On the role of regularization in Direct Data-Driven LQR Control

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On the Role of Regularization in Direct Data-Driven LQR Control

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Abstract— **The linear quadratic regulator (LQR) problem is a cornerstone of control theory and a widely studied benchmark problem. When a system model is not available, the conventional approach to LQR design is indirect, i.e., based on a model identified from data. Recently a suite of direct data-driven LQR design approaches has surfaced by-passing explicit system identification (SysID) and based on ideas from subspace methods and behavioral systems theory. In either approach, the data underlying the design can be taken at face value (certainty-equivalence) or the design is robustified to account for noise. An emerging topic in direct data-driven LQR design is to regularize the optimal control objective to account for implicit SysID (in a least-square or low-rank sense) or to promote robust stability. These regularized formulations are flexible, computationally attractive, and theoretically certifiable; they can interpolate between direct vs. indirect and certainty-equivalent vs. robust approaches; and they can be blended resulting in remarkable empirical performance. This manuscript reviews and compares different approaches to regularized direct data-driven LQR.**

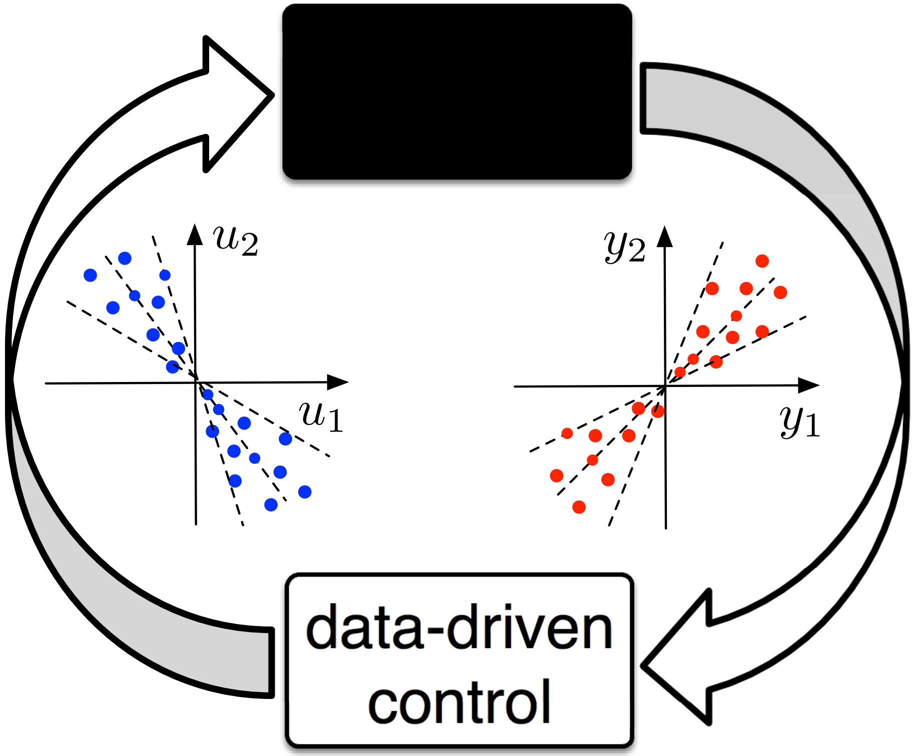
problems when identifying models from data. They facilitate finding solutions to optimization problems by rendering them unique or speeding up algorithms. Aside from such numerical advantages, a Bayesian interpretation of regularizations is that they condition models on prior knowledge [26], and they robustify problems to uncertainty [27], [28].

An emergent approach to data-driven control is borne out of the intersection of behavioral systems theory and subspace methods [29]. In particular, the so-called *Fundamental Lemma* characterizes the behavior of an LTI system by the range space of matrix time series data [30]. This perspective gave rise to direct data-driven predictive and explicit feedback control formulations [14]–[17], [24], [31], [32]. Both lines of work emphasize robustness to noisy data.

