

Voltage Stability of Droop-Controlled Microgrids* (Extended Abstract)

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I. INTRODUCTION

As the deployment of distributed generation continues, an increasing number of *DC/AC power inverters* will be interfaced into the electric grid. Groupings of these power inverters, along with distributed storage and load, are referred to as *microgrids*. Microgrids offer the potential for unprecedented reliability and flexibility, but also pose new control challenges. Inverters in microgrids must be controlled not only to maintain network stability but also to achieve ancillary objectives such as frequency regulation and power sharing. A simple microgrid is depicted in Figure 1.

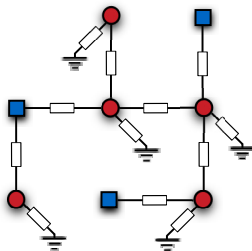


Fig. 1. A simple non-parallel microgrid consisting of five loads ● and three inverters ■.

One of the most fundamental requirements in microgrids is voltage stability, which is the requirement of ensuring the existence and stability of a power flow solution with high voltage magnitudes. In this presentation, we discuss the problem of voltage stabilization through droop control, where the inverter voltage magnitude is controlled as a function of the reactive power injection. Here, we improve upon the conventional linear droop control by introducing the *quadratic droop controller*, which respects the asymmetric and quadratic nature of the reactive power flow. As a result, the closed-loop can be analyzed using circuit-theoretic methods. We provide parametric conditions — some necessary and some sufficient — for solvability of the closed-loop circuit equations and stability of a high-voltage solution. These conditions have the form that the loads should not

be overly inductive while the fictitious (controller-internal) voltages should be sufficiently high.

II. PROBLEM SETUP

In inductive networks, the existence and stability of a power flow solution with high voltage magnitudes is associated with the solvability of the reactive power flow equations given by

$$Q_i(E, \theta) = - \sum_{j=1}^n E_i E_j B_{ij} \cos(\theta_i - \theta_j). \quad (1)$$

Here $Q_i \in \mathbb{R}$ is the reactive power injection at the i^{th} node in the network, $E_i \geq 0$ is the i^{th} voltage magnitude, $\theta_i \in \mathbb{S}$ is the voltage phase angle, and $B \in \mathbb{R}^{n \times n}$ is the network susceptance matrix. As solutions in practical networks typically have the property that $|\theta_i - \theta_j| \ll 1$ for all branches $\{i, j\}$, the equations (1) are generally insensitive to changes in the angular differences, and are classically approximated by the *decoupled reactive power flow equations*

$$Q_i(E) = - \sum_{j=1}^n B_{ij} E_i E_j, \quad i \in \{1, \dots, n\}. \quad (2)$$

Defining $L_{ij} \triangleq -B_{ij}$, in the absence of shunt loads it holds that $L \geq 0$ is a Laplacian matrix, and the set of n equations (2) can be rewritten in vector form as

$$Q = \text{diag}(E) L E, \quad (3)$$

where $E \in \mathbb{R}^n$ is the vector of voltage magnitudes. In the following we partition the nodes of the network into loads and inverters as $\{1, \dots, n\} = \{\mathcal{V}_L, \mathcal{V}_I\}$. The inverters voltage magnitudes are typically controlled via the *voltage-droop* controller [1]

$$\tau_i \dot{E}_i = -\tilde{C}_i (E_i - E_i^*) - Q_i(E), \quad i \in \mathcal{V}_I, \quad (4)$$

where $C_i > 0$ is the droop gain and E_i^* is the nominal voltage magnitude for the inverter. In steady-state, this controller imposes a linear relationship between voltage magnitude and reactive power magnitude, with the goal of sharing the reactive power load in the network among the inverters while simultaneously maintaining voltage stability. Meanwhile, the loads must satisfy the algebraic power balance equations

$$Q_i^*(E_i) - Q_i(E) = 0, \quad i \in \mathcal{V}_L, \quad (5)$$

where $Q_i^*(E_i)$ is a model of the local reactive power load. In this presentation, we consider the most difficult case of stiff constant-power loads $Q_i^*(E_i) = Q_i^* \in \mathbb{R}$. All of the results in our presentation can be easily extended to constant-current, constant-impedance, and general “ZIP” loads.

