Distributed Frequency Control for Stability and Economic Dispatch in Power Networks

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Abstract— **We explore two different frequency control strategies to ensure stability of power networks and achieve economic dispatch between generators and controllable loads. We first show the global asymptotic stability of a completely decentralized frequency integral control. Then we design a distributed averaging-based integral (*DAI*) control which operates by local frequency sensing and neighborhood communication. Equilibrium analysis shows that *DAI* recovers nominal frequency with minimum total generation cost and user disutility for load control after a change in generation or load. Local asymptotic stability of *DAI* is established with a Lyapunov method. Simulations demonstrate improvement in both transient and steady-state performance achieved by the proposed control strategies, compared to droop control.**

I. INTRODUCTION

Maintaining frequency tightly around the nominal value is important for power grids since frequency excursions degrade power quality and may damage facilities. Frequency control is traditionally performed by adjusting real power generation to balance the load. This traditional scheme has a hierarchical structure composed of three layers working in concert, i.e., primary (droop control), secondary (automatic generation control) and tertiary (economic dispatch), from fast to slow timescales [1], [2].

The integration of distributed renewable generation, like solar and wind power, introduces larger and faster fluctuations in power supply and frequency. Hence relying purely on generator-side frequency control requires more fast-acting generators as spinning reserves, which are expensive and produce high emissions [3]. As a supplement to generator-side frequency control, distributed load-side frequency control has been extensively studied [4]–[8]. These studies have shown significant performance improvement mainly due to fast-acting capability of frequency-responsive loads and reduction in the need for generation reserves. On the other hand, the distributed energy resources, which generate either DC or variable frequency AC power, are interfaced with the main grid via power electronic DC/AC inverters. These inverters are typically designed to emulate droop control [1], [9]. Different from bulk generators, the controllable loads and inverters usually have low or no inertia. Hence, a structure preserving model [10] with both positive and zero inertia buses is suitable for design and stability analysis of frequency control in a power network with bulk generators, controllable loads and distributed energy resources interfaced via inverters.

Previous work on frequency control focuses on two issues. The first issue is stability of closed-loop power networks, which has been studied for different generator-side frequency control schemes [11]–[14], and for networks with linear frequency dependent loads [10], [15]. All these studies use network models with nonlinear power flows, which are more realistic than linearized models. Global asymptotic stability is usually established with complicated control schemes which require some physical parameters to be known, which is hard in practice. Otherwise, simple decentralized droop control only guarantees local asymptotic stability, and does not recover frequency to the nominal value [10]. The second issue is incorporating economic dispatch with frequency control at a fast (seconds) timescale, which breaks the traditional hierarchy of frequency control. Existing work on this issue ranges from generator side [16]–[21] or load side [22]–[25] to microgrids [1], [26]. A common feature of these studies is that, while variations of economic dispatch are solved over the entire network, the control schemes are decentralized (in that only local sensing and feedback is required) or distributed (in that moderate communication between neighboring controlled units is required) to ensure scalability to future power grids with a large number of actively controlled endpoints.

In this paper, we explore two different frequency control strategies, both operating jointly from the generator and load sides in a network with positive and zero inertia buses, to address the two issues above. Applying a Lyapunov method to the structure preserving model with nonlinear power flows, we first establish the *global* asymptotic stability of a simple, completely decentralized frequency integral control. Then, to achieve economic dispatch, we modify the decentralized integral control by adding neighborhood communication, and hence get a distributed averaging-based integral (*DAI*) control. The *DAI* control recovers nominal frequency with minimum total cost of generation and user disutility for participating in load control, after a change in generation or load. Local asymptotic stability of *DAI* is proved using a Lyapunov method. Simulations of the IEEE 39-bus test system demonstrate improvement in both transient and steady-state performance achieved by using the two proposed control strategies, compared to the traditional droop control.

The rest of this paper is organized as follows. Section II

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describes the structure preserving model, introduces the control objective of economic dispatch, and connects equilibria of the system to the solutions of economic dispatch. Section III shows global asymptotic stability of the completely decentralized frequency integral control. Section IV proposes the *DAI* control and proves its local stability. Section V is a simulation-based case study to show the performance of the proposed control strategies. Section VI concludes the paper and discusses future work.

II. SYSTEM MODEL AND PROBLEM FORMULATION

For a set $\mathcal{N}$, let $|\mathcal{N}|$ denote its cardinality. A variable with an underscore and a set as the subscript denotes a vector with appropriate components, e.g., $\underline{\omega}_{\mathcal{G}} = (\omega_j, j \in \mathcal{G}) \in \mathbb{R}^{|\mathcal{G}|}$. A variable with a set as the subscript but without an underscore denotes a diagonal matrix with appropriate diagonal entries, e.g., $K_{\mathcal{G}} = \mathrm{diag}(K_j, j \in \mathcal{G}) \in \mathbb{R}^{|\mathcal{G}|\times|\mathcal{G}|}$. The subscript may also be omitted when it denotes the set of all the nodes or lines in the network. We use 1_n or 0_n to denote the n-dimensional vector whose components are all 1 or all 0, where the subscript n may be omitted when the number of dimension is clear. Let A^T denote the transpose of a matrix A. The expression $A \succ 0$ ($A \prec 0$) means the square matrix A is positive (negative) definite. For a signal $\omega(t)$ of time t, let $\dot{\omega}$ denote its time derivative $d\omega/dt$. The time index t is usually dropped from equations when the meaning is clear.

Our analysis is based on the structure preserving model [10]. The power network is modeled as an *undirected* graph $(\mathcal{N}, \mathcal{E})$ where $\mathcal{N} = \{1, \ldots, |\mathcal{N}|\}$ is the set of buses (nodes) and $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$ is the set of lines connecting those buses. We use either (j, k) or (k, j) to denote the line connecting buses j and k, i.e., if $(j, k) \in \mathcal{E}$ then $(k, j) \notin \mathcal{E}$. Notice that the pair $(j, k) \in \mathcal{E}$ implicitly assumes a direction from j to k. However, such orientation is arbitrary and does not affect the results of this paper. We assume the graph $(\mathcal{N}, \mathcal{E})$ is connected, and make the following assumptions which are well-justified for transmission networks [27]:

- Bus voltage magnitudes $|V_j| = 1$ pu for $j \in \mathcal{N}$.
- Lines $(j, k) \in \mathcal{E}$ are lossless and characterized by their susceptances $B_{jk} = B_{kj} > 0$. Let $B_{jk} = B_{kj} = 0$ when $(k, j) \notin \mathcal{E}$ and $(j, k) \notin \mathcal{E}$.
- Reactive power flows do not affect bus voltage phase angles and frequencies.

A subset $\mathcal{G} \in \mathcal{N}$ of the buses are fictitious buses representing the internal of generators. Hence we call $\mathcal{G}$ the set of generators and $\mathcal{L} := \mathcal{N} \setminus \mathcal{G}$ the set of load buses. We label all the buses so that $\mathcal{G} = \{1, \ldots, |\mathcal{G}|\}$ and $\mathcal{L} = \{|\mathcal{G}| + 1, \ldots, |\mathcal{N}|\}$.