This manuscript presents a tutorial review of regularized direct data-driven LQR [16], [33], which touches upon all of the above. As a baseline, indirect CE data-driven LQR

Data-driven control dichotomy

- **indirect data-driven control** via models:
  data $\xrightarrow{\text{ID}}$ model + uncertainty → control
- growing trend **direct data-driven control**
  by-passing models … (again) hyped, why?



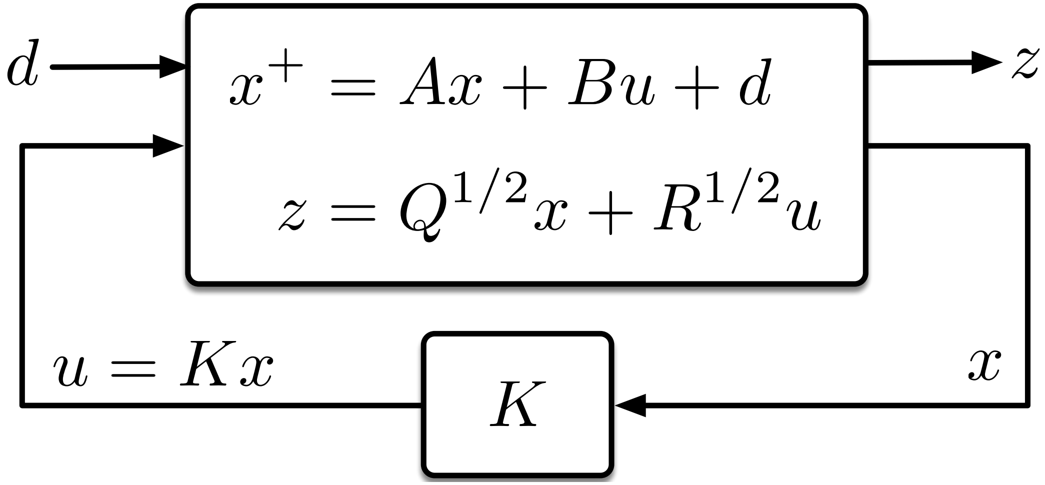
The direct approach is a **viable alternative**

- for some **applications** : models (plant, environments, or sensing modalities) too complex to be useful (e.g., wind farm, soft robotics)
- due to (well-known) **shortcomings of ID** → cumbersome, models not identified for control, incompatible uncertainty estimates, ...
- when sufficient **brute force** data / compute / storage is available

- **trade-offs**
  - (non)modular
  - (in)tractable
  - (sub)optimal
  - data richness

today: give explicit answers for **LQR**

LQR



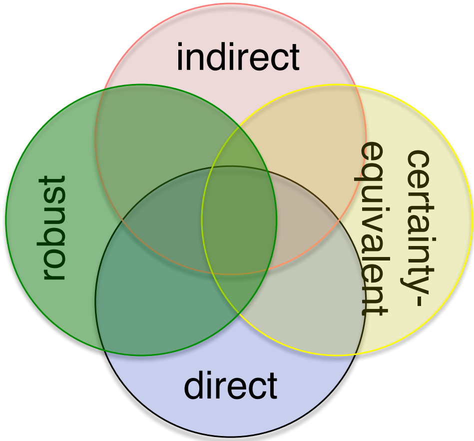
- **cornerstone** of automatic control

- $\mathcal{H}_2$ **parameterization**

  (can be posed as convex SDP)

$$\begin{aligned} \underset{P \succeq I, K}{\text{minimize}} \quad & \text{trace}\,(QP) + \text{trace}\left(K^\top RKP\right) \\ \text{subject to} \quad & (A+BK)P(A+BK)^\top - P + I \preceq 0 \end{aligned}$$

- <u>**the**</u> **benchmark** for all data-driven control approaches in last decades(!)



