On Reactive Power Flow and Voltage Stability in Microgrids

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Abstract— **This paper focuses on reactive power flow and voltage stability in electrical grids. We provide novel analytical understanding of the solutions to the classic nonlinear polynomial equations describing the decoupled reactive power flow. As of today, solutions to these equations can be found only via numerical methods. Yet an analytical understanding would be beneficial to the rigorous design of future electrical grids. This paper has two main contributions. First, for sufficiently high reference voltages, we guarantee the existence of a high-voltage solution for the reactive power flow equations and provide its approximate analytical expression. We bound the approximation error in terms of network topology and parameters. Second, we consider a recently proposed droop control strategy for voltage stabilization in a microgrid equipped with inverters. For sufficiently high reference voltages, we prove the existence and the exponential stability of a high-voltage fixed point of the closed-loop dynamics. We provide an approximate expression for this fixed point and bound the approximation error in terms of the network topology and parameters. Finally, we validate the accuracy of our approximations through numerical simulation of the IEEE 37 standard test case.**

I. Introduction

The power flow equations model the relationships among bus power injections, power demands, and bus voltages and angles in a power network. They are the heart of most system-planning and operational studies and also the starting point for transient and dynamic stability studies. They constitute a set of coupled equations with trigonometric and polynomial nonlinearities, and the solution space admits a rich and complex phenomenology [1], [2]. Conditions for the existence and exact expression of the solutions have been derived for the case of a radial grid [3], while for a general network only conservative conditions have been proposed [4]–[6]. The work of [7] establishes instead a sufficient condition for the insolvability of the power flow equations, by considering an associated convex optimization problem. The lack of sharp results on solutions to the power flow has motivated the interest in approximate solutions. Of particular interest is [8], where an approximate solution to the power flow equations was developed for electrical networks connected to a larger parent grid at a single Point of Common Coupling, such as typical distribution networks. While this analytic approximation is potentially powerful, it can not be used in more general electrical networks where multiple fixed-voltage buses are present. Also due to the lack of analytic results, the current standard for power flow solution is numerical simulation [9], and the power industry invests considerable effort in simulating thousands of power flow equations for large grids. This motivates the importance of a deeper analytic insight into the problem.

A classic approach [4], [5] to the analysis of the power flow equations is to study the active and the reactive power equations separately under mild decoupling assumptions which are usually satisfied under regular system operation [10]. After the decoupling, the phase angles become the only variables appearing in the active power equations, while the voltage magnitudes become the only variables in the reactive ones. We focus our attention on the resulting reactive power flow equations: these are a system of quadratic equations in the voltage magnitudes at the buses. Despite the simpler problem formulation, no sharp analytic answers pertaining to the existence of solutions are known to date [4]–[6].

The first contribution of this paper is the extension of the approximate load flow solution proposed in [8] to networks with multiple fixed-voltage buses. In particular, we present the result as an approximate solution to the decoupled reactive power flow equations. The resulting solution can be viewed as the reactive power counterpart of the DC load flow approximation for the active power flow [11]. The classic DC load flow approximation expresses the solution to the non-linear active flow equations as a linear combination of the active powers at the buses. The linear coefficients only depend on the network parameters. The approximate solution that we propose for the reactive power flow is the sum of two main terms: the first one is similar to the DC approximation, as it is a linear combination of the reactive powers at the buses; the linear coefficients only depend on the network parameters. The second term consists of a constant high-voltage value for each bus, and it is related to the general and well accepted idea that strongly-clustered high-voltage solutions of the reactive flow equations are the desired stable solutions [12].

In the second part of the paper we focus on the stability of a droop control strategy in an islanded microgrid. Microgrids are low-voltage electrical distribution networks, heterogeneously composed of distributed generation, load, and managed autonomously from the larger primary network. Power sources in microgrids generate either variable frequency AC power or DC power, and are interfaced with a synchronous AC microgrid via power electronic DC/AC

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inverters. In islanded operation, it is through these inverters that actions must be taken to ensure synchronization, voltage stability, power balance and load sharing in the network [13].

