Folk theorems, myths, & conjectures in complex oscillator networks

NetSci 2015 Satellite Symposium

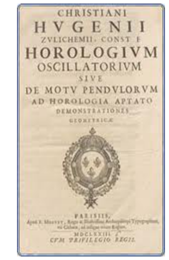
Florian Dörfler

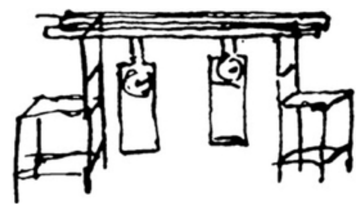
A brief history of sync

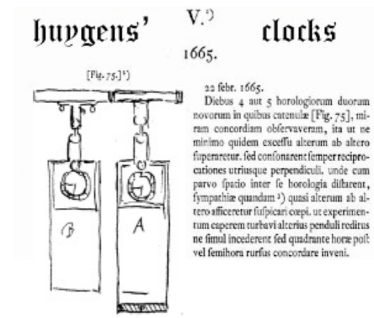
Christiaan Huygens (1629 – 1695)

- physicist & mathematician
- engineer & horologist

observed *"an odd kind of sympathy"*
[Letter to Royal Society of London, 1665]



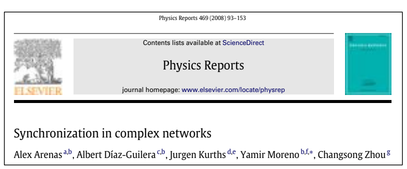


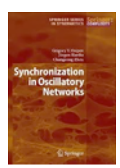


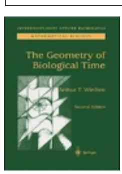
Recent reviews, experiments, & analysis
[M. Bennet et al. '02, M. Kapitaniak et al. '12]

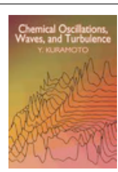
A field was born

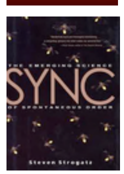
- sync in mathematical biology [A. Winfree '80, S.H. Strogatz '03, ...]
- sync in physics and chemistry [Y. Kuramoto '83, M. Mézard et al. '87...]
- sync in neural networks [F.C. Hoppensteadt and E.M. Izhikevich '00, ...]
- sync in complex networks [C.W. Wu '07, S. Bocaletti '08, ...]
- ...and numerous technological applications (reviewed later)

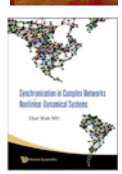






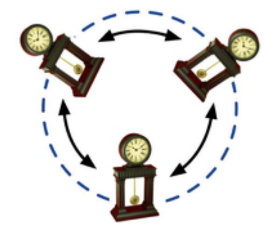






Coupled phase oscillators

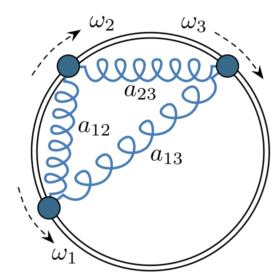
$\exists$ various models of oscillators & interactions



Today: **coupled phase oscillator model**
[A. Winfree '67, Y. Kuramoto '75]

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

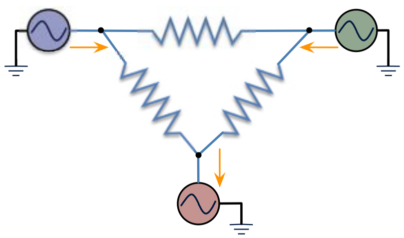
- n **oscillators** with phase $\theta_i \in \mathbb{S}^1$
- **non-identical** natural frequencies $\omega_i \in \mathbb{R}^1$
- elastic **coupling** with strength $a_{ij} = a_{ji}$
- undirected & connected **graph** $G = (\mathcal{V}, \mathcal{E}, A)$

