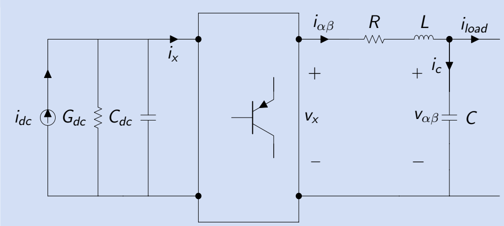


standard power electronics control would continue by

1. constructing voltage/current/power **references** (e..g, droop, synchronous machine emulation, etc.)
2. **tracking** these references at the converter terminals typically by means of cascaded PI controllers

let's do **something different** (smarter?) today ...

See the similarities & the differences ?



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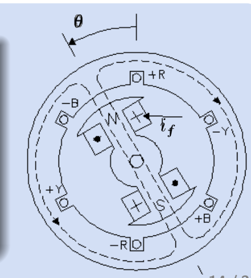
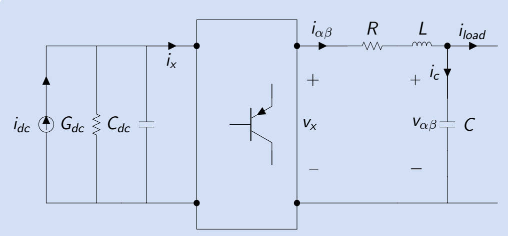


Model matching ($\neq$ emulation) as inner control loop



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matching control: $\dot{\theta} = \eta \cdot v_{dc}\ ,\ m = \mu \cdot \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$ with $\eta, \mu > 0$

$\Rightarrow$ **pros:** is balanced, uses natural storage, & based on DC measurement

$\Rightarrow$ **virtual machine** with $M = \frac{C_{dc}}{\eta^2}$, $D = \frac{G_{dc}}{\eta^2}$, $\tau_m = \frac{i_{dc}}{\eta}$, $i_f = \frac{\mu}{\eta L_m}$

$\Rightarrow$ base for **outer controls** via i_{dc} & μ, e.g., virtual torque, PSS, & inertia

Some properties & different viewpoints

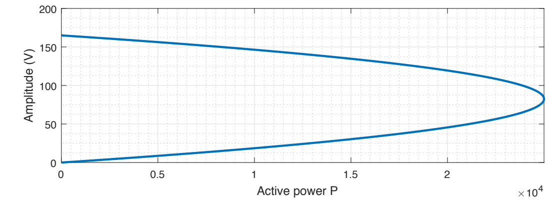
1. **quadratic curves** for stationary P vs. $(|V|, \omega)$

$\Rightarrow$ $P \leq P_{max} = i_{dc}^2/4G_{dc}$

$\Rightarrow$ reactive power not directly affected

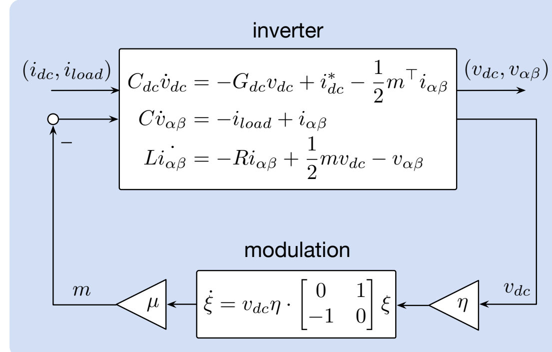
$\Rightarrow$ (P, ω)-droop $\approx 1/\eta$

$\Rightarrow$ $(P, |V|)$-droop $\approx 1/\mu$

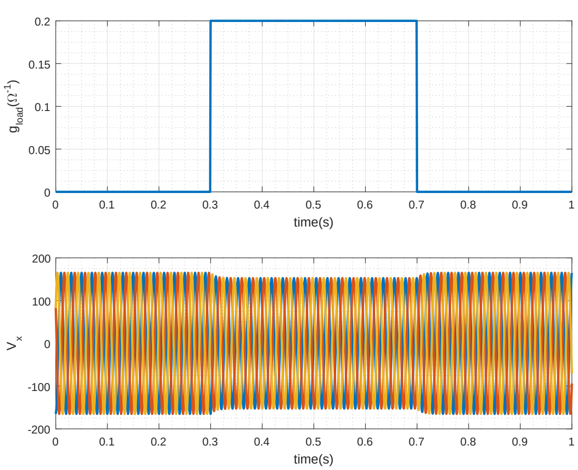


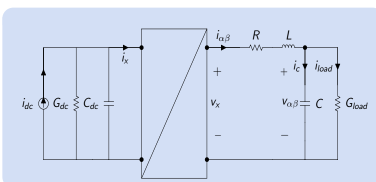
2. reformulation as virtual & adaptive **oscillator**