We consider the problem of voltage stabilization; that is, keeping the average voltage level in the network high, and keeping the total voltage profile roughly uniform. This is a crucial aspect of microgrid control, as the relatively low voltage levels and uncompensated loads in microgrids put the network at risk for voltage instability and collapse [2]. In the last two decades the $E-Q$ voltage-droop controller has become the tool commonly used for these tasks [14]. Despite the wide-spread adoption of the $E-Q$ voltage-droop controller, few analytic results are available about its closed-loop performance. Specifically, to the best of our knowledge, no results are available on the existence and locations of the equilibria of the closed-loop network.

This paper considers the *quadratic droop controller* proposed by [15]. This modified version of the standard $E-Q$ droop controller reproduces the inherently quadratic and asymmetric nature of the reactive power flow equations and facilitates an analytic treatment. Our previous work [15] characterizes the existence, stability and location of the equilibrium point for a purely-inductive (lossless) network with parallel topology. In this work, we consider networks with arbitrary topology and we make the important assumption of negligible voltage angle differences; by applying the approximation method proposed for the reactive power flow equations, this paper establishes the existence and the stability of a high-voltage fixed point and provides an approximate expression for its location.

This paper is organized as follows. In the remainder of this section we introduce some preliminaries and recall the reactive power flow equations. In Section II we give the approximate solution to the reactive power flow equations, and in Section III we apply the results to study voltage stability in a droop-controlled microgrid. Section IV reports a numerical study of the accuracy of the proposed approximation. Finally, Section V contains our concluding remarks.

Notation and Network Modeling

Given a finite set $\mathcal{V}$, let $|\mathcal{V}|$ denote its cardinality. Let $\mathbb{1}$ denote the vector of all ones, $\mathbf{0}$ a matrix of all zeros; their respective dimensions are determined by context. Let $[x_i]_{i\in\mathcal{V}}$ be an alternative notation for the vector x, with indices in the set $\mathcal{V}$. Let $\mathrm{diag}(x)$ denote the diagonal matrix whose main diagonal is the vector x and $\mathrm{diag}^{-1}(x)$ its inverse, when defined. Given the vectors x and y, we write $x > y$ (resp. $x \geq y$) if $x_i > y_i$ (resp. $x_i \geq y_i$), for all $i \in I$. For $a \in \mathbb{C}$, a^* denotes the complex-conjugate of a.

We model a power network in synchronous steady-state as a connected, undirected and complex-weighted graph $G(\mathcal{V}, \mathcal{E}, Y)$, where $\mathcal{V} = \{1, \ldots, n\}$ is the set of nodes (or *buses*) and $\mathcal{E} \subset \mathcal{V}\times\mathcal{V}$ is the set of edges (or *branches*). Since the graph is undirected, if $(i, j) \in \mathcal{E}$, then it is also $(j, i) \in \mathcal{E}$. The weight of edge (i, j) is its admittance $W_{ij} = W_{ji} \in \mathbb{C}$. To the weighted graph G we associate the *admittance matrix* $Y \in \mathbb{C}^{|\mathcal{V}|\times|\mathcal{V}|}$, defined element-wise by $Y_{ij} = -W_{ij},\ i \neq j$, with diagonal elements $Y_{ii} = -\sum_{i\neq j} Y_{ij}$. To each node $i \in \mathcal{V}$ we assign a phasor voltage $U_i = E_i e^{j\theta_i} \in \mathbb{C}$, a phasor current $I_i \in \mathbb{C}$, and a power injection $S_i = P_i + jQ_i \in \mathbb{C}$, whose real part $P_i \in \mathbb{R}$ is the *active power* and imaginary part $Q_i \in \mathbb{R}$ is the *reactive power*. In vector notation, Kirchoff's current law and Ohm's law give the current-balance relation $I = YU$. Moreover, power, voltage and current at each node are related through: $S_i = U_i I_i^*$. Combining the last two equations in vector notation results in