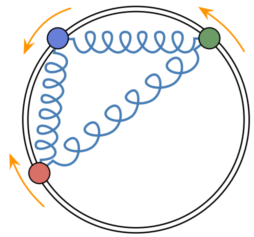


Note: can be derived as **canonical** coupled limit-cycle oscillator model

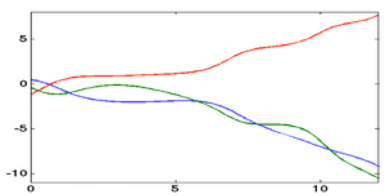
My application of interest: sync in AC power networks

- sync is **crucial for AC power grids** – a coupled oscillator analogy

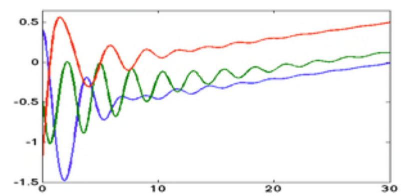




- sync is a **trade-off**



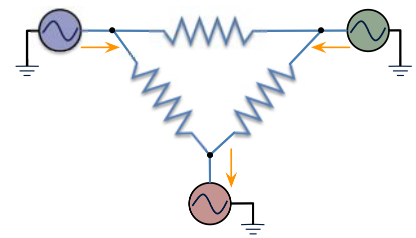
weak coupling & heterogeneous

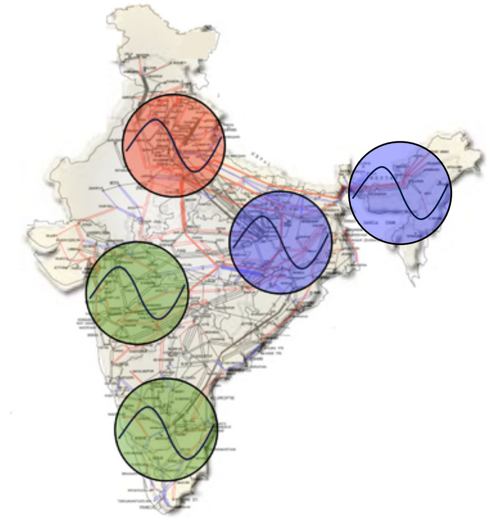


strong coupling & homogeneous

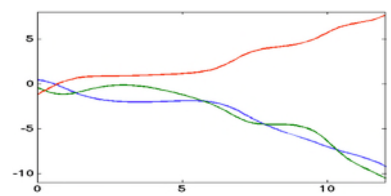
My application of interest: sync in AC power networks

- sync is **crucial for AC power grids**





- sync is a **trade-off**

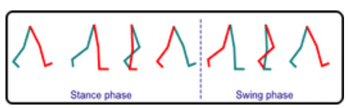


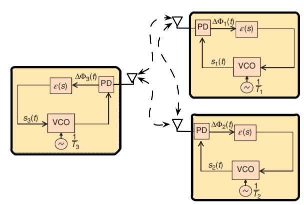
weak coupling & heterogeneous

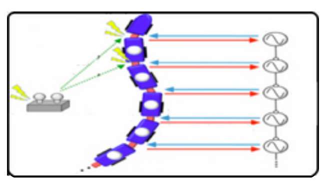
Blackout India July 30/31 2012

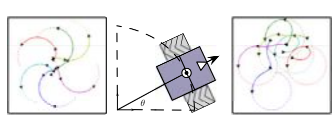
Other technological applications of phase oscillators

- particle filtering to estimate limit cycles [A. Tilton & P. Mehta et al. '12]
- clock synchronization over networks [Y. Hong & A. Scaglione '05, O. Simeone et al. '08, Y. Wang & F. Doyle et al. '12]
- central pattern generators and robotic locomotion [J. Nakanishi et al. '04, S. Aoi et al. '05, L. Righetti et al. '06]
- decentralized maximum likelihood estimation [S. Barbarossa et al. '07]
- carrier sync without phase-locked loops [M. Rahman et al. '11]
- robotic vehicle coordination [R. Sepulchre et al. '07, D. Klein et al. '09]