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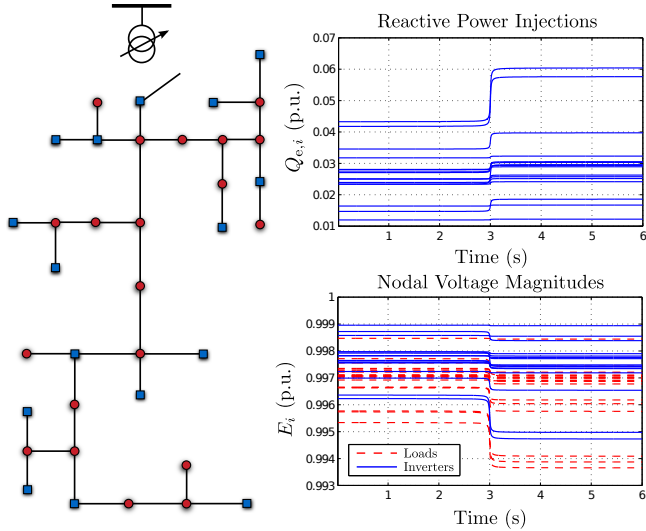


Fig. 2. IEEE 37 bus distribution network containing loads \bullet and inverters \blacksquare , along with simulation results when controlled with the quadratic droop controller (6).

III. QUADRATIC DROOP CONTROL

The conventional droop-controller (4) is based on the linearized behavior of the reactive power flow (2) near an operating point, but it does not take into account the true nonlinear nature of power flow. This conflict between linearization and nonlinear power flow complicates the closed-loop analysis of the system (4)–(5). In keeping with the network physics, we propose the *quadratic droop controller* [2]

$$\tau_i \dot{E}_i = -C_i E_i (E_i - E_i^*) - Q_i(E), \quad i \in \mathcal{V}_I, \quad (6)$$

where $C_i > 0$. The quadratic droop controller (6) respects the quadratic and inherently asymmetric nature of the reactive power flow equations (2). In steady-state, the controller action can be physically interpreted as connecting a fictitious node with voltage E_i^* to node i through a susceptance with value $-C_i$. Figure 2 shows a typical simulation of the quadratic droop controller implemented in a modified version of the IEEE37 distribution network.

We now ask the following question [3], [4]: “under what conditions on the network topology, susceptances, controller gains, and loads does the closed-loop system (5)–(6) possess a stable, high-voltage equilibrium point?”

To answer this question, we partition the network and voltage vector according to loads and inverters as

$$L = \begin{pmatrix} L_{LL} & L_{LI} \\ L_{IL} & L_{II} \end{pmatrix}, \quad E = (E_L, E_I)^T.$$

Let C_I be the diagonal matrix of controller gains, let E_I^* be the vector of nominal inverter voltages, and define the

shorthands

$$L_{\text{red}} \triangleq L_{LL} - L_{LI} (C_I + L_{II})^{-1} L_{IL} \in \mathbb{R}^{|\mathcal{V}_L| \times |\mathcal{V}_L|}, \quad (7)$$

$$W_1 \triangleq -L_{\text{red}}^{-1} L_{LI} (L_{II} + C_I)^{-1} C_I \in \mathbb{R}^{|\mathcal{V}_L| \times |\mathcal{V}_I|}, \quad (8)$$

$$E_{\text{avg}}^* \triangleq W_1 E_I^* \in \mathbb{R}_{>0}^{|\mathcal{V}_L|}, \quad (9)$$

$$W_2 \triangleq (L_{II} + C_I)^{-1} (-L_{IL} \quad C_I) \in \mathbb{R}^{|\mathcal{V}_I| \times n}. \quad (10)$$

With this notation, the equilibria of the closed-loop system (5)–(6) can be shown to be in one-to-one correspondence with the solutions of the following reduced power flow equation

$$Q_L = \text{diag}(E_L) L_{\text{red}} (E_L - E_{\text{avg}}^*), \quad (11)$$

where Q_L is the vector of reactive power loads, with the vector of inverter voltages being given by

$$E_I = W_2 \begin{pmatrix} E_L \\ E_I^* \end{pmatrix} \in \mathbb{R}_{>0}^{|\mathcal{V}_I|}. \quad (12)$$

The matrices W_1 and W_2 can be shown to be row-stochastic, which highlights the underlying *averaging* behavior of the network. Hence, the elements of E_{avg}^* are convex combinations of the inverter set points, and the reduced power flow equation (11) expresses the trade off between loading and the zero-load voltage profile E_{avg}^* . This reduction process is shown pictorially in Figure 3. A recently derived necessary condition for solvability of systems of the form (11) is that [5]

$$\sum_{j \in \mathcal{V}_L} Q_j^* \geq -\frac{1}{4} (E_{\text{avg}}^*)^T L_{\text{red}} E_{\text{avg}}^*. \quad (13)$$

In words, the total load should not be overly inductive, and the fictitious voltages should be sufficiently high.