The voltage phase angle of bus $j \in \mathcal{N}$, with respect to the rotating framework of nominal frequency $\omega_0 = 2\pi \cdot 60$ Hz, is denoted by θ_j. Then

$$\dot{\theta}_j = \omega_j \qquad j \in \mathcal{N} \tag{1}$$

is the frequency deviation from the nominal value on bus j. The network dynamics are described by the swing equations

$$M_j \dot{\omega}_j = -D_j\omega_j + p_j + u_j - \sum_{k\in\mathcal{N}} B_{jk} \sin(\theta_j - \theta_k) \quad j \in \mathcal{G} \tag{2}$$

$$0 = -D_j\omega_j + p_j + u_j - \sum_{k\in\mathcal{N}} B_{jk} \sin(\theta_j - \theta_k) \quad j \in \mathcal{L} \tag{3}$$

where $M_j > 0$ are moments of inertia of generators, $D_j > 0$ are droop coefficients of generators when $j \in \mathcal{G}$ or linear frequency dependent loads when $j \in \mathcal{L}$. The exogenous input $(p_j, j \in \mathcal{N})$ are uncontrollable real power injections from, e.g., uncontrollable loads and renewable generation. The control variables $(u_j, j \in \mathcal{N})$ are mechanic power injections to generators for $j \in \mathcal{G}$ and additive inverses of controllable loads for $j \in \mathcal{L}$. The real power flow from buses j to k is $B_{jk} \sin(\theta_j - \theta_k)$. Aside from frequency dependent loads, the dynamics (3) also occur in low-inertia DC or variable frequency AC power sources that are interfaced with the network through droop-controlled inverters [1], [26].

We are interested in frequency-synchronized solutions of the model (1)–(3) satisfying $\dot{\theta}_j = \omega_j = \omega^*$ for some $\omega^* \in \mathbb{R}$. Summing over equations (2)(3) and evaluating $\omega_j = \omega^*$, we obtain the synchronization frequency

$$\omega^* = \frac{\sum_{j\in\mathcal{N}} p_j + u_j}{\sum_{j\in\mathcal{N}} D_j} \tag{4}$$

which implies that there is an equilibrium satisfying $\omega^* = 0$ only if all power injections are balanced across the entire network, i.e., $\sum_{j\in\mathcal{N}} p_j + u_j = 0$.

Our objective is, given exogenous input $\underline{p} \in \mathbb{R}^{|\mathcal{N}|}$ to the system (1)–(3), to design control law for $\underline{u}$ based on feedback of states $(\underline{\theta}, \underline{\omega})$, such that the system converges to an equilibrium $(\underline{\theta}^*, \underline{\omega}^* = 0, \underline{u}^*)$ which is at the same time a solution to the following economic dispatch problem:

Economic Dispatch (ED):

$$\min_{\underline{\theta}, \underline{u}} \quad \sum_{j\in\mathcal{N}} \frac{1}{2} a_j u_j^2 \tag{5}$$

subject to

$$p_j + u_j - \sum_{k\in\mathcal{N}} B_{jk} \sin(\theta_j - \theta_k) = 0 \quad j \in \mathcal{N} \tag{6}$$

$$|\theta_j - \theta_k| \le \gamma_{jk} < \pi/2 \qquad (j, k) \in \mathcal{E}. \tag{7}$$

The terms $\frac{1}{2}a_j u_j^2$ in (5) are generation cost if $j \in \mathcal{G}$ and user disutility for participating in load control if $j \in \mathcal{L}$, where $a_j > 0$ are constant coefficients. Indeed, quadratic generation cost or user disutility functions are widely used, e.g., in [1], [2], [16]–[18], [21], [28]. The flow balance constraint (6) ensures $\omega^* = 0$ as well as the existence of a synchronized solution to the system (1)–(3). The thermal limit constraint (7) restricts the line flows in the network. We shall make the following assumption on the thermal limit constraint.

Assumption 1 (Strict Feasibility of ED): Any optimal solution $(\underline{\theta}^*, \underline{u}^*)$ of ED satisfies the constraint (7) strictly, i.e.,

$$|\theta_j^* - \theta_k^*| < \gamma_{jk} < \frac{\pi}{2} \qquad (j, k) \in \mathcal{E}. \tag{8}$$

Assumption 1 essentially implies that the network is sufficiently meshed, the transfer capacities are sufficiently large, and generation and load are sufficiently well distributed so that *no line congestion* occurs. In this case, the inequality constraint (7) may be dropped, and (by summing over all equality constraints (6)) we conclude that if $(\underline{\theta}^*, \underline{u}^*)$ is an optimal solution of ED, then u^* is a feasible solution for the *Reduced Economic Dispatch (RED):*

$$\min_{\underline{u}} \quad \sum_{j\in\mathcal{N}} \frac{1}{2} a_j u_j^2 \tag{9}$$

subject to

$$\sum_{j\in\mathcal{N}} p_j + u_j = 0. \tag{10}$$

Notice that RED is a quadratic program subject to linear constraint and thus convex. A comparison of the optimality conditions for ED and RED leads to the following result for strictly feasible solutions of ED.

Lemma 1 (Conditions for optimality): Under Assumption 1, any strictly feasible solution $(\underline{\theta}^*, \underline{u}^*)$ of ED is an optimal solution of ED, if and only if it has identical marginal costs

$$a_j u_j^* = a_k u_k^* \qquad j, k \in \mathcal{N}. \tag{11}$$

Proof: ($\Longrightarrow$): By Assumption 1, we disregard (7). Let λ_j denote the Lagrange multiplier of the equality constraint (6). Using the necessary Karush-Kuhn-Tucker (KKT) conditions for optimality [29], any primal-dual optimal solution $(\underline{\theta}^*, \underline{u}^*; \underline{\lambda}^*)$ must satisfy

- Primal feasibility (6) and dual feasibility ($\underline{\lambda}^* \in \mathbb{R}^{|\mathcal{N}|}$)
- Stationarity:

$$\underline{\lambda}^{*T} \nabla_\theta L(\underline{\theta}^*, \underline{u}^*; \underline{\lambda}^*) = 0 \tag{12}$$

and

$$\nabla_u L(\underline{\theta}^*, \underline{u}^*; \underline{\lambda}^*) = 0, \tag{13}$$

where

$$L(\underline{\theta}, \underline{u}; \underline{\lambda}) = \sum_{j\in\mathcal{N}} \frac{1}{2} a_j u_j^2 + \lambda_j \left(p_j + u_j - \sum_{k\in\mathcal{N}} B_{jk} \sin(\theta_j - \theta_k) \right)$$

is the Lagrangian of ED with constraint (6) and $\nabla_\theta L$ (resp. $\nabla_u L$) is its gradient with respect to $\underline{\theta}$ (resp. $\underline{u}$). Equation (13) implies

$$a_j u_j^* = \lambda_j^* \qquad j \in \mathcal{N}.$$

Now using (12) we get $\underline{\lambda}^{*T} L_B(\underline{\theta}^*) = 0$ where