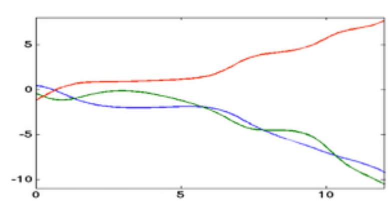


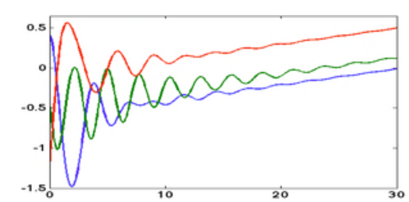


Phenomenology and challenges in synchronization

many fundamental questions are still open

Transition to synchronization is a **trade-off:** coupling vs. heterogeneity





Some **central questions:**

(still after 45 years of work)

- quantify "coupling" vs. "heterogeneity"
- multiple sync'd states & their sync basin
- interplay of network & dynamics

In more technical terms: existence, uniqueness, & stability of equilibria and their basin of attraction . . . as a function of network topology & parameters

Outline

Introduction

Synchronization Threshold

Equilibrium Landscape

Almost Global Synchronization

Conclusions

I try to shed light on some fundamental yet poorly understood questions.

Main references today

Automatica 50 (2014) 1539–1564

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Survey paper

Synchronization in complex networks of phase oscillators: A survey

Florian Dörfler[a,1], Francesco Bullo[b]

[a] Automatic Control Laboratory, ETH Zürich, Switzerland

[b] Department of Mechanical Engineering, University of California Santa Barbara, USA

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Algebraic geometrization of the Kuramoto model: Equilibria and stability analysis

Dhagash Mehta,[1,a)] Noah S. Daleo,[2,b)] Florian Dörfler,[3,c)] and Jonathan D. Hauenstein[1,d)]

[1] Department of Applied and Computational Mathematics and Statistics, University of Notre Dame, Notre Dame, Indiana 46556, USA

[2] Department of Mathematics, North Carolina State University, Raleigh, North Carolina 27695, USA

[3] Automatic Control Laboratory, Swiss Federal Institute of Technology (ETH) Zürich, 8092 Zürich, Switzerland

Models & sync notion

finite dimensional & heterogeneous

uniform all-to-all **Kuramoto model**

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} \frac{K}{n} \sin(\theta_i - \theta_j)$$

where $K > 0$ is the coupling strength among the oscillators

general **coupled oscillator model**

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

where $a_{ij} = a_{ji} \geq 0$ induces a connected and undirected graph

Frequency synchronization: $\dot{\theta}_i = \omega_{\text{sync}} \in \mathbb{R}$ for all $i \in \{1, \ldots, n\}$

Lemma: if there is a frequency-sync'd solution, then $\omega_{\text{sync}} = \sum_{i=1}^{n} \omega_i / n$

$\Longrightarrow$ frequency-synchronized solutions are equilibria in rotating coordinates

the synchronization threshold

or existence, uniqueness, &

local stability of equilibria

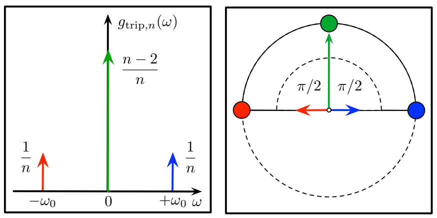
Synchronization threshold for the complete graph

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} \frac{K}{n} \sin(\theta_i - \theta_j)$$

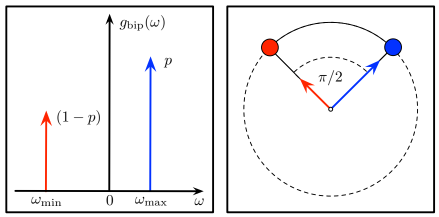
synchronization if $K > K_{\text{crit}}(\omega)$

$\Rightarrow$ necessary & tight lower bound [Chopra & Spong '09] $K_{\text{crit}} \geq \max_{i,j} \frac{1}{2} |\omega_i - \omega_j|$