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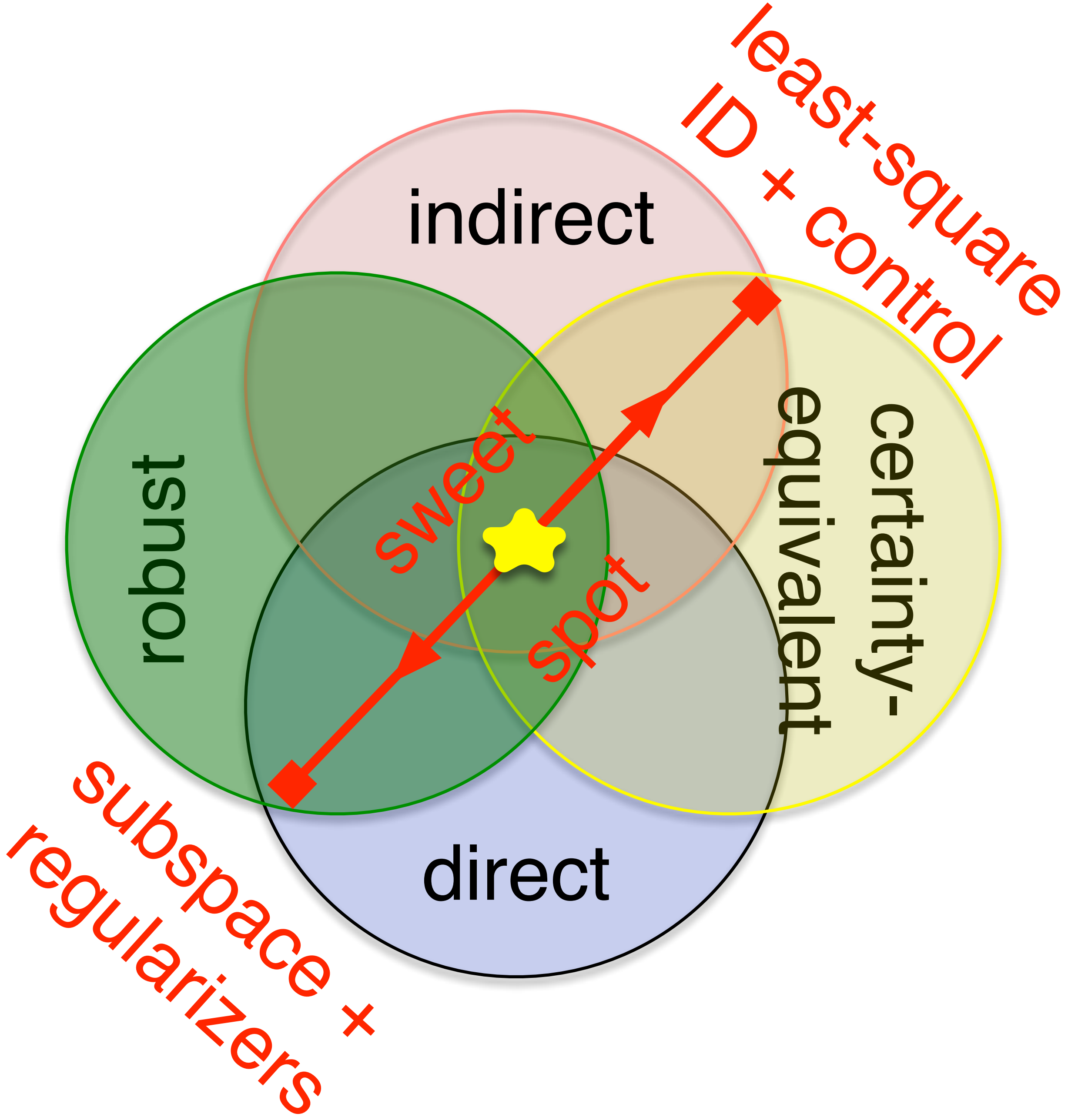


least-square
ID + control
indirect
robust
sweet
spot
certainty-
equivalent
direct
subspace +
regularizers

Indirect & certainty-equivalence LQR

- collect **I/O data:** D_0 unknown & PE: $\operatorname{rank} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} = n + m$

$$U_0 := \begin{bmatrix} u(0) & u(1) & \dots & u(T-1) \end{bmatrix}$$
$$D_0 := \begin{bmatrix} d(0) & d(1) & \dots & d(T-1) \end{bmatrix}$$

$$X_1 = AX_0 + BU_0 + \cancel{D_0}$$

$$X_0 := \begin{bmatrix} x(0) & x(1) & \dots & x(T-1) \end{bmatrix}$$
$$X_1 := \begin{bmatrix} x(1) & x(2) & \dots & x(T) \end{bmatrix}$$