$$P + jQ = \mathrm{diag}(U)(YU)^*\,, \tag{1}$$

which in components reads

$$\begin{aligned} P_i = &\sum\nolimits_{j=1}^{n} \mathfrak{Im}(Y_{ij}) E_i E_j \sin(\theta_i - \theta_j) \\ &+ \sum\nolimits_{j=1}^{n} \mathfrak{Re}(Y_{ij}) E_i E_j \cos(\theta_i - \theta_j)\,, \quad i \in \mathcal{V}\,, \end{aligned} \tag{2}$$

and

$$\begin{aligned} Q_i = &-\sum\nolimits_{j=1}^{n} \mathfrak{Im}(Y_{ij}) E_i E_j \cos(\theta_i - \theta_j) \\ &+ \sum\nolimits_{j=1}^{n} \mathfrak{Re}(Y_{ij}) E_i E_j \sin(\theta_i - \theta_j)\,, \quad i \in \mathcal{V}. \end{aligned} \tag{3}$$

During regular power system operation the solutions to (1) usually satisfy $|\theta_i - \theta_j| \ll 1$ for each $(i, j) \in \mathcal{E}$ [10], [11]. We assume from now on that $\theta_i - \theta_j = 0$ for each $(i, j) \in \mathcal{E}$. Under this condition, we can decouple equations (2) and (3) and write the reactive power flow equations (RPFE) (3) in compact vector notation as

$$Q = \mathrm{diag}(E) L E\,, \tag{4}$$

where $L = -\mathfrak{Im}(Y)$ is a Laplacian matrix (therefore having non-positive off-diagonal terms and zero row sums), since the susceptance of each resistive and inductive line is negative. Finally, as standard in load flow analysis, we model loads as stiff constant-power demands [10].

II. Approximate solution to the reactive power flow equations

In this section we partition the network nodes as $\mathcal{V} = \{\mathcal{V}_L, \mathcal{V}_S\}$ corresponding to loads and sources (or generators). The source buses have the property that their voltage magnitudes are regulated to constant, predetermined values. The typical example of such a network is a transmission-level grid consisting of loads and PE-generation sources such as synchronous generators. The voltage magnitude vector and the Laplacian L inherit the partitioning as

$$E = \begin{bmatrix} E_L \\ E_S \end{bmatrix}\,, \qquad L = \begin{bmatrix} L_{LL} & L_{LS} \\ L_{SL} & L_{SS} \end{bmatrix}.$$

With this in mind, equation (4) becomes

$$\begin{bmatrix} Q_L \\ Q_S \end{bmatrix} = \mathrm{diag}(E_L, E_S) \begin{bmatrix} L_{LL} & L_{LS} \\ L_{SL} & L_{SS} \end{bmatrix} \begin{bmatrix} E_L \\ E_S \end{bmatrix}. \tag{5}$$

Besides assuming that the source voltages E_S are fixed, no constraints are imposed here on the sources power injections Q_S, that is, the sources are PE-buses [10]. Hence, the second block of equations in (5) can be thought of as determining Q_S as a function of the load voltages E_L. Thus, the equations (5) reduce to their first block:

$$Q_L = \operatorname{diag}(E_L) \begin{bmatrix} L_{LL} & L_{LS} \end{bmatrix} \begin{bmatrix} E_L \\ E_S \end{bmatrix} . \qquad (6)$$

The variables in the $|\mathcal{V}_L|$ equations (6) are the $|\mathcal{V}_L|$ load voltages E_L. In other words, these equations, if solvable, determine E_L as a function of the remaining constant source voltages and network parameters.

In general the system of quadratic equations (6) is not solvable analytically. The classic example of a two node network (see [16] for a detailed analysis) nicely illuminates some of the general features of these equations and motivates our subsequent approximation.