$\Rightarrow$ sufficient & tight upper bound [FD & Bullo '11] $K_{\text{crit}} \leq \max_{i,j} |\omega_i - \omega_j|$



tight lower bound



tight upper bound

Synchronization threshold for the complete graph – cont'd

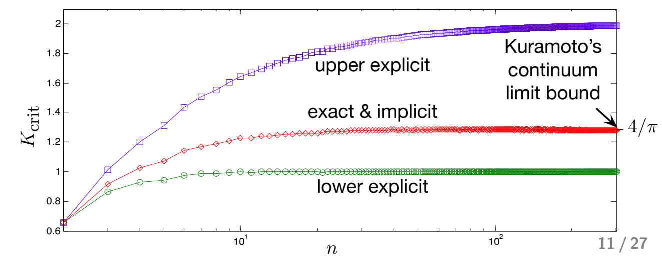
❶ explicit & tight lower/upper bounds [Chopra & Spong '09, FD & Bullo '11]

$$\frac{1}{2} \max_{i,j} |\omega_i - \omega_j| \leq K_{\text{crit}} \leq \max_{i,j} |\omega_i - \omega_j|$$

❷ exact & implicit [Aeyels & Rogge '04, Mirollo & Strogatz '05, Verwoerd & Mason '08]

$K_{\text{crit}} = \frac{nu^*}{\sum_{i=1}^{n} \sqrt{1-(\omega_i/u^*)^2}}$ where $u^* \in [\|\omega\|_\infty, 2\|\omega\|_\infty]$ is the unique solution to the equation $2\sum_{i=1}^{n} \sqrt{1-(\omega_i/u)^2} = \sum_{i=1}^{n} 1/\sqrt{1-(\omega_i/u)^2}$.

comparison of bounds for uniform distribution $g_{\text{unif}}(\omega) \in [-1, +1]$



there's nothing more to say

for the complete uniform graph

. . . so let's move on

Primer on algebraic graph theory

Laplacian matrix L = "degree matrix" − "adjacency matrix"

$$L = L^T = \begin{bmatrix} \vdots & \ddots & \vdots & \iddots & \vdots \\ -a_{i1} & \cdots & \sum_{j=1}^{n} a_{ij} & \cdots & -a_{in} \\ \vdots & \iddots & \vdots & \ddots & \vdots \end{bmatrix} \geq 0$$

Notions of connectivity

- spectral: 2nd smallest eigenvalue of L is "algebraic connectivity" $\lambda_2(L)$
- topological: degree $\sum_{j=1}^{n} a_{ij}$ or degree distribution

Notions of heterogeneity

$\|\omega\|_{\mathcal{E},\infty} = \max_{\{i,j\}\in\mathcal{E}} |\omega_i - \omega_j|$, $\qquad \|\omega\|_{\mathcal{E},2} = \left(\sum_{\{i,j\}\in\mathcal{E}} |\omega_i - \omega_j|^2\right)^{1/2}$

Synchronization threshold in sparse networks

a brief overview on theoretical guarantees

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

1. **necessary sync condition**: $\sum_{j=1}^{n} a_{ij} \geq |\omega_i| \quad \Leftarrow \quad$ sync
   [C. Tavora and O.J.M. Smith '72]

2. **sufficient sync condition**: $\lambda_2(L) > \|\omega\|_{\mathcal{E},2} \quad \Rightarrow \quad$ sync
   [FD and F. Bullo '12]

$\Rightarrow$ $\exists$ similar conditions with diff. metrics on coupling & heterogeneity

$\Rightarrow$ **Problem:** sharpest general conditions are conservative

Nearly exact synchronization threshold [FD, Chertkov, & Bullo '12]

$\left\|L^{\dagger}\omega\right\|_{\mathcal{E},\infty} < 1 \quad \Longrightarrow \quad$ locally exponentially stable synchronization for

1) **extremal topologies:** acyclic, complete graphs, or $\{3,4\}$ rings
2) **extremal parameters:** $L^{\dagger}\omega$ is bipolar, small, or symmetric (for rings)
3) arbitrary one-connected **combinations** of 1) and 2)
4) with high probability, accuracy, & confidence **"for almost all"** G & ω