- **indirect & certainty-equivalence** LQR
  (in special cases optimal
  in MLE sense but not robust)

$$\begin{aligned} \underset{P \succeq I,\, K}{\text{minimize}} \quad & \operatorname{trace}(QP) + \operatorname{trace}\left(K^\top R K P\right) \\ \text{subject to} \quad & (\hat{A} + \hat{B}K)P(\hat{A} + \hat{B}K)^\top - P + I \preceq 0 \end{aligned}$$

certainty-equivalent LQR

$$\begin{bmatrix} \hat{B} & \hat{A} \end{bmatrix} = \arg\min_{B,A} \left\| X_1 - \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} \right\|_F$$

least squares SysID

Direct approach from subspace relations in data

- **data:** $\operatorname{rank}\begin{bmatrix} U_0 \\ X_0 \end{bmatrix} = n + m \;\Rightarrow\; \forall K \;\exists G \text{ s.t. } \begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G$

$$U_0 := \begin{bmatrix} u(0) & u(1) & \dots & u(T-1) \end{bmatrix}$$
$$D_0 := \begin{bmatrix} d(0) & d(1) & \dots & d(T-1) \end{bmatrix}$$
$$\to \boxed{X_1 = AX_0 + BU_0 + D_0} \to$$
$$X_0 := \begin{bmatrix} x(0) & x(1) & \dots & x(T-1) \end{bmatrix}$$
$$X_1 := \begin{bmatrix} x(1) & x(2) & \dots & x(T) \end{bmatrix}$$

- **subspace relations** $A + BK = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G = (X_1 - D_0) G$

- **data-driven LQR** LMIs by substituting $A + BK = (X_1 - D_0)G$

  → certainty equivalence by neglecting noise D_0: $\boxed{A + BK = X_1 G}$

Equivalence: direct + *xxx* ⇔ indirect

- **direct** approach
  → optimizer has nullspace $\ker \begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$
  → orthogonality constraint

$$\begin{aligned} \underset{P \succeq I, K, G}{\text{minimize}} \quad & \operatorname{trace}(QP) + \operatorname{trace}\left(K^\top R K P\right) \\ \text{subject to} \quad & X_1 G P G^\top X_1^\top - P + I \preceq 0 \\ & \begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G \\ & \left(I - \begin{bmatrix} U_0 \\ X_0 \end{bmatrix}^\dagger \begin{bmatrix} U_0 \\ X_0 \end{bmatrix}\right) G = 0 \end{aligned}$$

$$G = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix}^\dagger \begin{bmatrix} K \\ I \end{bmatrix}$$

equivalent constraints:

$$\left(X_1 \begin{bmatrix} U_0 \\ X_0 \end{bmatrix}^\dagger \begin{bmatrix} K \\ I \end{bmatrix}\right) P \left(\ldots\right)^\top$$
$$-P + I \preceq 0$$

- **indirect** approach

$$\begin{aligned} \underset{P \succeq I, K}{\text{minimize}} \quad & \operatorname{trace}(QP) + \operatorname{trace}\left(K^\top R K P\right) \\ \text{subject to} \quad & (\hat{A} + \hat{B}K) P (\hat{A} + \hat{B}K)^\top - P + I \preceq 0 \\ & \begin{bmatrix} \hat{B} & \hat{A} \end{bmatrix} = \underset{B, A}{\arg\min} \left\| X_1 - \begin{bmatrix} B & A \end{bmatrix} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} \right\|_F \end{aligned}$$

$$\begin{bmatrix} \hat{B} & \hat{A} \end{bmatrix} = X_1 \begin{bmatrix} U_0 \\ X_0 \end{bmatrix}^\dagger$$