Example II.1 (Two node network). Consider a network with two nodes connected through an inductive line with susceptance $-\ell$. One node is a load with reactive power demand q, while the other node is a source with fixed voltage magnitude $E_N > 0$. If we denote by e the voltage magnitude at the load, equation (6) reduces to

$$q = \ell e(e - E_N) . \qquad (7)$$

If

$$q \geq -q_{\text{crit}} := -\frac{1}{4}\ell E_N^2 , \qquad (8)$$

then equation (7) admits two real-valued solutions, given by

$$e_{1,2} = E_N \left(\frac{1}{2} \pm \frac{1}{2}\sqrt{1 + \frac{q}{q_{\text{crit}}}} \right) . \qquad (9)$$

If $|q/q_{\text{crit}}| \ll 1$, the first-order Taylor series expansion ($\sqrt{1+x} \simeq 1 + \frac{1}{2}x$) leads to the approximate expressions:

$$e_1 \simeq E_N + \frac{q}{\ell E_N} , \quad e_2 \simeq -\frac{q}{\ell E_N} . \qquad (10)$$

The solution e_1 is the desired one in practice, as it corresponds to a high-voltage low-current configuration for the network, resulting in low power losses. In particular, we can interpret the solution as being roughly E_N, with a correction term linear in the power demand, scaled inversely by both E_N and the line susceptance. □

We now build further on the motivation of Example II.1 and offer some intuitive derivations on how to generalize the example. We set $E_N := \min_{i \in \mathcal{V}_S} E_i$ and define the vector η so that the voltages can be decomposed into $E = E_N(\mathbb{1}+\eta)$. As in the example, we are interested in the high-voltage solution to the power flow equations and, moreover, we are interested in solutions with uniform voltages. High and uniform voltages correspond to the regime where $E_N \gg 1$ and $\eta \ll 1$. In this regime, equation (6) becomes

$$\begin{aligned} Q_L &= E_N \operatorname{diag}(\mathbb{1} + \eta_L) \begin{bmatrix} L_{LL} & L_{LS} \end{bmatrix} E_N(\mathbb{1} + \eta) \\ &= E_N^2 \Big(\begin{bmatrix} L_{LL} & L_{LS} \end{bmatrix} \eta + \operatorname{diag}(\eta_L) \begin{bmatrix} L_{LL} & L_{LS} \end{bmatrix} \eta \Big) \\ &\simeq E_N^2 (L_{LL}\eta_L + L_{LS}\eta_S) , \end{aligned} \qquad (11)$$

where the second equality holds because $\mathbb{1}$ is in the kernel of L, and the last approximation neglects the quadratic term in η. Solving (11) for η_L, we obtain the following approximate solution

$$\begin{aligned} E_L &= E_N(\mathbb{1} + \eta_L) \\ &\simeq E_N \mathbb{1} - E_N L_{LL}^{-1} L_{LS} \eta_S + \frac{1}{E_N} L_{LL}^{-1} Q_L . \end{aligned} \qquad (12)$$

Looking back at Example II.1, we see how the first order expansion that in the two-node network led to the solution e_1 in equation (10) corresponds exactly to the approximation (12).

Building on the intuitive derivations leading to (12), we now state our first rigorous result, which extends the work carried out in [8] to transmission-level networks with multiple generating sources. The proof of the following theorem extends the proof strategy in [8] and uses arguments of multivariate analysis along with the implicit function theorem. Due to space constraints, we do not report the proof here.