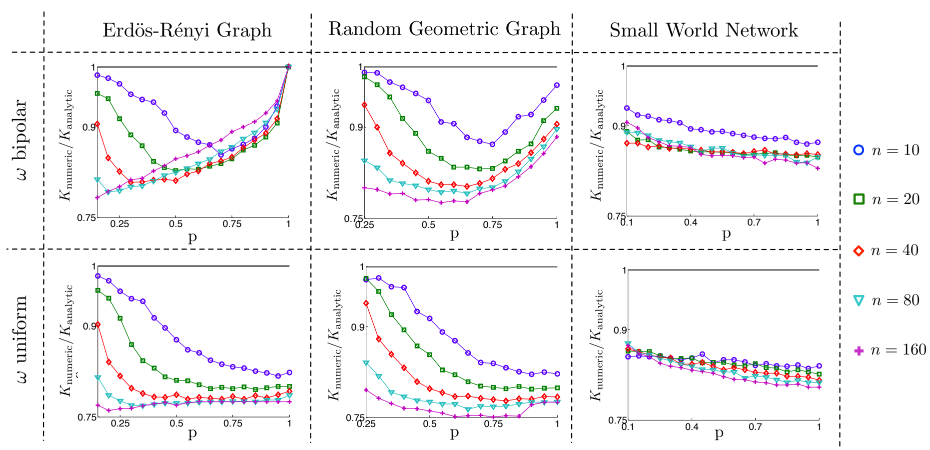
---

intuition: cond' $\left\|L^{\dagger}\omega\right\|_{\mathcal{E},\infty} < 1$ includes previous λ_2, degree, & complete:

$$\left\| \left[\text{eigenvectors of L}\right] \begin{bmatrix} 0 & 0 & \dots & \dots & 0 \\ 0 & 1/\lambda_2(L) & 0 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & 1/\lambda_n(L) \end{bmatrix} \left[\text{eigenvectors of L}\right]^T \omega \right\|_{\mathcal{E},\infty} < 1$$

Nearly exact synchronization threshold – cont'd

Comparison with numerical K_{crit} for $\dot{\theta}_i = \omega_i - K \cdot \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$



$\Rightarrow$ condition $\left\|L^{\dagger}\omega\right\|_{\mathcal{E},\infty} < 1$ is highly accurate & always guarantees sync

The synchronization threshold

Conjecture 1: $\left\|L^{\dagger}\omega\right\|_{\mathcal{E},\infty} < 1 \;\Rightarrow\;$ exists locally exponentially stable sync

☺ **Monte Carlo**: $\left\|L^{\dagger}\omega\right\|_{\mathcal{E},\infty} < 1 \;\Longrightarrow\;$ sync "for almost all" G & ω

☹ thin 0.03% set of **counter-examples** with $\mathcal{O}(10^{-4})$ error

☹ **analytic counter-example** with a large ring [FD, Chertkov, & Bullo '12]

Many related problems are actually **NP-hard**:

- throughput maximization in capacitated network flow [A. Verma, '09]
- power dispatch optimization [K. Lehmann, A. Grastien, & P. Van Hentenryck, '14]
- finding non-zero stable equilibria of the Kuramoto model [R. Taylor, '15]
- finding stable equilibria of the repulsive Kuramoto model [A. Sarlette, '11]

The conjecture is **rejected**. The sync threshold remains open & hard(?)...

The problem may be hopeless … but the bounds ain't bad

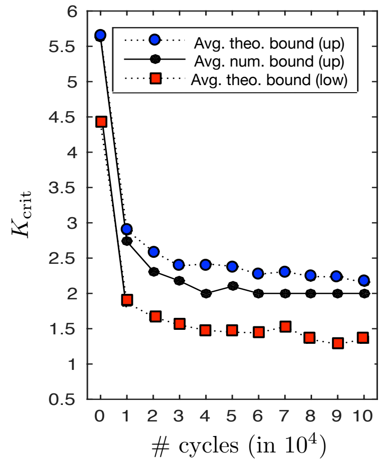
$$\dot{\theta}_i = \omega_i - K \cdot \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

necessary & "sufficient" sync bounds:

$$\max_i \frac{\sum_{j=1}^{n} a_{ij}}{|\omega_i|} \leq K_{\text{crit}} \leq \left\| L^{\dagger} \omega \right\|_{\mathcal{E},\infty}$$

(exact for acyclic and tight for complete)