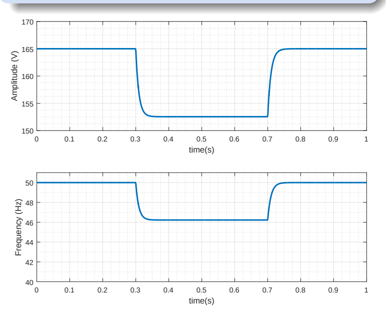
3. remains **passive**: $(i_{dc}, i_{load}) \to (v_{dc}, v_{\alpha\beta})$



Eye candy: response to a load step







optimal placement of virtual inertia

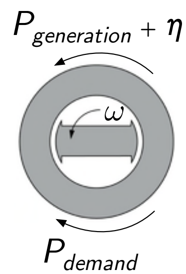
Linearized & Kron-reduced swing equation model

$m_i \ddot{\theta}_i + d_i \dot{\theta}_i = p_{in,i} - p_{e,i}$

generator swing equations

$p_{e,i} \approx \sum_{j \in \mathcal{N}} b_{ij}(\theta_i - \theta_j)$

linearized power flows



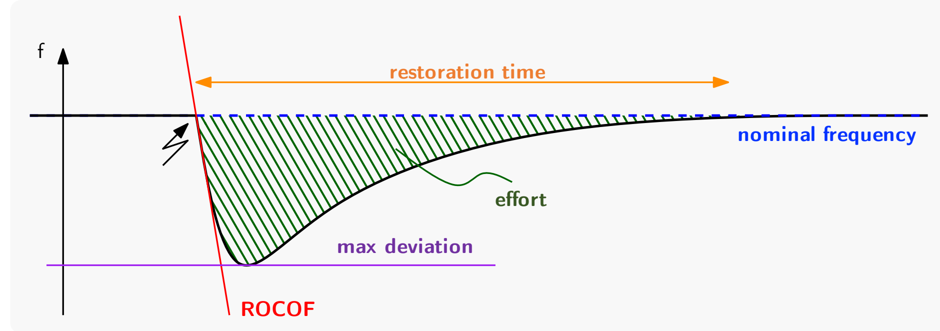
likelihood of **disturbance** at $\#i$: $t_i \geq 0$

state space representation:

$$\begin{bmatrix} \dot{\theta} \\ \dot{\omega} \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & I \\ -M^{-1}L & -M^{-1}D \end{bmatrix}}_{A} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} T^{1/2}}_{B} \eta$$

where $M = \mathrm{diag}(m_i)$, $D = \mathrm{diag}(d_i)$, $T = \mathrm{diag}(t_i)$, & $L = L^{\mathsf{T}}$ (Laplacian)

Performance metric for emulation of rotational inertia



System norm:
amplification of
disturbances: impulse (fault), step (loss of unit), white noise (renewables)
to
performance outputs: integral, peak, ROCOF, restoration time, ...

Coherency performance metric & $\mathcal{H}_2$ norm

Energy expended by the system to return to synchronous operation:

$$\int_0^\infty \sum_{\{i,j\} \in \mathcal{E}} a_{ij}(\theta_i(t) - \theta_j(t))^2 + \sum_{i=1}^{n} s_i\, \omega_i^2(t)\, dt$$

$\mathcal{H}_2$ norm interpretation:

1. associated **performance output**: $y = \begin{bmatrix} Q_1^{1/2} & 0 \\ 0 & Q_2^{1/2} \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix}$
2. **impulses** (faults) $\longrightarrow$ output energy $\int_0^\infty y(t)^{\mathsf{T}} y(t)\, dt$
3. **white noise** (renewables) $\longrightarrow$ output variance $\lim_{t \to \infty} \mathbb{E}\left(y(t)^{\mathsf{T}} y(t)\right)$