Regularized, direct, & certainty-equivalent LQR

- orthogonality constraint

$$\Pi = I - \begin{bmatrix} U_0 \\ X_0 \end{bmatrix}^{\dagger} \begin{bmatrix} U_0 \\ X_0 \end{bmatrix}$$

lifted to regularizer

$$\begin{aligned} \underset{P \succeq I, K, G}{\text{minimize}} \quad & \text{trace}\,(QP) + \text{trace}\left(K^\top RKP\right) + \lambda \cdot \|\Pi G\| \\ \text{subject to} \quad & X_1 GPG^\top X_1^\top - P + I \preceq 0 \\ & \begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G \end{aligned}$$

- **equivalent** to indirect certainty-equivalent LQR design for λ suff. large
- λ **interpolates** between direct & indirect approaches
- **multi-criteria interpretation**: λ interpolates control & SysID objectives
- however, certainty-equivalence formulation may not be **robust (?)**

Robustness-promoting regularization

- **effect of noise** entering data: $X_1 = AX_0 + BU_0 + D_0$
  Lyapunov constraint $X_1 G P G^\top X_1^\top - P + I \preceq 0$
  becomes $(X_1 - D_0) G P G^\top (X_1 - D_0)^\top - P + I \preceq 0$

for robustness $GPG^\top$ **or** $\|G\|$ **should be small**

- previous **certainty-equivalence regularizer** $\|\Pi G\|$ achieves small $\|G\|$

- **robustness-promoting regularizer** [de Persis & Tesi, '21]

$$
\begin{aligned}
\underset{P \succeq I, K, G}{\text{minimize}} \quad & \operatorname{trace}(QP) + \operatorname{trace}\left(K^\top R K P\right) \\
& + \rho \cdot \operatorname{trace}\left(G P G^\top\right) \\
\text{subject to} \quad & X_1 G P G^\top X_1^\top - P + I \preceq 0 \\
& \begin{bmatrix} K \\ I \end{bmatrix} = \begin{bmatrix} U_0 \\ X_0 \end{bmatrix} G
\end{aligned}
$$

Performance & robustness analysis

- **SNR** (signal-to-noise-ratio) $\frac{\sigma_{min}([X_0 \ U_0])}{\sigma_{max}(D_0)}$

- **relative performance** metric

realized cost from regularized design with λ & ρ — if exact system matrices *A* and *B* were known

$$\frac{\{\text{regularized data-driven LQR performance}\} - \{\text{ground-truth performance}\}}{\{\text{ground-truth performance}\}}$$

- **certificate**: optimal control problem is **always feasible & stabilizing** for suff. large SNR & **relative performance** $\sim \mathcal{O}\left(\text{SNR}^{-1}\right) + \text{const.} \cdot \rho$ robust reg.

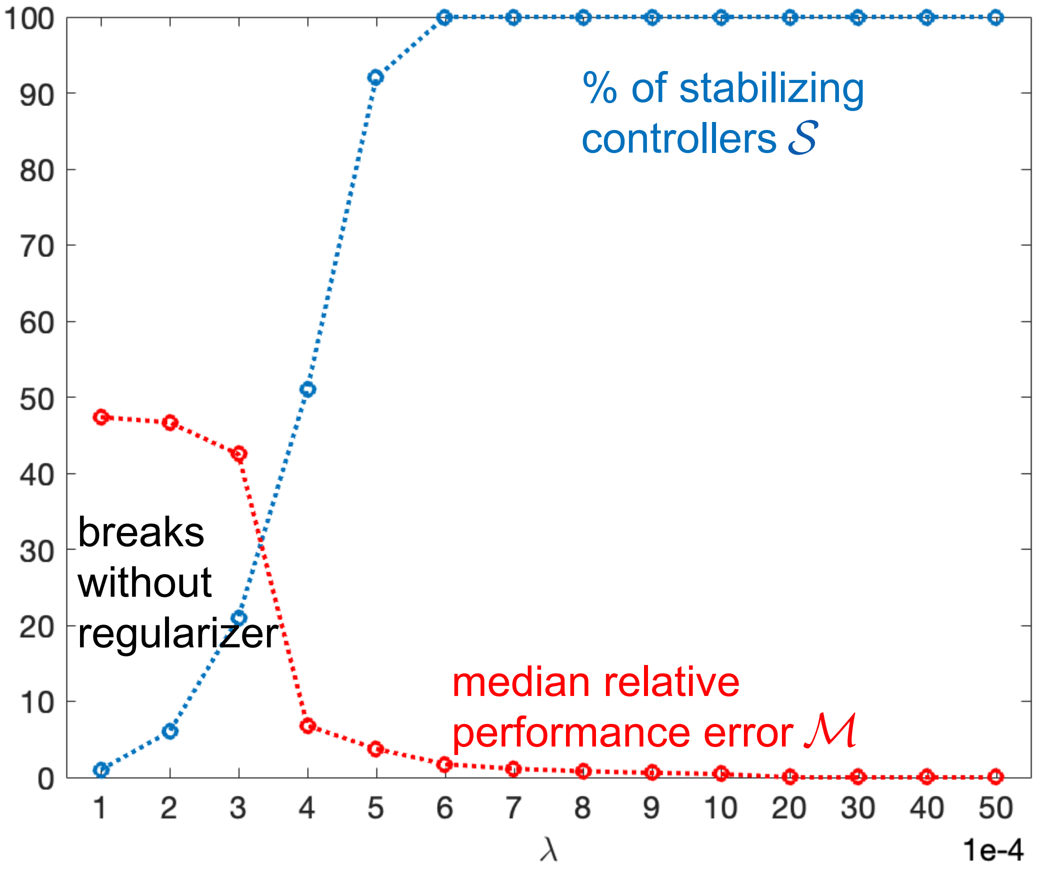
proof bounds Lyapunov constraint $(X_1 - D_0)GPG^\top(X_1 - D_0)^\top - P + I \preceq 0$