⇒ comparison w/ coarse numerical K_{crit}



working horse: **algebraic geometry** [D. Mehta, N. Daleo, FD, & J. Hauenstein, '15]

$$\omega_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \quad \overset{s_i = \sin(\theta_i)}{\underset{c_i = \cos(\theta_i)}{\Longleftrightarrow}} \quad \omega_i = \sum_{j=1}^{n} a_{ij} (s_i c_j - s_j c_i)$$

more fun with **stable equilibria**

Systems without stable equilibria

Conjecture 2: if there are any equilibria, then at least one must be stable

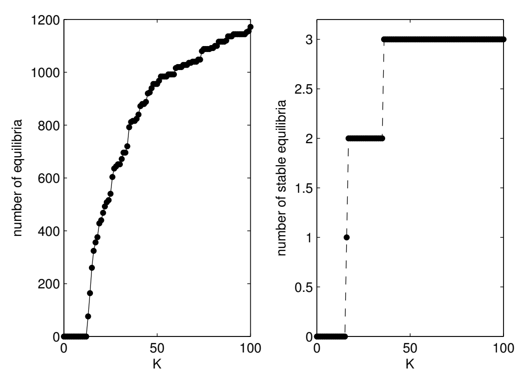
equilibria of a ring graph with $n = 10$ & $\omega_i \in [-1, 1]$ uniformly

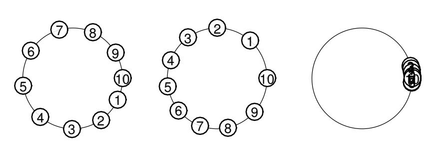
$$\dot{\theta}_i = \omega_i - K \sin(\theta_i - \theta_{i-1}) - K \sin(\theta_i - \theta_{i+1})$$

⇒ **multi-stable** cases

⇒ **all unstable** for $K = 13 - 15$

⇒ analytic counterexample by [A. Arapostathis et al., '81]



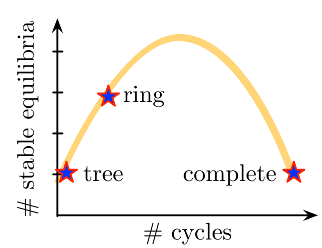


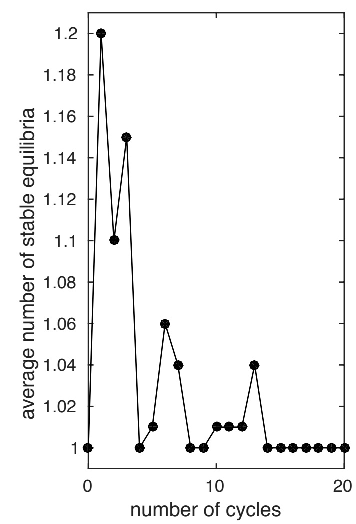
The conjecture is **rejected**.

How many stable equilibria are there?

- **acyclic graphs** have a single stable equilibrium [FD, Chertkov, & Bullo '12]
- previously: **rings** have multi-stable equilibrium landscapes
- **complete graphs** have a single stable equilibrium [Aeyels & Rogge '04]

Conjecture 3: the plot of # stable equilibria vs. cycles is a concave curve





The conjecture is **rejected** & the problem is more puzzling.

A popular folk theorem about the "$\pi/2$-box"

Stable $\pi/2$-box: any equilibrium in $\{\theta \in \mathbb{T}^n : |\theta_i - \theta_j| < \pi/2 \; \forall \{i,j\} \in \mathcal{E}\}$ is locally exponentially stable (modulo rotational symmetry).