Algebraic characterization of the $\mathcal{H}_2$ norm

Lemma: via observability Gramian

$$\|G\|_2^2 = \mathrm{Trace}(B^{\mathsf{T}} P B)$$

where P is the observability Gramian $P = \int_0^\infty e^{A^{\mathsf{T}} t} C^{\mathsf{T}} C e^{At}\, dt$

- P solves a Lyapunov equation: $P A + A^{\mathsf{T}} P + Q = 0$
- A has a zero eigenvalue $\rightarrow$ restricts choice of Q

$$y = \begin{bmatrix} Q_1^{1/2} & 0 \\ 0 & Q_2^{1/2} \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} \qquad Q_1^{1/2} \mathbb{1} = \mathbb{0}$$

- P is unique for $P\,[\mathbb{1}\ \mathbb{0}] = [\mathbb{0}\ \mathbb{0}]$

Problem formulation

$$\begin{aligned}
&\underset{P,\, m_i}{\text{minimize}} && \|G\|_2^2 = \mathrm{Trace}(B^\top P B) && \rightarrow \text{performance metric}\\
&\text{subject to} && \textstyle\sum_{i=1}^{n} m_i \le m_{\mathrm{bdg}} && \rightarrow \text{budget constraint}\\
& && \underline{m_i} \le m_i \le \overline{m_i}\,, \quad i \in \{1,\dots,n\} && \rightarrow \text{capacity constraint}\\
& && P A + A^\top P + Q = 0 && \rightarrow \text{observability Gramian}\\
& && P\,[\mathbb{1}\ \mathbb{0}] = [\mathbb{0}\ \mathbb{0}] && \rightarrow \text{uniqueness}
\end{aligned}$$

Insights

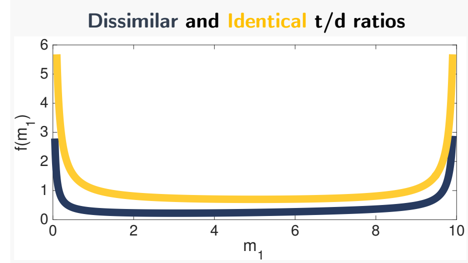
1. m appears as m^{-1} in system matrices A, B
2. product of $B(m)$ & P in the objective
3. product of $A(m)$ & P in the constraint

$\Rightarrow$ **large-scale & non-convex**

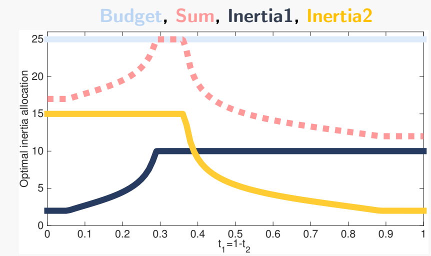
Building the intuition: results for two-area networks

Fundamental learnings

1. explicit closed-form solution is rational function
2. sufficiently uniform $(t/d)_i$ $\rightarrow$ strongly **convex** & fairly **flat** cost
3. non trivial in the presence of capacity constraints



performance metric



optimal inertia allocation

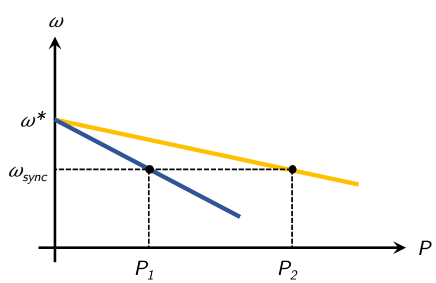
Closed-form results for cost of primary control

P/$\dot\theta$ primary droop control

$$(\omega_i - \omega^*) \propto (P_i^* - P_i(\theta))$$

$$\Updownarrow$$

$$D_i \dot\theta_i = P_i^* - P_i(\theta)$$



Primary control effort $\rightarrow$ accounted for by integral quadratic cost

$$\int_0^\infty \dot\theta(t)^\top D\,\dot\theta(t)\,dt$$

which is the $\mathcal{H}_2$ performance for the penalties $Q_1^{1/2} = 0$ and $Q_2^{1/2} = D$

Primary Control . . . cont'd

Theorem: the primary control effort optimization reads equivalently as

$$\begin{aligned}
&\underset{m_i}{\text{minimize}} && \textstyle\sum_{i=1}^{n} \frac{t_i}{m_i}\\
&\text{subject to} && \textstyle\sum_{i=1}^{n} m_i \le m_{\mathrm{bdg}}\\
& && \underline{m_i} \le m_i \le \overline{m_i}, \quad i \in \{1,\dots,n\}
\end{aligned}$$