$$L_B(\underline{\theta}) := CB \cdot \text{diag}\left(\underline{\cos}(C^T \underline{\theta})\right) \cdot C^T \tag{14}$$

is the Laplacian matrix of the graph $(\mathcal{N}, \mathcal{E})$ with weights $B_{jk} \cos(\theta_j^* - \theta_k^*)$. The function $\underline{\cos} : \mathbb{R}^{|\mathcal{E}|} \to \mathbb{R}^{|\mathcal{E}|}$ is defined such that if $\underline{y} = \underline{\cos}(\underline{\delta})$ then $y_e = \cos(\delta_e)$ for $e \in \mathcal{E}$. The incidence matrix $C \in \mathbb{R}^{|\mathcal{N}| \times |\mathcal{E}|}$ of $(\mathcal{N}, \mathcal{E})$ has $C_{je} = 1$ if $e = (j, k) \in \mathcal{E}$ for some $k \in \mathcal{N}$ and $C_{je} = -1$ if $e = (i, j) \in \mathcal{E}$ for some $i \in \mathcal{N}$, and $C_{je} = 0$ otherwise. The diagonal matrix $B = \text{diag}(B_{jk}, (j, k) \in \mathcal{E}) \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$ has its diagonal entries arranged in the same order as the columns of C. Since the graph is connected, using Assumption 1, it follows that the weights of $L_B(\underline{\theta})$ are strictly positive and therefore we must have $\lambda_j^* = \lambda^*$ for all $j \in \mathcal{N}$ and some $\lambda^* \in \mathbb{R}$. Therefore (12) is necessary for optimality.

($\Longleftarrow$): By adding (6) over $j \in \mathcal{N}$ we have (10). It follows that any feasible solution $(\underline{\theta}, \underline{u})$ of ED is also a feasible solution of RED. If we let opt(ED) and opt(RED) be the optimal values of ED and RED respectively, it follows that

$$\text{opt(RED)} \leq \text{opt(ED)}.$$

However, since $(\underline{\theta}^*, \underline{u}^*)$ satisfies (12), it is easy to show (by invoking the KKT conditions for RED) that $\underline{u}^*$ is an optimal solution of the convex RED problem. We also know that $(\underline{\theta}^*, \underline{u}^*)$ is strictly feasible for ED. Therefore $(\underline{\theta}^*, \underline{u}^*)$ is an optimal solution of ED. ∎

The next proposition relates the system dynamics (1)–(3) with the ED problem (5)–(7).

Proposition 1 (Optimality condition of equilibria): Under Assumption 1, a frequency-synchronized solution $(\underline{\theta}^*, \underline{\omega}^*, \underline{u}^*)$ of the system (1)–(3) is optimal for ED if and only if the following conditions are satisfied:

$$\omega_j^* = 0 \qquad j \in \mathcal{N} \tag{15a}$$

$$|\theta_j^* - \theta_k^*| < \gamma_{jk} < \frac{\pi}{2} \qquad (j, k) \in \mathcal{E} \tag{15b}$$

$$a_j u_j^* = a_k u_k^* \qquad j, k \in \mathcal{N}. \tag{15c}$$

Proof: ($\Longrightarrow$): Suppose $(\underline{\theta}^*, \underline{\omega}^*, \underline{u}^*)$ is a frequency-synchronized solution such that $(\underline{\theta}^*, \underline{u}^*)$ is an optimal solution of ED. Then the primal feasibility condition (6) holds for $(\underline{\theta}^*, \underline{u}^*)$, which implies (15a). Under Assumption 1, we have (15b). By Lemma 1 we have (15c).

($\Longleftarrow$): Now suppose there is a frequency-synchronized solution $(\underline{\theta}^*, \underline{\omega}^*, \underline{u}^*)$ satisfying (15). By (15a), the primal feasibility condition (6) is satisfied by $(\underline{\theta}^*, \underline{u}^*)$. This, together with (15b), guarantees the strict feasibility of $(\underline{\theta}^*, \underline{u}^*)$. By Lemma 1, $(\underline{\theta}^*, \underline{u}^*)$ is optimal for ED since (15c) holds. ∎

The remainder of the paper focuses on the following question: how to achieve frequency recovery (15a) while simultaneously achieving economic optimality (15c)?

III. COMPLETELY DECENTRALIZED FREQUENCY INTEGRAL CONTROL

We first look at frequency integral control

$$u_j = -K_j s_j \tag{16a}$$

$$\dot{s}_j = \omega_j \tag{16b}$$

which is completely decentralized in that every generator and controllable load only needs to take integral of the frequency deviation measured on its local bus without communication with other buses. The parameters $K_j > 0$ for $j \in \mathcal{N}$ are constant control gains. Without loss of generality, we take $s_i(0) = 0$, which allows us to rewrite (16) as

$$u_j(t) = -K_j \int_0^t \omega_j(\tau) d\tau \qquad j \in \mathcal{N}. \tag{17}$$

We select arbitrary parameters $K \succ 0$ and input $\underline{p}$, and fix them in the rest of this section. Define

$$\underline{F}(\underline{\theta}) := \underline{p} - K(\underline{\theta} - \underline{\theta}_0) - CB\underline{\sin}(C^T \underline{\theta}) \tag{18}$$

where $\underline{\theta}_0 := (\theta_j(0), j \in \mathcal{N})$ is a fixed parameter vector, the matrices C and B are again the incidence matrix and the diagonal matrix of B_{jk}, respectively, and the function $\underline{\sin} : \mathbb{R}^{|\mathcal{E}|} \to \mathbb{R}^{|\mathcal{E}|}$ is defined such that if $\underline{y} = \underline{\sin}(\underline{\delta})$ then

$y_e = \sin(\delta_e)$ for $e \in \mathcal{E}$. Then the set of equilibria of the closed-loop system (1)–(3) and (17) is

$$\Theta^* := \left\{ (\underline{\theta}, \underline{\omega}) \in \mathbb{R}^{2|\mathcal{N}|} \mid \underline{\omega} = 0, \underline{F}(\underline{\theta}) = 0 \right\}. \quad (19)$$

Theorem 1 below states the existence and global convergence to this set of closed-loop equilibria.

Theorem 1: The set Θ^* of equilibria is nonempty, and every trajectory $(\underline{\theta}(t), \underline{\omega}(t))$ of the closed-loop system (1)–(3) and (17) globally converges to Θ^* as $t \to +\infty$.

Proof: By (3)(17)(18) we have

$$\underline{\omega}_{\mathcal{L}}(t) \equiv D_{\mathcal{L}}^{-1} \underline{F}_{\mathcal{L}}(\underline{\theta}(t)), \quad (20)$$

i.e., $\underline{\omega}_{\mathcal{L}}(t)$ is a continuous function of $\underline{\theta}(t)$ for all t. Therefore we only need to show that

$$\Theta^*_{\mathcal{G}} := \left\{ (\underline{\theta}, \underline{\omega}_{\mathcal{G}}) \in \mathbb{R}^{|\mathcal{N}|+|\mathcal{G}|} \mid \underline{\omega}_{\mathcal{G}} = 0, \underline{F}(\underline{\theta}) = 0 \right\} \quad (21)$$

is nonempty and every trajectory $(\underline{\theta}(t), \underline{\omega}_{\mathcal{G}}(t))$ globally converges to $\Theta^*_{\mathcal{G}}$ as $t \to +\infty$.