Numerical case study

- **case study** [Dean et al. '19]: discrete-time marginally unstable Laplacian system subject to noise of variance $\sigma^2 = 0.01$

$$A = \begin{bmatrix} 1.01 & 0.01 & 0 \\ 0.01 & 1.01 & 0.01 \\ 0 & 0.01 & 1.01 \end{bmatrix}, \quad B = I$$

- **take-home message** 1:
  regularization is needed !
  prior work without regularizer has no robustness margin



Numerical case study cont'd

- **take-home message 2**: different regularizers promote different features: robustness vs. certainty-equivalence (performance)

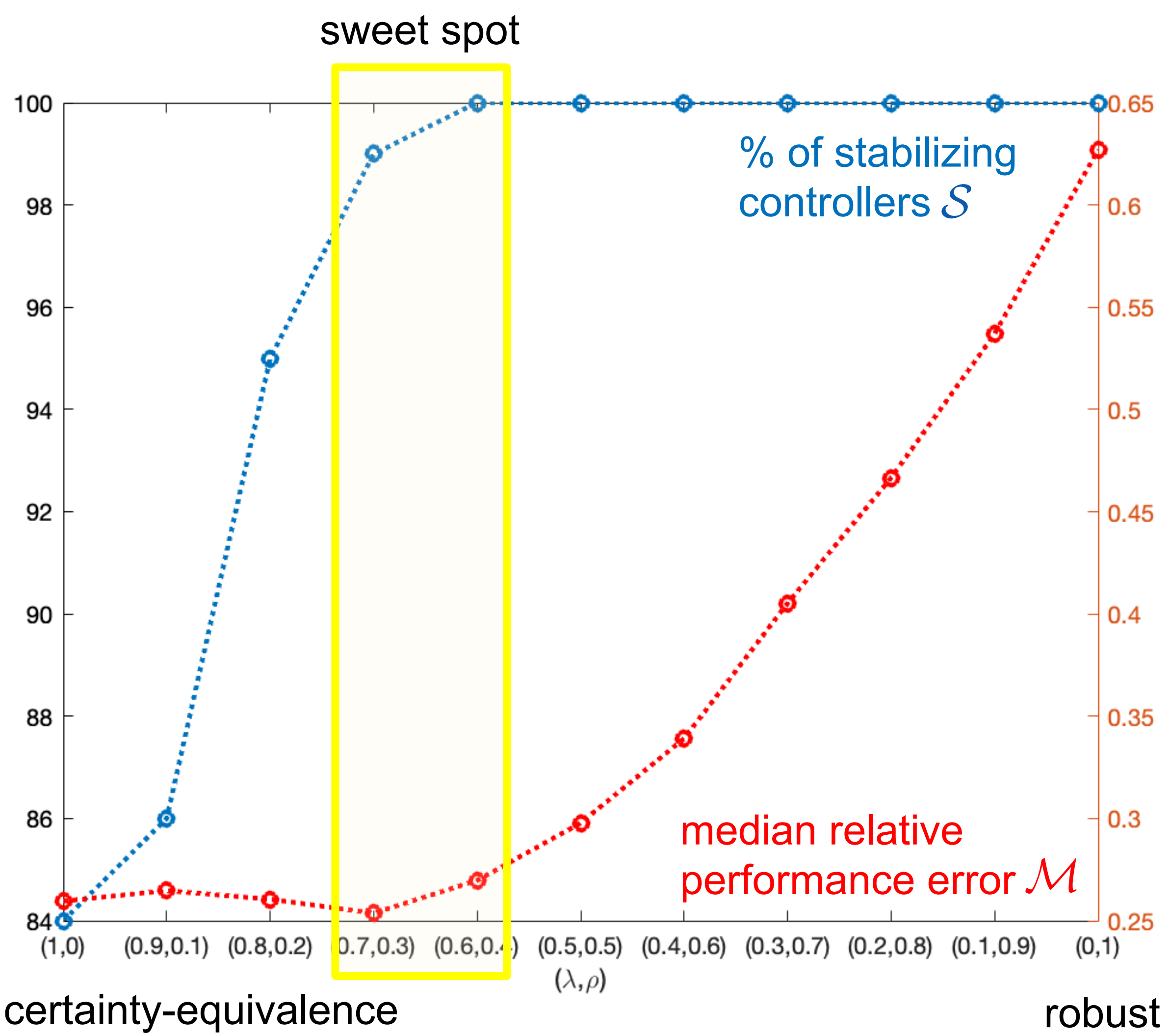
	$\sigma = 0.01$ (SNR > 15dB)	$\sigma = 0.1$ (SNR $\in [5, 10]$dB)	$\sigma = 0.3$ (SNR $\in [0, 5]$dB)	$\sigma = 0.7$ (SNR ≈ 0dB)	$\sigma = 1$ (SNR < -5dB)
Certainty-equivalence ($\lambda = 1, \rho = 0$)	$\mathcal{S} = 100\%$ $\mathcal{M} = 2.5599e\text{-}05$	$\mathcal{S} = 100\%$ $\mathcal{M} = 0.0026$	$\mathcal{S} = 100\%$ $\mathcal{M} = 0.0237$	$\mathcal{S} = 97\%$ $\mathcal{M} = 0.1366$	$\mathcal{S} = 84\%$ $\mathcal{M} = 0.2596$