Key take-aways:

- optimal solution independent of network topology
- allocation $\propto \sqrt{t_i}$ or $m_i = \min\{m_{\mathrm{bdg}}, \overline{m_i}\}$

Location & **strength** of disturbance are **crucial** solution ingredients

numerical method for the general case

Taylor & power series expansions

Key idea: expand the performance metric as a power series in m

$$\|G\|_2^2 = \text{Trace}(B(m)^\mathsf{T}\mathbf{P}(m)B(m))$$

Motivation: scalar series expansion at m_i in direction μ_i:

$$\frac{1}{(m_i + \delta\mu_i)} = \frac{1}{m_i} - \frac{\delta\mu_i}{m_i^2} + \mathcal{O}(\delta^2)$$

Expand system matrices as **Taylor series** in direction μ:

$$\mathbf{A}(m + \delta\mu) = \mathbf{A}^{(0)}_{(m,\mu)} + \mathbf{A}^{(1)}_{(m,\mu)}\delta + \mathcal{O}(\delta^2)$$

$$\mathbf{B}(m + \delta\mu) = \mathbf{B}^{(0)}_{(m,\mu)} + \mathbf{B}^{(1)}_{(m,\mu)}\delta + \mathcal{O}(\delta^2)$$

Expand the observability Gramian as a **power series** in direction μ:

$$\mathbf{P}(m + \delta\mu) = \mathbf{P}^{(0)}_{(m,\mu)} + \mathbf{P}^{(1)}_{(m,\mu)}\delta + \mathcal{O}(\delta^2)$$

Explicit gradient computation

Expansion of system matrices & Gramian ⇒ **match coefficients** . . .

Formula for gradient at m in direction μ

1. nominal Lyapunov equation for $\mathcal{O}(\delta^0)$:

$$\mathbf{P}^{(0)} = \mathbf{Lyap}(\mathbf{A}^{(0)}, \mathbf{Q})$$

2. perturbed Lyapunov equation for $\mathcal{O}(\delta^1)$ terms:

$$\mathbf{P}^{(1)} = \mathbf{Lyap}(\mathbf{A}^{(0)}, \mathbf{P}^{(0)}\mathbf{A}^{(1)} + \mathbf{A}^{(1)^\mathsf{T}}\mathbf{P}^{(0)})$$

3. expand objective in direction μ:

$$\|G\|_2^2 = \text{Trace}(B(m)^\mathsf{T}\mathbf{P}(m)B(m)) = \text{Trace}((\dots) + \delta(\dots)) + \mathcal{O}(\delta^2)$$

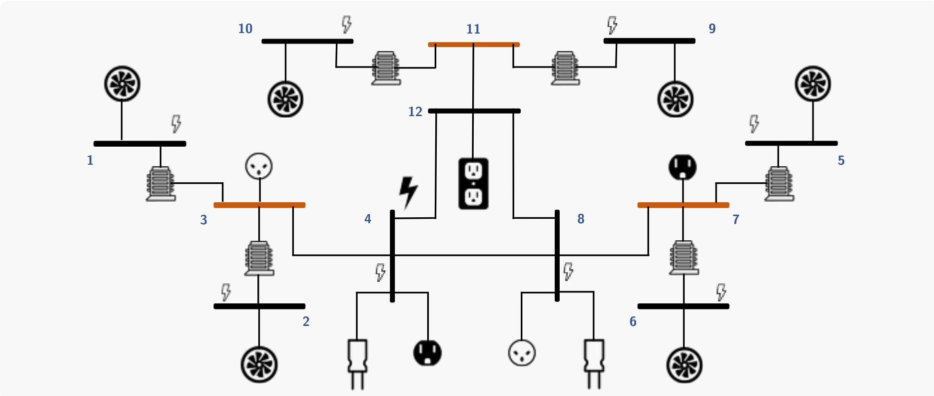
4. gradient: $\text{Trace}(2 * B^{(1)^\mathsf{T}} P^{(0)} B^{(0)} + B^{(0)^\mathsf{T}} P^{(1)} B^{(0)})$