Consider the Lyapunov function[1]

$$V(\underline{\theta}, \underline{\omega}_{\mathcal{G}}) = \frac{1}{2} \underline{\omega}_{\mathcal{G}}^T M_{\mathcal{G}} \underline{\omega}_{\mathcal{G}} + U(\underline{\theta}) + \sum_{j \in \mathcal{N}} K_j \theta_j \left(\frac{\theta_j}{2} - \theta_{0,j}\right) \quad (22)$$

where the open-loop potential energy is

$$U(\underline{\theta}) := \sum_{(j,k) \in \mathcal{E}} B_{jk}(1 - \cos(\theta_j - \theta_k)) - \sum_{j \in \mathcal{N}} p_j \theta_j. \quad (23)$$

The derivative of V along any trajectory is

$$\begin{aligned} \dot{V}(\underline{\theta}, \underline{\omega}_{\mathcal{G}}) &= \underline{\omega}_{\mathcal{G}}^T M_{\mathcal{G}} \dot{\underline{\omega}}_{\mathcal{G}} + \sum_{(j,k) \in \mathcal{E}} B_{jk} \sin(\theta_j - \theta_k)(\omega_j - \omega_k) \\ &\quad - \sum_{j \in \mathcal{N}} p_j \omega_j + \sum_{j \in \mathcal{N}} K_j (\theta_j - \theta_{0,j}) \omega_j \\ &= -\underline{\omega}_{\mathcal{G}}^T D_{\mathcal{G}} \underline{\omega}_{\mathcal{G}} + \underline{\omega}_{\mathcal{G}}^T \underline{F}_{\mathcal{G}}(\underline{\theta}) - \underline{\omega}^T \underline{F}(\underline{\theta}) \quad (24) \\ &= -\underline{\omega}_{\mathcal{G}}^T D_{\mathcal{G}} \underline{\omega}_{\mathcal{G}} - \underline{\omega}_{\mathcal{L}}^T(\underline{\theta}) D_{\mathcal{L}} \underline{\omega}_{\mathcal{L}}(\underline{\theta}) \le 0 \quad (25) \end{aligned}$$

where the equalities in (24) and (25) result from (2) and (3) respectively, and $\underline{\omega}_{\mathcal{L}}(\cdot)$ as a function of $\underline{\theta}$ is defined in (20). Take arbitrary $(\underline{\theta}(0), \underline{\omega}_{\mathcal{G}}(0)) \in \mathbb{R}^{|\mathcal{N}|+|\mathcal{G}|}$. Then the set

$$\Omega := \left\{ (\underline{\theta}, \underline{\omega}_{\mathcal{G}}) \mid V(\underline{\theta}, \underline{\omega}_{\mathcal{G}}) \le V(\underline{\theta}(0), \underline{\omega}_{\mathcal{G}}(0)) \right\} \quad (26)$$

is compact. Indeed Ω is closed due to continuity of V, and is bounded since V is radially unbounded (thanks to the dominating quadratic terms in θ_j and ω_j). Also Ω is invariant since $\dot{V} \le 0$. Define

$$\begin{aligned} E &:= \left\{ (\underline{\theta}, \underline{\omega}_{\mathcal{G}}) \in \mathbb{R}^{|\mathcal{N}|+|\mathcal{G}|} \mid \dot{V}(\underline{\theta}, \underline{\omega}_{\mathcal{G}}) = 0 \right\} \\ &= \left\{ (\underline{\theta}, \underline{\omega}_{\mathcal{G}}) \in \mathbb{R}^{|\mathcal{N}|+|\mathcal{G}|} \mid \underline{\omega}_{\mathcal{G}} = 0, \underline{\omega}_{\mathcal{L}}(\underline{\theta}) = 0 \right\} \quad (27) \end{aligned}$$

by (25). Let $E_\Omega := E \cap \Omega$. Let L^+_Ω be the largest invariant subset of E_Ω. Then by LaSalle's theorem [31, Theorem 4.4], the trajectory starting from $(\underline{\theta}(0), \underline{\omega}_{\mathcal{G}}(0))$ converges to a *nonempty*, compact, invariant limit set which is a subset of L^+_Ω. Hence it is sufficient to show $L^+_\Omega \subseteq \Theta^*_{\mathcal{G}}$.

Consider any point $(\underline{\theta}', \underline{\omega}'_{\mathcal{G}}) \in L^+_\Omega$. Due to the invariance of L^+_Ω, the trajectory $(\underline{\theta}(\tau), \underline{\omega}_{\mathcal{G}}(\tau))$ starting from $(\underline{\theta}', \underline{\omega}'_{\mathcal{G}})$ must stay in L^+_Ω and hence must stay in E_Ω. Therefore, by (27) we have $\underline{\omega}_{\mathcal{G}}(\tau) \equiv 0$ and hence $\dot{\underline{\omega}}_{\mathcal{G}}(\tau) \equiv 0$, and $\underline{\omega}_{\mathcal{L}}(\underline{\theta}(\tau)) \equiv 0$. It follows that $\underline{F}(\underline{\theta}(\tau)) \equiv 0$ and in particular, $(\underline{\theta}', \underline{\omega}'_{\mathcal{G}}) \in \Theta^*_{\mathcal{G}}$. Hence $L^+_\Omega \subseteq \Theta^*_{\mathcal{G}}$ as we wanted to prove. ■

Theorem 1 shows that the closed-loop system with controller (17) globally converges to the set Θ^* even in the case where the open-loop system (1)–(3) (with $\underline{u} = 0$) does not have an equilibrium. When Θ^* is composed by a finite number of isolated equilibria, which occurs with measure one on the set of system parameters [32], Theorem 1 implies that the system will always converge to one of them.

Unfortunately, it is in general not possible to control the final equilibrium to which the system will settle. In the next theorem, we show that if certain conditions on the gains K_j and line susceptances B_{jk} are satisfied, the set Θ^* contains a unique equilibrium which is globally asymptotically stable.

Theorem 2: If $K_j > 2 \sum_{k \in \mathcal{N}} B_{jk}$ for all $j \in \mathcal{N}$, then the closed-loop system (1)–(3) and (17) has a unique and globally asymptotically stable equilibrium.