A. Parallel Network Topology

The reduced power flow equation (11) can be solved exactly for a simple parallel (or star) topology in which there is one load fed by multiple inverters. In this case, the necessary condition (13) is also sufficient for solvability of the equations and the existence and stability of a high voltage equilibrium given by [2]

$$E_L^+ = \frac{E_{\text{avg}}^*}{2} \left(1 + \sqrt{1 - \frac{Q_L}{Q_{\text{crit}}}} \right), \quad (14a)$$

$$E_i^+ = \frac{C_i E_i^* + |b_{iL}| E_L^+}{C_i + |b_{iL}|}, \quad i \in \mathcal{V}_I. \quad (14b)$$

where $Q_{\text{crit}} < 0$ is given by the right hand side of (13), and b_{iL} is the susceptance connecting the i th inverter and the load. The solution has the additional property that it is strictly lower-bounded component-wise by the strictly positive vector

$$E_{\text{crit}} \triangleq \left(\frac{E_{\text{avg}}^*}{2}; W_2 \begin{pmatrix} E_{\text{avg}}^* \\ E_I^* \end{pmatrix} \right).$$

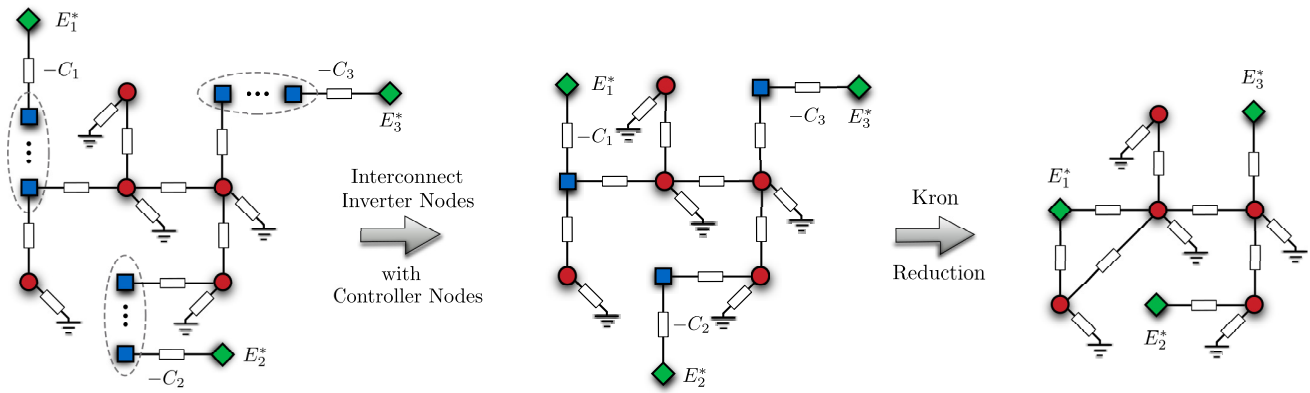


Fig. 3. Diagram showing network augmentation and reduction. First, each inverter node \blacksquare of the network in Figure ?? is interconnected with a two-node controller circuit, consisting of an inverter node \blacksquare and fictitious node \blacklozenge at constant voltage E_i^* . The inverter nodes are eliminated via Kron reduction, leaving a reduced network with only fixed voltage nodes \blacklozenge and load nodes \bullet .

B. General Network Topologies

For general network topologies, the fixed point equation (11) possesses multiple solutions which cannot be determined analytically. However, using recent approximation techniques [6], [7], an approximate solution can be computed. Indeed, if the minimum nominal inverter voltage $\min_{i \in \mathcal{V}_I} E_i^*$ is sufficiently high, then a locally exponentially stable high-voltage equilibrium can be shown to exist. Strictly for simplicity, assume that $E_i^* = E_N > 0$ for all $i \in \mathcal{V}_I$. Then for sufficiently large E_N , the stable fixed point exists and has the form [8]

$$\begin{bmatrix} E_L \\ E_I \end{bmatrix} = E_N \mathbb{1}_n + \frac{1}{E_N} \begin{bmatrix} X_{LL} \\ X_{IL} \end{bmatrix} Q_L + e_{\text{eq}}, \quad (15)$$

where

$$X = \begin{bmatrix} X_{LL} & X_{LI} \\ X_{IL} & X_{II} \end{bmatrix} \triangleq \begin{bmatrix} L_{LL} & L_{LI} \\ L_{IL} & L_{II} + C_I \end{bmatrix}^{-1}, \quad (16)$$

and where the “error” term e_{eq} goes to zero cubically as the source baseline voltage E_N increases:

$$\lim_{E_N \rightarrow \infty} E_N^3 e_{\text{eq}} = \text{const.} \quad (17)$$

Equation (15) expresses the linear relationship between voltage and reactive power demand in the high-voltage regime. This analysis can also be extended to the case of multiple different and sufficiently high source voltages E_i^* [8].

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