⇒ use favorite method for reduced optimization problem

results

Modified Kundur case study: 3 regions & 12 buses

transformer reactance 0.15 p.u., line impedance (0.0001+0.001i) p.u./km

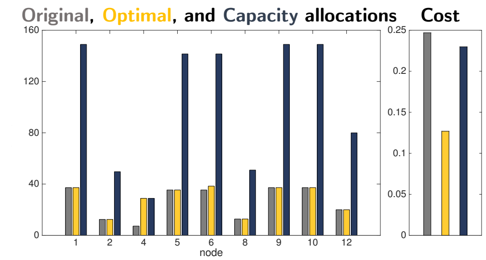


Heuristics outperformed by $\mathcal{H}_2$ - optimal allocation

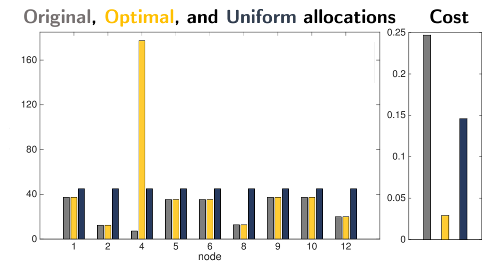
Scenario: disturbance at #4

- locally optimal solution **outperforms heuristic** max/uniform allocation
- optimal allocation ≈ **matches disturbance**
- inertia emulation at all undisturbed nodes is actually **detrimental**

⇒ **location** of disturbance & inertia emulation matters

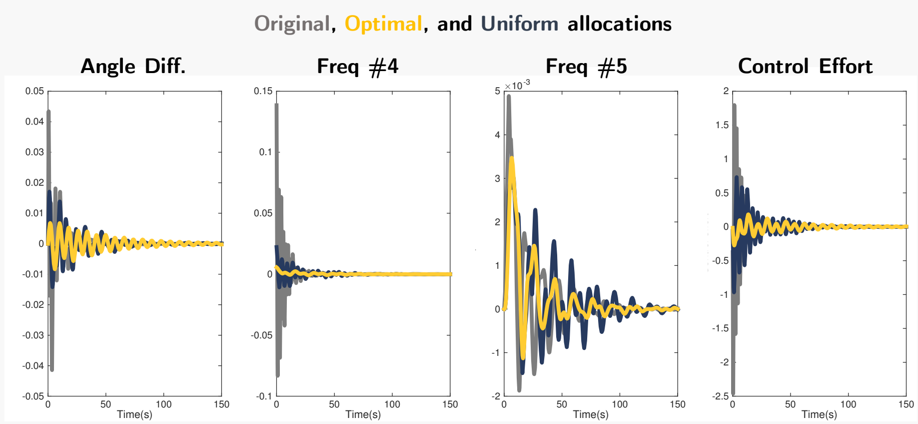


allocation subject to capacity constraints



allocation subject to the budget constraint

Eye candy: time-domain plots of post fault behavior



Take-home messages:

best oscillation performance	smallest peak frequency at #4	undisturbed sites are irrelevant	minimal control effort $m_i \cdot \ddot{\theta}_i$

conclusions

Conclusions on virtual inertia emulation

Where to do it?

1. $\mathcal{H}_2$-optimal (non-convex) allocation
2. closed-form results for cost of primary control
3. numerical approach via gradient computation

How to do it?

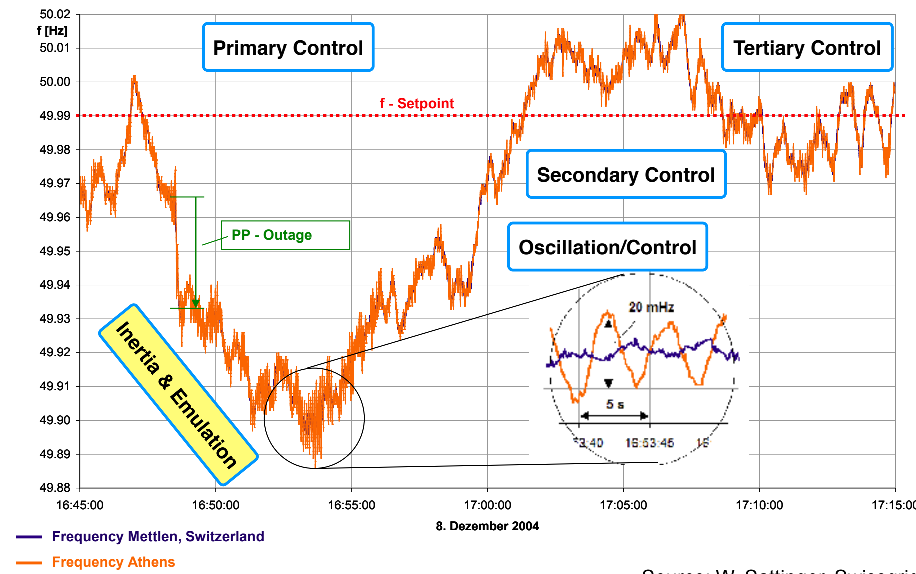
1. down-sides of naive inertia emulation
2. novel machine matching control

What else to do? Inertia emulation is . . .