Proof: The Jacobian matrix of function $\underline{F}$ is

$$\frac{\partial F}{\partial \underline{\theta}}(\underline{\theta}) = -K - L_B(\underline{\theta}) \quad (28)$$

where $L_B(\underline{\theta})$ is the Laplacian matrix defined in (14). We drop the argument $\underline{\theta}$ of $\frac{\partial F}{\partial \underline{\theta}}$, and denote the (j,k)-th entry of $\frac{\partial F}{\partial \underline{\theta}}$ by $\left(\frac{\partial F}{\partial \underline{\theta}}\right)_{jk}$. By Gershgorin circle theorem [33], for any eigenvalue λ of $\frac{\partial F}{\partial \underline{\theta}}$, there exists $j \in \mathcal{N}$ such that

$$\left| \lambda - \left(\frac{\partial F}{\partial \underline{\theta}}\right)_{jj} \right| \le \sum_{k \in \mathcal{N}, k \ne j} \left| \left(\frac{\partial F}{\partial \underline{\theta}}\right)_{jk} \right|. \quad (29)$$

On the other hand, by assumption we have

$$K_j > 2 \sum_{k \in \mathcal{N}} B_{jk} \ge \sum_{k \in \mathcal{N}} B_{jk}(|\cos \theta_{jk}| - \cos \theta_{jk}) \qquad j \in \mathcal{N}$$

where $\theta_{jk} := \theta_j - \theta_k$, which, by (28), implies

$$\left(\frac{\partial F}{\partial \underline{\theta}}\right)_{jj} + \sum_{k \in \mathcal{N}, k \ne j} \left| \left(\frac{\partial F}{\partial \underline{\theta}}\right)_{jk} \right| < 0 \qquad j \in \mathcal{N}. \quad (30)$$

By (29)(30), we have $\frac{\partial F}{\partial \underline{\theta}}(\underline{\theta}) \prec 0$ for all $\underline{\theta} \in \mathbb{R}^{|\mathcal{N}|}$.

Now suppose there are $\underline{\theta}, \underline{\theta}' \in \mathbb{R}^{|\mathcal{N}|}$ such that $\underline{\theta} \ne \underline{\theta}'$ and $\underline{F}(\underline{\theta}') = \underline{F}(\underline{\theta}) = 0$. Then we have, by the fundamental theorem of calculus [34], that

$$\begin{aligned} 0 &= \underline{F}(\underline{\theta}') - \underline{F}(\underline{\theta}) \\ &= \left[\int_0^1 \frac{\partial F}{\partial \underline{\theta}}(\underline{\theta} + h\Delta\underline{\theta}) dh \right] \Delta\underline{\theta} \quad (31) \end{aligned}$$

where $\Delta\underline{\theta} := \underline{\theta}' - \underline{\theta} \ne 0$. Notice that the integral term in (31), denoted by int_F, is negative definite since $\frac{\partial F}{\partial \underline{\theta}}(\underline{\theta} + h\Delta\underline{\theta}) \prec 0$ for all $h \in [0,1]$. Hence we have $\Delta\underline{\theta}^T \cdot \text{int}_F \cdot \Delta\underline{\theta} < 0$. However we have from (31) that $\Delta\underline{\theta}^T \cdot \text{int}_F \cdot \Delta\underline{\theta} = 0$ which

[1] We remark that one can construct a strictly decreasing Lyapunov function (outside equilibria) by applying Chetaev's trick [30] and adding the cross-term $\varepsilon \left(\nabla_{\mathcal{G}} U(\underline{\theta}) - \nabla_{\mathcal{G}} U(\underline{\theta}^*)\right)^T M_{\mathcal{G}} \underline{\omega}_{\mathcal{G}}$ to $V(\underline{\theta}, \underline{\omega}_{\mathcal{G}})$.

leads to a contradiction. Therefore, by the nonemptiness of Θ^*, there is a unique equilibrium $(\underline{\theta}^*, \underline{\omega}^* = 0) \in \Theta^*$.

Using again the function $V(\underline{\theta}, \underline{\omega}_{\mathcal{G}})$ as defined in (22), the compactness of (26) together with $\dot{V} \leq 0$ implies stability, and by Theorem 1, the trajectories converge *globally* to the unique equilibrium point in Θ^*. Thus this equilibrium is globally asymptotically stable. ■

The completely decentralized integral control successfully achieves *global* asymptotic stability without assuming knowledge of the system parameters in the controller design. To the best of our knowledge there is no other decentralized control strategy for structure-preserving power network models that leads to a globally convergent closed-loop system.

However, the resulting equilibrium may be neither an optimal nor a feasible solution of ED in Section II. Additionally, our theoretical results require controllers at every bus, and Theorem 2 requires large gains K_j, which may be impractical and lead to large control actions.

While having ubiquitous controllers is still a limitation of our design, in the next section we remedy the remaining disadvantages by introducing a distributed control action that corrects the steady-state solution and recovers optimality.

IV. DISTRIBUTED AVERAGING-BASED INTEGRAL CONTROL

To simultaneously address the objectives of frequency regulation and economic dispatch, we merge the integral control (17) with a distributed consensus filter. Consider the following distributed averaging-based integral (*DAI*) control

$$
\begin{aligned}
u_j &= -K_j s_j - R_j q_j & j \in \mathcal{N} \quad &(32a)\\
\dot{s}_j &= \omega_j & j \in \mathcal{N} \quad &(32b)\\
\dot{q}_j &= Q_j \sum_{k\in\mathcal{N}} Y_{jk}(a_j u_j - a_k u_k) & j \in \mathcal{N} \quad &(32c)
\end{aligned}
$$

where $K_j, R_j, Q_j > 0$ for $j \in \mathcal{N}$ are control gains, and the weights $Y_{jk} \geq 0$ for $j, k \in \mathcal{N}$ induce an *undirected* and connected communication graph, i.e., $Y_{jk} = Y_{kj} > 0$ when the local controllers at buses j and k communicate, otherwise $Y_{jk} = Y_{kj} = 0$, and $Y_{jj} = 0$ for $j \in \mathcal{N}$.

It is easy to see that any point $(\underline{\theta}^*, \underline{\omega}^*, \underline{u}^*) \in \mathbb{R}^{3|\mathcal{N}|}$ is an equilibrium of the closed-loop system (1)–(3) and (32) if and only if it satisfies

$$
\begin{aligned}
\underline{\omega}^* &= 0 & (33a)\\
\nabla U(\underline{\theta}^*) &= \underline{u}^* & (33b)\\
\underline{u}^* &= \gamma A^{-1} 1_{|\mathcal{N}|} & (33c)
\end{aligned}
$$

where U is defined in (23) and ∇U is its gradient, $\gamma := -\sum_{j\in\mathcal{N}} p_j / \sum_{j\in\mathcal{N}} a_j^{-1}$ by summing over all the equations in (33b), and $A := \text{diag}(a_j, j \in \mathcal{N})$. Hence $(\underline{\omega}^*, \underline{u}^*)$ exists and is unique. We make the following assumption regarding the existence of $\underline{\theta}^*$ and its *strict* feasibility for ED.

Assumption 2: Assume that the closed-loop system (1)–(3) and (32) features a set of equilibria $(\underline{\theta}^*, \underline{\omega}^*, \underline{u}^*)$ that satisfy (33) and (8).

In simulations we observe that the *DAI* control (32) is stable for an arbitrary positive choice of control gains. To simplify the following presentation, we choose simple control gains for stability analysis.

Assumption 3: We choose simple control gains: $Q = A$ and $K = R = T^{-1}A^{-1}$ with T being an arbitrary diagonal and positive definite matrix.