In order to formulate the theorem, we consider the reactive power balance equation (6), we define $E_N := \min_{i \in \mathcal{V}_S} E_i$ as the *source baseline voltage*, and we let η_S be the *source voltage spread*, such that $E_S = E_N(\mathbb{1} + \eta_S)$. Define the *approximate load voltage*

$$\begin{aligned} E_{L,\text{approx}} &:= E_N(\mathbb{1} - L_{LL}^{-1} L_{LS} \eta_S) + \\ &+ \frac{1}{E_N} \big(L_{LL}^{-1} \operatorname{diag}^{-1}(\mathbb{1} - L_{LL}^{-1} L_{LS}\eta_S) Q_L \big) . \end{aligned} \qquad (13)$$

Then the following holds.

Theorem II.2 (Approximate solution to the RPFE). *There exists a minimum source baseline voltage $E_N^{\min}$ such that, for all $E_N > E_N^{\min}$, a high-voltage solution of the decoupled reactive power flow equation* (6) *exists and is given by*

$$E_L = E_{L,\text{approx}} + \frac{1}{E_N^3} k , \qquad (14)$$

where the term k satisfies

$$\|k\|_2 \leq \gamma , \qquad (15)$$

with γ depending only on the network parameters L, Q_L and η_S.

Remark II.3 We point out that the existence of the threshold above which a solution exists is a generalization of inequality (8). Note that the approximate solution (13) differs from the intuitively derived solution (12) by an additional term proportional to $\frac{1}{E_N}$. In Section IV we will show by numerical comparison of the two approximations that this additional and perhaps unexpected term drastically increases the numerical accuracy of the approximation. The intuition now suggests that in general the expansion of the load flow solution consists only in odd powers of E_N, as we could already notice in the two-node example by Taylor-expanding the square root in (9).

From equations (13) and (14) one sees that as the source baseline voltage E_N becomes large and the source voltage spread η_S diminishes, the load voltage solution tends to the source baseline E_N. This regime of parameters is the one practically relevant for regular power system operation [10]. While the explicit expression (not reported here) of the bound γ in (15) is quite conservative for any given network, the numerical simulations reported in Section IV indicate that the error term k is much smaller than the theoretical upper bound (15), and thus the approximation (13) is extremely accurate. □

III. Application to the quadratic droop controller

In this section, we consider the problem of voltage stabilization in an inverter-based microgrid. We partition the set of nodes in the microgrid as $\mathcal{V} = \{\mathcal{V}_L, \mathcal{V}_I\}$, where $\mathcal{V}_L$ are loads and $\mathcal{V}_I$ are inverters. The E-Q voltage droop controller specifies the voltage magnitudes E_i at the inverters by [17, Chapter 19]

$$C_i(E_i - E_i^R) = -Q_i\,, \qquad i \in \mathcal{V}_I\,, \tag{16}$$

where E_i^R is the fixed voltage reference and C_i is a fixed control parameter for inverter i. One can easily see that if an inverter injects a non-zero amount of reactive power Q_i, its voltage deviates from the reference E_i^R. For comparison purposes, it is convenient to add an integral channel to the controller (16) yielding the conventional first-order droop controller

$$\tau_i \dot{E}_i = -C_i(E_i - E_i^R) - Q_i\,, \qquad i \in \mathcal{V}_I\,, \tag{17}$$

where $\tau_i > 0$. Note that solutions of (16), correspond to steady-states of (17).

Instead of using the standard E-Q controller, we consider here a modified version of it, namely the *quadratic droop controller* recently proposed by [15], which reproduces the quadratic nature of the reactive power flow equation. This controller adjusts the inverter voltage magnitude according to

$$\tau_i \dot{E}_i = -C_i E_i(E_i - E_i^R) - Q_i\,, \qquad i \in \mathcal{V}_I\;, \tag{18}$$

where $\tau_i, C_i > 0$ are fixed controller parameters, and E_i^R is a fixed reference voltage. If the inverter i injects no reactive power, the equilibrium voltage of (18) is again E_i^R. By combining the reactive power flow equation at the load (6) and the controller (18), we obtain the differential-algebraic system