☺ decentralized, plug-and-play (passive), grid-friendly, user-friendly, . . .

☹ suboptimal, wasteful in control effort, & need for new actuators

Recall: operation centered around (virtual) sync generators



Source: W. Sattinger, Swissgrid

A control perspective of power system operation

Conventional strategy: **emulate generator physics & control**

$$\underbrace{M\dot{\omega}(t)}_{\text{(virtual) inertia}} = \underbrace{P_{\text{mech}}}_{\text{tertiary control}} - \underbrace{D\omega(t)}_{\text{primary control}} - \underbrace{\int_0^t \omega(\tau)\, d\tau}_{\text{secondary control}} - P_{\text{elec}}$$

Essentially all **PID + setpoint control** (simple, robust, & scalable)

$$\underbrace{M\dot{\omega}(t)}_{\text{D}} = \underbrace{P}_{\text{set-point}} - \underbrace{D\omega(t)}_{\text{P}} - \underbrace{\int_0^t \omega(\tau)\, d\tau}_{\text{I}} - P_{\text{elec}}$$

Control engineers should be able to do better . . .

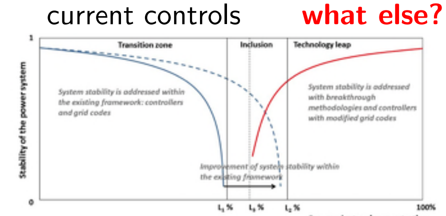
This *" what else? "* has been broadly recognized

by TSOs, device manufacturers, academia, etc.

Massive **Inte**GRAT**ion of power **E**lectronic devices

"The question that has to be examined is: how much power electronics can the grid cope with?" (European Commission)

current controls **what else?**



all options are on the table and keep us busy . . .

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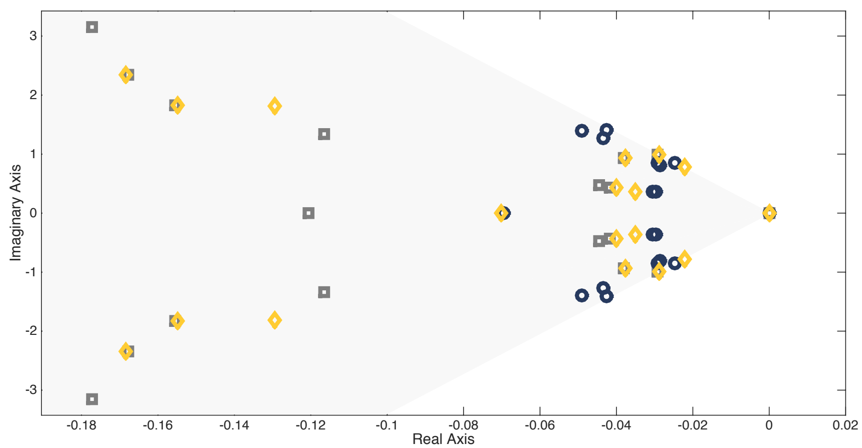
appendix

Spectral perspective on different inertia allocations

Cone, Original, Optimal, and Uniform allocations



- $\mathbf{m} = \underline{\mathbf{m}}$ → best damping asymptote & best damping ratio
- Spectrum holds only **partial information !!**

The planning problem

sparse allocation of limited resources

ℓ_1-regularized inertia allocation (promoting a sparse solution):

$$\begin{aligned} \underset{P,\, m_i}{\text{minimize}} \quad & \mathbf{J}_\gamma(\mathbf{m}, \mathbf{P}) = \|G\|_2^2 + \gamma \|\mathbf{m} - \underline{\mathbf{m}}\|_1 \\ \text{subject to} \quad & \textstyle\sum_{i=1}^{n} m_i \leq m_{\text{bdg}} \\ & \underline{m_i} \leq m_i \leq \overline{m_i} \quad i \in \{1, \dots, n\} \\ & P\,A + A^{\mathsf{T}} P + Q = 0 \\ & P[\mathbb{1}\ \mathbb{0}] = [\mathbb{0}\ \mathbb{0}] \end{aligned}$$