We remark that with this choice of gains, the *DAI* control (32) includes with the *DAPI* control proposed in [1], [26]. The controller in [1], [26] makes the additional parametric assumption $D = A^{-1}$ and merges the variables $s_j + q_j$.

Theorem 3: Suppose the ED problem in (5)–(7) satisfies Assumption 1. Suppose the closed-loop system (1)–(3) and (32) has a nonempty set of equilibria as given in Assumption 2, and the control gains are selected as in Assumption 3. Then these equilibria are locally asymptotically stable and optimal for ED.

Proof: We partition the states $\underline{\theta}$ and $\underline{u}$ as well as the matrix D of droop coefficients according to $\mathcal{N} = \mathcal{G} \cup \mathcal{L}$. Consider the auxiliary variable

$$\underline{y} = -\underline{u}.$$

Under Assumptions 2 and 3, we obtain the following state space model for the incremental closed-loop system:

$$
\begin{aligned}
M_{\mathcal{G}}\dot{\underline{\omega}}_{\mathcal{G}} &= -D_{\mathcal{G}}\underline{\omega}_{\mathcal{G}} - (\nabla_{\mathcal{G}}U(\underline{\theta}) - \nabla_{\mathcal{G}}U(\underline{\theta}^*)) - (\underline{y}_{\mathcal{G}} - \underline{y}_{\mathcal{G}}^*) & (34a)\\
0 &= -D_{\mathcal{L}}\dot{\underline{\theta}}_{\mathcal{L}} - (\nabla_{\mathcal{L}}U(\underline{\theta}) - \nabla_{\mathcal{L}}U(\underline{\theta}^*)) - (\underline{y}_{\mathcal{L}} - \underline{y}_{\mathcal{L}}^*) & (34b)\\
\dot{\underline{\theta}}_{\mathcal{G}} &= \omega_{\mathcal{G}} & (34c)\\
T\dot{\underline{y}} &= A^{-1}\dot{\underline{\theta}} - L_Y A(\underline{y} - \underline{y}^*) & (34d)
\end{aligned}
$$

where L_Y is the Laplacian matrix of the communication graph. To analyze the system (34) we choose the following incremental Lyapunov function candidate inspired by [21]:

$$
\begin{aligned}
V(\underline{\theta}, \underline{\omega}_{\mathcal{G}}, \underline{y}) = &\frac{1}{2}\underline{\omega}_{\mathcal{G}}^T M_{\mathcal{G}}\underline{\omega}_{\mathcal{G}} + U(\underline{\theta}) - U(\underline{\theta}^*)\\
&-\nabla U(\underline{\theta}^*)(\underline{\theta} - \underline{\theta}^*) + \frac{1}{2}(\underline{y} - \underline{y}^*)^T AT(\underline{y} - \underline{y}^*) \quad (35)
\end{aligned}
$$

For an equilibrium $\underline{\theta}^*$ satisfying Assumptions 1, $|\theta_j^* - \theta_k^*| < \pi/2$ for all $(j, k) \in \mathcal{E}$, the Hessian $\nabla^2 U(\underline{\theta}^*)$ is positive semidefinite with the only nullspace $1_{|\mathcal{N}|}$ corresponding to the rotational symmetry of equilibria $\underline{\theta}^*$. It follows that the incremental Lyapunov function $V(\underline{\theta}, \underline{\omega}_{\mathcal{G}}, \underline{y})$ in (35) is locally positive definite with respect to the equilibrium subspace.

The derivative of $V(\underline{\theta}, \underline{\omega}_{\mathcal{G}}, \underline{y})$ along trajectories of (34) is

$$
\begin{aligned}
&\dot{V}(\underline{\theta}, \underline{\omega}_{\mathcal{G}}, \underline{y})\\
&= \underline{\omega}_{\mathcal{G}}^T M_{\mathcal{G}}\dot{\underline{\omega}}_{\mathcal{G}} + (\nabla_{\mathcal{L}}U - \nabla_{\mathcal{L}}U^*)^T\, \dot{\underline{\theta}}_{\mathcal{L}}\\
&\quad + (\nabla_{\mathcal{G}}U - \nabla_{\mathcal{G}}U^*)^T\, \dot{\underline{\theta}}_{\mathcal{G}} + (\underline{y} - \underline{y}^*)^T AT\dot{\underline{y}}\\
&= -\underline{\omega}_{\mathcal{G}}^T D_{\mathcal{G}}\underline{\omega}_{\mathcal{G}} - \underline{\omega}_{\mathcal{G}}^T (\nabla_{\mathcal{G}}U - \nabla_{\mathcal{G}}U^*) - \underline{\omega}_{\mathcal{G}}^T(\underline{y}_{\mathcal{G}} - \underline{y}_{\mathcal{G}}^*)\\
&\quad - (\nabla_{\mathcal{L}}U - \nabla_{\mathcal{L}}U^*)^T D_{\mathcal{L}}^{-1} (\nabla_{\mathcal{L}}U - \nabla_{\mathcal{L}}U^*)\\
&\quad - (\nabla_{\mathcal{L}}U - \nabla_{\mathcal{L}}U^*)^T D_{\mathcal{L}}^{-1}(\underline{y}_{\mathcal{L}} - \underline{y}_{\mathcal{L}}^*)\\
&\quad + (\nabla_{\mathcal{G}}U - \nabla_{\mathcal{G}}U^*)^T\, \underline{\omega}_{\mathcal{G}} + (\underline{y}_{\mathcal{G}} - \underline{y}_{\mathcal{G}}^*)^T \underline{\omega}_{\mathcal{G}}\\
&\quad - (\underline{y}_{\mathcal{L}} - \underline{y}_{\mathcal{L}}^*)^T D_{\mathcal{L}}^{-1} (\nabla_{\mathcal{L}}U - \nabla_{\mathcal{L}}U^*)
\end{aligned}
$$

$$
\begin{aligned}
&-(\underline{y}_{\mathcal{L}}-\underline{y}_{\mathcal{L}}^*)^T D_{\mathcal{L}}^{-1}(\underline{y}_{\mathcal{L}}-\underline{y}_{\mathcal{L}}^*)-(\underline{y}-\underline{y}^*)^T A L_Y A(\underline{y}-\underline{y}^*)\\
=& -\underline{\omega}_{\mathcal{G}}^T D_{\mathcal{G}}\underline{\omega}_{\mathcal{G}} - (\underline{y}-\underline{y}^*)^T A L_Y A(\underline{y}-\underline{y}^*)\\
&-\left(\nabla_{\mathcal{L}}U-\nabla_{\mathcal{L}}U^*+\underline{y}_{\mathcal{L}}-\underline{y}_{\mathcal{L}}^*\right)^T D_{\mathcal{L}}^{-1}\\
&\cdot\left(\nabla_{\mathcal{L}}U-\nabla_{\mathcal{L}}U^*+\underline{y}_{\mathcal{L}}-\underline{y}_{\mathcal{L}}^*\right)
\end{aligned}
\tag{36}
$$

where ∇U and ∇U^* denote $\nabla U(\underline{\theta})$ and $\nabla U(\underline{\theta}^*)$ respectively. Hence, the Lyapunov function $V(\underline{\theta}, \underline{\omega}_{\mathcal{G}}, \underline{y})$ is non-increasing. We construct a strictly decreasing Lyapunov function by applying Chetaev's trick [30] and adding the cross-term $\varepsilon\left(\nabla_{\mathcal{G}}U(\underline{\theta})-\nabla_{\mathcal{G}}U(\underline{\theta}^*)\right)^T M_{\mathcal{G}}\underline{\omega}_{\mathcal{G}}$ to the Lyapunov function for some sufficiently small $\varepsilon > 0$.