Robust approach ($\lambda = 0, \rho = 1$)	$\mathcal{S} = 100\%$ $\mathcal{M} = 0.0035$	$\mathcal{S} = 100\%$ $\mathcal{M} = 0.0074$	$\mathcal{S} = 100\%$ $\mathcal{M} = 0.0369$	$\mathcal{S} = 100\%$ $\mathcal{M} = 0.2350$	$\mathcal{S} = 100\%$ $\mathcal{M} = 0.6270$

- **take-home message 3**: mixed regularization achieves best of both

Mixed regularization ($\lambda = \rho = 0.5$)	$\mathcal{S} = 100\%$ $\mathcal{M} = 0.0010$	$\mathcal{S} = 100\%$ $\mathcal{M} = 0.0035$	$\mathcal{S} = 100\%$ $\mathcal{M} = 0.0235$	$\mathcal{S} = 100\%$ $\mathcal{M} = 0.1262$	$\mathcal{S} = 100\%$ $\mathcal{M} = 0.2978$

Conclusions

- **interpolation** of different regularizers with high noise: $\sigma^2 = 1$ (SNR< -5db)
- **flexible multi-criteria formulation** trading off different objectives by regularizers (best of all is attainable)
- **classification direct vs. indirect** is less relevant: λ interpolates
- simple & **extendable(?)** framework



thanks