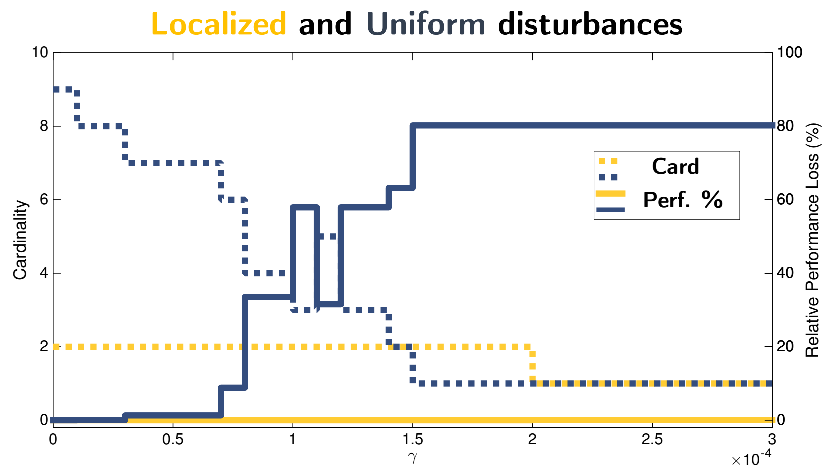
where $\gamma \geq 0$ trades off sparsity penalty and the original objective

Highlights:

1. regularization term is linear & **differentiable**
2. gradient computation algorithm can be used with some tweaking

Relative performance loss (%) as a function of γ

0% $\to$ optimal allocation, **100%** $\to$ no additional allocation



1. uniform disturbance $\Rightarrow \exists\,\gamma$ **1.3%** loss $\equiv$ **(9 $\to$ 7)**
2. localized disturbance $\Rightarrow$ **(2 $\to$ 1)** without affecting performance

Uniform disturbance to damping ratio

power sharing $\to$ $\mathbf{d} \propto P^*$, assuming $\mathbf{t} \propto$ source rating P^*

Theorem: for $t_i/d_i = t_j/d_j$ the allocation problem reads equivalently as

$$\begin{aligned} \underset{m_i}{\text{minimize}} \quad & \sum\nolimits_{i=1}^{n} \frac{s_i}{m_i} \\ \text{subject to} \quad & \sum\nolimits_{i=1}^{n} m_i \leq m_{\text{bdg}} \\ & \underline{m_i} \leq m_i \leq \overline{m_i}, \quad i \in \{1, \ldots, n\} \end{aligned}$$

Key takeaways:

- optimal solution independent of network topology
- allocation $\propto \sqrt{s_i}$ or $m_i = \min\{m_{\text{bdg}}, \overline{m_i}\}$

What if **freq. penalty $\propto$ inertia**? $\to$ norm **independent** of inertia

Taylor & power series expansions

Key idea: expand the performance metric as a power series in m

$$\|G\|_2^2 = \text{Trace}(B(m)^{\mathsf{T}} \mathbf{P}(\mathbf{m}) B(m))$$

Motivation: scalar series expansion at m_i in direction μ_i:

$$\frac{1}{(m_i + \delta \mu_i)} = \frac{1}{m_i} - \frac{\delta \mu_i}{m_i^2} + \mathcal{O}(\delta^2)$$

Expand system matrices in direction μ, where $\Phi = \text{diag}(\mu)$:

$$\mathbf{A}^{(0)}_{(\mathbf{m},\boldsymbol{\mu})} = \begin{bmatrix} 0 & I \\ -M^{-1}L & -M^{-1}D \end{bmatrix}, \quad \mathbf{A}^{(1)}_{(\mathbf{m},\boldsymbol{\mu})} = \begin{bmatrix} 0 & 0 \\ \Phi M^{-2} L & \Phi M^{-2} D \end{bmatrix}$$

$$\mathbf{B}^{(0)}_{(\mathbf{m},\boldsymbol{\mu})} = \begin{bmatrix} 0 \\ M^{-1} T^{1/2} \end{bmatrix}, \quad \mathbf{B}^{(1)}_{(\mathbf{m},\boldsymbol{\mu})} = \begin{bmatrix} 0 \\ -\Phi M^{-2} T^{1/2} \end{bmatrix}$$