Proof: linearization is $\dot{\theta} = -L(\theta^*) \cdot \theta$ where $L(\theta^*)$ is a Laplacian:

$$L(\theta^*) = \begin{bmatrix} \vdots & \ddots & \vdots & \iddots & \vdots \\ -a_{i1}\cos(\theta_i^* - \theta_1^*) & \cdots & \sum_{j=1}^n a_{ij}\cos(\theta_i^* - \theta_j^*) & \cdots & -a_{in}\cos(\theta_i^* - \theta_n^*) \\ \vdots & \iddots & \vdots & \ddots & \vdots \end{bmatrix}$$

$\Longrightarrow$ a major part of the literature focuses on the $\pi/2$-box

Conjecture 4: there is at most one equilibrium in the $\pi/2$-box

has been proved ... at least on $\mathbb{R}^n$ [A. Arapostathis et al., '81, K. Dvijotham et al., '15]

The "$\pi/2$-box" does not guarantee uniqueness on $\mathbb{T}^n$

Stable $\pi/2$-box: any equilibrium in $\{\theta \in \mathbb{T}^n : |\theta_i - \theta_j| < \pi/2 \; \forall \{i,j\} \in \mathcal{E}\}$ is locally exponentially stable (modulo rotational symmetry).

Conjecture 4: there is at most one equilibrium in the $\pi/2$-box

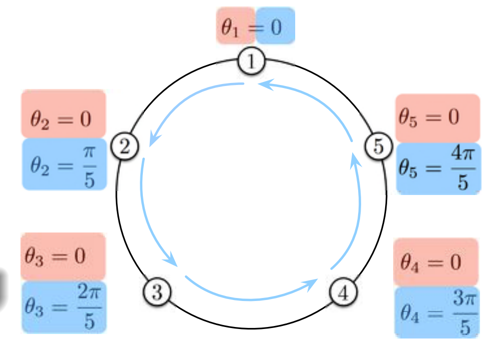
Homogeneous **counterexample**

$\dot{\theta}_i = -\sin(\theta_i - \theta_{i-1}) - \sin(\theta_i - \theta_{i+1})$

admits two equilibria in $\pi/2$-box

(does not work in $\mathbb{R}^n$)

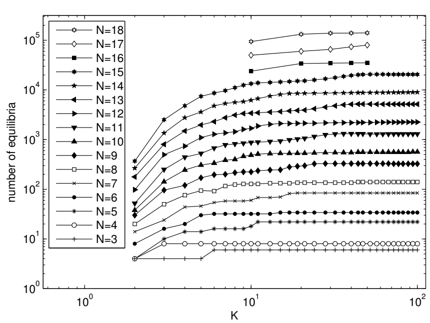
The conjecture is **rejected** on $\mathbb{T}^n$.



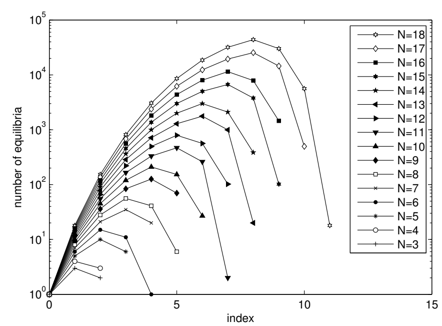
Equilibrium indices in the Kuramoto model

Equilibria of the Kuramoto model & their indices (# stable eigenvalues)

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^n \frac{K}{n} \sin(\theta_i - \theta_j)$$



converge to 2^n as $K \to \infty$



for K = 100

Conjecture 5 (open): for n & K large, there are $\binom{n}{j}$ equilibria of index j

(almost) global stability

=

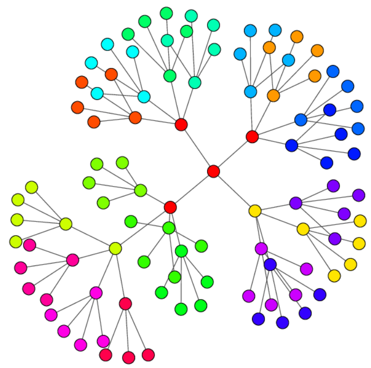
sync basin is almost all of $\mathbb{T}^n$

Conjecture for acyclic & undirected networks

Conjecture 6 for acyclic networks: if there is a locally exponentially stable equilibrium, then it is almost globally stable.

Partial proof: conjecture is true for homogeneous ω_i [P. Monzon, '06] & can be extended to weakly heterogeneous cases via ISS [Angeli & Praly,'11].

Numerics: randomized simulations apparently always confirm conjecture.