Some straightforward calculations give the following expression for the time-derivative of the additional cross-term:

$$
\begin{aligned}
&\frac{d}{dt}\varepsilon\left(\nabla_{\mathcal{G}}U-\nabla_{\mathcal{G}}U^*\right)^T M_{\mathcal{G}}\underline{\omega}_{\mathcal{G}}\\
=&\ \varepsilon\left(\nabla_{\mathcal{G}}U-\nabla_{\mathcal{G}}U^*\right)^T\Big(-D_{\mathcal{G}}\underline{\omega}_{\mathcal{G}}-(\nabla_{\mathcal{G}}U-\nabla_{\mathcal{G}}U^*)\\
&-(\underline{y}_{\mathcal{G}}-\underline{y}_{\mathcal{G}}^*)\Big)+\varepsilon\underline{\omega}_{\mathcal{G}}^T L_{\mathcal{G}}\underline{\omega}_{\mathcal{G}}\,.
\end{aligned}
$$

Here $L_{\mathcal{G}}$ is a non-symmetric Laplacian matrix associated to the graph with state-dependent weights $M_j B_{jk}\cos(\theta_j-\theta_k)$. Note that the weights are strictly lower and upper bounded. Consider now the augmented incremental Lyapunov function

$$
\tilde{V}(\underline{\theta}, \underline{\omega}_{\mathcal{G}}, \underline{y}) = V(\underline{\theta}, \underline{\omega}_{\mathcal{G}}, \underline{y}) + \varepsilon\left(\nabla_{\mathcal{G}}U-\nabla_{\mathcal{G}}U^*\right)^T M_{\mathcal{G}}\underline{\omega}_{\mathcal{G}}\,.
$$

Its time-derivative along trajectories of (34) is

$$
\dot{\tilde{V}}(\underline{\theta}, \underline{\omega}_{\mathcal{G}}, \underline{y}) = -\begin{bmatrix}\underline{\omega}_{\mathcal{G}}\\ \nabla_{\mathcal{G}}U-\nabla_{\mathcal{G}}U^*\\ \underline{y}-\underline{y}^*\\ \nabla_{\mathcal{L}}U-\nabla_{\mathcal{L}}U^*\end{bmatrix}^T Q \begin{bmatrix}\underline{\omega}_{\mathcal{G}}\\ \nabla_{\mathcal{G}}U-\nabla_{\mathcal{G}}U^*\\ \underline{y}-\underline{y}^*\\ \nabla_{\mathcal{L}}U-\nabla_{\mathcal{L}}U^*\end{bmatrix}
$$

where the matrix Q is defined in (37).

To verify that the Lyapunov function is strictly decreasing outside equilibria, observe the following: (i) The diagonal upper-left block of Q is positive definite for $\varepsilon > 0$ sufficiently small. Indeed, εI is positive definite, and the Schur complement $D_{\mathcal{G}} - \varepsilon^2 D_{\mathcal{G}}^2/8(L_{\mathcal{G}}+L_{\mathcal{G}}^T)$ is positive definite as well for $\varepsilon > 0$ sufficiently small and any $\underline{\theta} \in \mathbb{R}^{|\mathcal{N}|}$. (ii) The Lyapunov function is decreasing for $\varepsilon > 0$ sufficiently small if it is decreasing with the off-diagonal (upper-right and lower-left) blocks of Q set to zero. (iii) Finally, the diagonal lower-right block of Q is positive semidefinite with nullspace

$$
\begin{bmatrix}\underline{y}_{\mathcal{G}}-\underline{y}_{\mathcal{G}}^*\\ \underline{y}_{\mathcal{L}}-\underline{y}_{\mathcal{L}}^*\\ \nabla_{\mathcal{L}}U-\nabla_{\mathcal{L}}U^*\end{bmatrix} \in \operatorname{span}\left(\begin{bmatrix}\mathbf{0}_{|\mathcal{G}|}\\ A^{-1}\mathbf{1}_{|\mathcal{L}|}\\ -A^{-1}\mathbf{1}_{|\mathcal{L}|}\end{bmatrix}\right). \tag{38}
$$

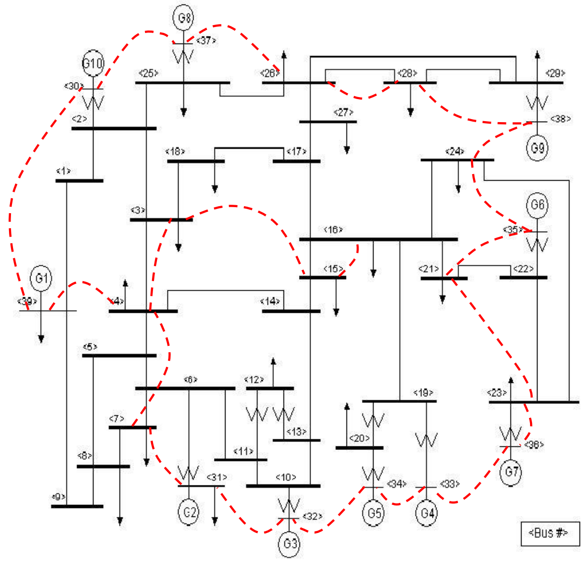


Fig. 1. IEEE New England test system [35]. The red dashed lines represent communication links between generators and controllable loads.

We conclude that $\tilde{V}(\underline{\theta}, \underline{\omega}_{\mathcal{G}}, \underline{y})$ is non-increasing, and its derivative is zero for (38) and for the equilibria

$$
\begin{bmatrix}\omega_{\mathcal{G}}\\ \nabla_{\mathcal{G}}U-\nabla_{\mathcal{G}}U^*\end{bmatrix} = \begin{bmatrix}\mathbf{0}_{|\mathcal{G}|}\\ \mathbf{0}_{|\mathcal{G}|}\end{bmatrix}. \tag{39}
$$

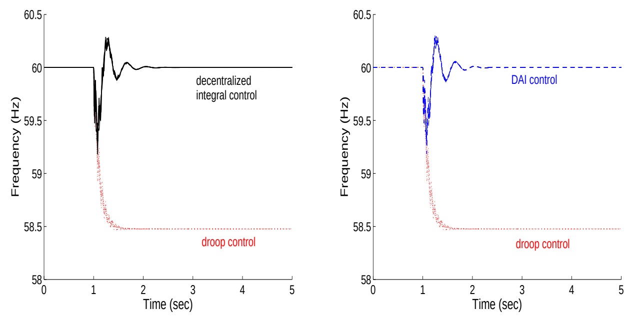
However, since $\nabla U - \nabla U^*$ is orthogonal to $\mathbf{1}_{|\mathcal{N}|}$ conditions (38) and (39) can be true simultaneously only if $\nabla_{\mathcal{L}}U - \nabla_{\mathcal{L}}U^* = \mathbf{0}_{|\mathcal{L}|}$ and $\underline{y}_{\mathcal{L}} - \underline{y}_{\mathcal{L}}^* = \mathbf{0}_{|\mathcal{L}|}$. We conclude that the Lyapunov function $\tilde{V}$ is strictly decreasing outside equilibria.