Taylor & power series expansions cont'd

Expand the observability Gramian as a power series in direction μ

$$\mathbf{P}(m) = \mathbf{P}(m + \delta\mu) = \mathbf{P}^{(0)}_{(\mathbf{m},\boldsymbol{\mu})} + \mathbf{P}^{(1)}_{(\mathbf{m},\boldsymbol{\mu})} \delta + \mathcal{O}(\delta^2)$$

Formula for gradient in direction μ

1. nominal Lyapunov equation for $\mathcal{O}(\delta^0)$: $\mathbf{P}^{(0)} = \mathbf{Lyap}(\mathbf{A}^{(0)}, \mathbf{Q})$
2. perturbed Lyapunov equation for $\mathcal{O}(\delta^1)$ terms:
$$\mathbf{P}^{(1)} = \mathbf{Lyap}(\mathbf{A}^{(0)}, \mathbf{P}^{(0)}\mathbf{A}^{(1)} + \mathbf{A}^{(1)\mathsf{T}}\mathbf{P}^{(0)})$$
3. expand objective in direction μ:
$$\|G\|_2^2 = \text{Trace}(B(m)^{\mathsf{T}} \mathbf{P}(m) B(m)) = \text{Trace}((\ldots) + \delta(\ldots)) + \mathcal{O}(\delta^2)$$
4. gradient: $\mathbf{Trace}(2 * \mathbf{B}^{(1)\mathsf{T}} \mathbf{P}^{(0)} \mathbf{B}^{(0)} + \mathbf{B}^{(0)\mathsf{T}} \mathbf{P}^{(1)} \mathbf{B}^{(0)})$

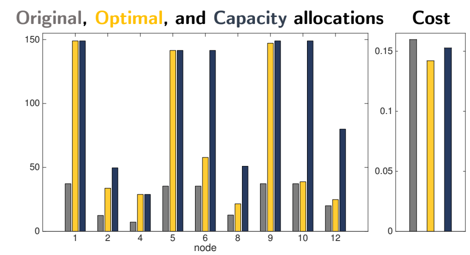
Gradient computation

Algorithm: Gradient computation & perturbation analysis

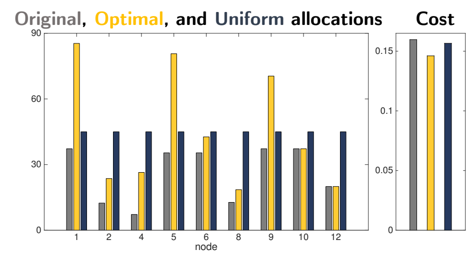
Input $\rightarrow$ current values of the decision variables $\mathbf{m_i}$
Output $\rightarrow$ numerically evaluated gradient ∇f of the cost function

1. Evaluate the system matrices $\mathbf{A}^{(0)}, \mathbf{B}^{(0)}$ based on current inertia
2. Solve for $\mathbf{P}^{(0)} = \mathbf{Lyap}(\mathbf{A}^{(0)}, \mathbf{Q})$ using a Lyapunov routine
3. For each node- obtain the perturbed system matrices $\mathbf{A}^{(1)}, \mathbf{B}^{(1)}$
4. Compute $\mathbf{P}^{(1)} = \mathbf{Lyap}(\mathbf{A}^{(0)}, \mathbf{P}^{(0)}\mathbf{A}^{(1)} + {\mathbf{A}^{(1)}}^{T}\mathbf{P}^{(0)})$
5. Gradient $\Rightarrow$ $\mathbf{Trace}(2 * {\mathbf{B}^{(1)}}^{T}\mathbf{P}^{(0)}\mathbf{B}^{(0)} + {\mathbf{B}^{(0)}}^{T}\mathbf{P}^{(1)}\mathbf{B}^{(0)})$

Heuristics outperformed also for uniform disturbance



allocation subject to capacity constraints



allocation subject to the budget constraint

Scenario: uniform disturbance

Heuristics for placement:

1. **max** allocation in case of capacity constraints
2. **uniform** allocation in case of budget constraint

Results & insights:

1. locally optimal solution **outperforms** heuristics
2. optimal solution $\neq$ **max** inertia at each bus