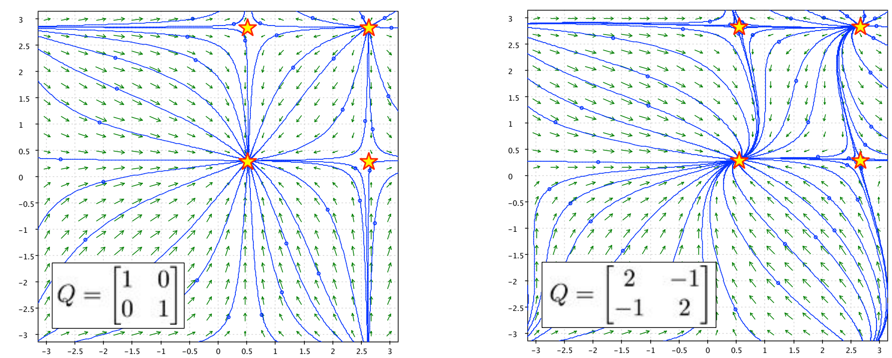


Non-rigorous reasoning for acyclic networks

Transformation to branch coordinates

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \quad \overset{\delta_{ij}=\theta_i-\theta_j}{\underset{\forall\{i,j\}\in\mathcal{E}}{\Longleftrightarrow}} \quad \begin{bmatrix} \vdots \\ \dot{\delta}_{ij} \\ \vdots \end{bmatrix} = Q \begin{bmatrix} \vdots \\ \tilde{\omega}_{ij} - \sin(\delta_{ij}) \\ \vdots \end{bmatrix},$$

where Q is a positive definite matrix distorting the decoupled vector field.

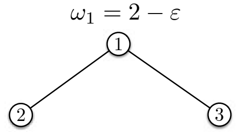


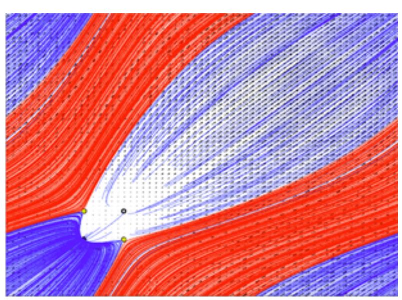
Conjecture for acyclic networks is partially rejected

Conjecture 6 for acyclic networks: if there is a locally exponentially stable equilibrium, then it is almost globally stable.

a 3-node **counterexample** by [A. Gushchin, E. Mallada, & A. Tang, '15]:

$$\dot{\theta}_i = \omega_i - 3 \cdot \sum_{j=1}^{3} a_{ij} \sin(\theta_i - \theta_j)$$





reveals continua of limit cycles

The conjecture is **rejected**, and the problem is now even more interesting due to partial proof for weakly heterogeneous oscillators. Possibly generic?

Complete & uniform (Kuramoto) networks

Conjecture 7 for complete networks: if there is a locally exponentially stable equilibrium, then it is almost globally stable.

$$\dot{\theta}_i = \omega_i - \frac{K}{n} \cdot \sum_{j=1}^{n} \sin(\theta_i - \theta_j)$$

Today the conjecture is still **open**.

Partial proofs: conjecture is true for homogeneous ω_i [P. Monzon, '06] & can be extended to weakly heterogeneous cases via ISS [Angeli & Praly,'11].

The semi-circle is know to be a subset of the sync basin [FD & F. Bullo,'11].

Numerics: randomized simulations apparently always confirm conjecture.

Plausible argument based on order parameter $re^{i\psi} = \sum_{j=1}^{n} \frac{1}{n} e^{i\theta_j}$

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j) \quad \Leftrightarrow \quad \dot{\theta}_i = \omega_i - Kr \sin(\theta_i - \psi)$$

This should essentially behave like a single oscillator system ...

conclusions

Summary and conclusions

We **rejected** some conjectures

- systems without stable equilibria
- non-unique equilibria in $\pi/2$-box
- non-trivial sync basin for trees
- synchronization threshold bounds

& found some **intriguing** problems:

- # stable equilibria vs. # cycles
- scaling of equilibrium indices
- almost global sync basin
- exact synchronization threshold

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"Surprisingly enough, this seemingly obvious fact seems difficult to prove."

[Y. Kuramoto, '84]