Since the Lyapunov function $\tilde{V}$ is also locally positive definite with respect to equilibrium set satisfying $|\theta_j^* - \theta_k^*| < \pi/2$ for all $(j,k) \in \mathcal{E}$, it follows that these equilibria are locally asymptotically stable. By Assumptions 1–2, $(\underline{\theta}^*, \underline{u}^*)$ is strictly feasible for ED and satisfies the optimality condition (12). Invoking Lemma 1, $(\underline{\theta}^*, \underline{u}^*)$ is optimal for ED. ■

V. SIMULATION CASE STUDY

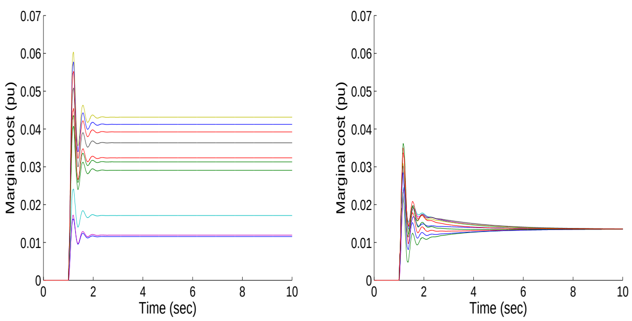
In this section we evaluate the performance of the proposed controllers using the IEEE New England test system shown in Fig. 1. This system has 10 generators and 39 buses, and serves a total load of about 6 GW. The generator inertia moments M_j and line susceptances B_{jk} are obtained from Power System Toolbox [35]. We choose uniform droop coefficients $D_j = 1$ pu for all the buses. Although the theoretical analysis requires controllers at every bus of the network, here we only control the generators and the loads on buses 3, 4, 7, 15, 16, 21, 23, 24, 26, 28, using uniform gains $K_j = 60$ pu and $R_j = 1$ pu. For the *DAI* control, a communication

$$
Q = \left[\begin{array}{cc|cc}
D_{\mathcal{G}}-\varepsilon(L_{\mathcal{G}}+L_{\mathcal{G}}^T)/2 & \varepsilon/2\cdot D_{\mathcal{G}} & 0 & 0\\
\varepsilon/2\cdot D_{\mathcal{G}} & \varepsilon I & \varepsilon/2\cdot\left[\begin{smallmatrix} I\\ 0\end{smallmatrix}\right] & 0\\
\hline
0 & \varepsilon/2\cdot\left[\begin{smallmatrix} I & 0\end{smallmatrix}\right] & AL_YA+\left[\begin{smallmatrix} 0 & 0\\ 0 & D_{\mathcal{L}}^{-1}\end{smallmatrix}\right] & \left[\begin{smallmatrix} 0\\ D_{\mathcal{L}}^{-1}\end{smallmatrix}\right]\\
0 & 0 & \left[\begin{smallmatrix} 0 & D_{\mathcal{L}}^{-1}\end{smallmatrix}\right] & D_{\mathcal{L}}^{-1}
\end{array}\right] \tag{37}
$$



(a) (b)

Fig. 2. Frequencies of generators 2, 4, 6, 8, 10, under droop control, the completely decentralized integral control, and *DAI*.



(a) (b)

Fig. 3. Marginal costs $a_j u_j$ for generators 2, 4, 6, 8, 10 and controllable loads on buses 4, 15, 21, 24, 28, under the completely decentralized integral control in (a) and the *DAI* control in (b).

graph connecting generators and controllable loads is shown in Fig. 1, with $Y_{jk} = 1$ for all connected pairs (j, k). We select controller gains $Q_j = 50/\text{degree}(j)$ where degree(j) denotes the degree of bus j in the communication graph. The coefficients a_j of economic dispatch are generated uniformly randomly from $[0, 1]$.

In the simulation, the system is initially at a supply-demand balanced setpoint with 60 Hz frequency. At time $t = 1$ second, buses 4, 12, 20 each makes a 33 MW step change in real power consumption, causing bus frequencies to drop. Figure 2 shows the frequencies of five generators, under cases with different control schemes: droop control, the completely decentralized integral control, and *DAI*. It can be seen that while droop control synchronizes bus frequencies to lower than 60 Hz, both the decentralized integral control and *DAI* recover bus frequencies to 60 Hz, with similar transients.

Figure 3 shows the trajectories of marginal costs $a_j u_j$, under the completely decentralized integral control and the *DAI* control. While at the equilibrium of the decentralized integral control the marginal costs are different across the generators and controllable loads, they are the same under *DAI*, which, by Proposition 1, implies that economic dispatch is solved by *DAI*. Moreover, for most of the displayed generators and controllable loads, *DAI* reduces both transient and steady-state control actions compared to the decentralized

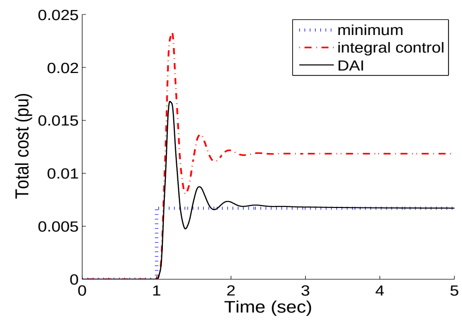


Fig. 4. Trajectories of economic dispatch objective for the completely decentralized integral control and *DAI*. The blue dotted line shows the minimum objective value of economic dispatch.

integral control.

In Fig. 4 we compare the objective values of economic dispatch, i.e., total costs of control and a measure for the control effort, along trajectories of control actions of the completely decentralized integral control and *DAI*, and compare them with the minimum objective value of economic dispatch for the given step change in load. We see that *DAI* achieves a better transient performance, a smaller total cost of control, as well as a more economic stead-steady state compared to the decentralized integral control, and indeed solves the economic dispatch problem at equilibrium.

VI. CONCLUSIONS

In this paper we proposed two control strategies–a completely decentralized integral control and a distributed averaging-based integral (*DAI*) control–that can be implemented using generators or loads. We showed that the decentralized integral control can achieve global asymptotic stability after arbitrary changes in generation or load. However, the resulting equilibrium may be neither optimal nor feasible for economic dispatch. Thus, we proposed the *DAI* control, for which local asymptotic stability of the closed-loop system was proved. Simulations demonstrated that *DAI* preserves similar convergence properties as the decentralized integral control, and achieves the desired economic dispatch performance.

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