$$\begin{bmatrix} \mathbf{0} \\ \tau \dot{E}_I \end{bmatrix} = \begin{bmatrix} Q_L \\ C\mathrm{diag}(E_I)(E_R - E_I) \end{bmatrix} - \mathrm{diag}(E)LE\;, \tag{19}$$

where $\tau = \mathrm{diag}([\tau_i]_{i\in\mathcal{V}_I})$ and $C = \mathrm{diag}([C_i]_{i\in\mathcal{V}_I})$ are diagonal matrices, while $E_R = [E_i^R]_{i\in\mathcal{V}_R}$ is the vector of the reference voltages. We point out that while in Section II the voltages E_S were considered to be fixed, now due to the introduction of the quadratic droop controller the voltages E_I in (19) are variables of the system; hence the variables are now E_L and E_I. The goal of this section is to study whether the differential-algebraic system (19) possesses a fixed point, to find an approximate expression for it, and to determine its stability properties.

Remark III.1 (Network interpretation of quadratic droop controller). If we compare the quadratic droop control law (18) and the right-hand side of (7), we can interpret the term $C_iE_i(E_i - E_i^R)$ in (18) as the reactive power injected from inverter i to a fictitious node of voltage E_i^R through a line of susceptance $-C_i$. Guided by this intuition, we consider an *extended network* (Figure 1) where we introduce the set of reference nodes $\mathcal{V}_R$ and we connect each node $i \in \mathcal{V}_I$ to the corresponding reference node $i \in \mathcal{V}_R$ (with voltage E_i^R) through a line of susceptance $-C_i$. The voltage vector and Laplacian matrix of the extended network are

$$\tilde{E} = \begin{bmatrix} E_L \\ E_I \\ E_R \end{bmatrix}, \quad \tilde{L} = \begin{bmatrix} L_{LL} & L_{LI} & \mathbf{0} \\ L_{IL} & L_{II} + C & -C \\ \mathbf{0} & -C & C \end{bmatrix}, \tag{20}$$

where the diagonal matrix C accounts for the new connections established between inverters $\mathcal{V}_I$ and reference nodes $\mathcal{V}_R$. From (20) we can compute the reactive power at the inverters in the extended network, which we denote by $\tilde{Q}_I$:

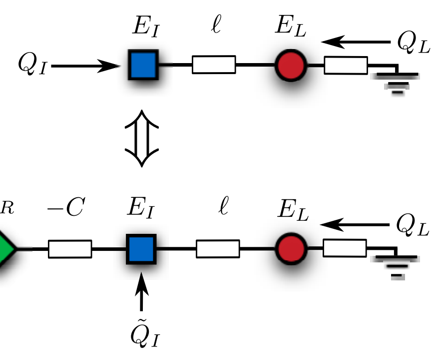

Fig. 1. The equivalence between the original network (top) which consists of an inverter (blue square) ■ feeding a load ●, and the extended network (bottom) with an additional fictitious node ◆ held at constant voltage E_R.

$$\begin{aligned} \tilde{Q}_I &= \mathrm{diag}(E_I)\begin{bmatrix} L_{IL} & L_{II} + C & -C \end{bmatrix}\tilde{E} \\ &= C\mathrm{diag}(E_I)(E_I - E_R) + Q_I\;, \end{aligned} \tag{21}$$

Inserting the expression of the reactive powers (21) in the quadratic droop controller (18), we can now write the control law as

$$\tau \dot{E}_I = -\tilde{Q}_I\;. \tag{22}$$

Using the compact expression (22) we can write the differential-algebraic system (19) on the extended network equivalently as

$$\begin{bmatrix} Q_L \\ \tilde{Q}_I \end{bmatrix} = \mathrm{diag}(E_L, E_I)\begin{bmatrix} L_{LL} & L_{LI} & \mathbf{0} \\ L_{IL} & L_{II} + C & -C \end{bmatrix}\tilde{E} \tag{23a}$$

$$\tau \dot{E}_I = -\tilde{Q}_I\;. \tag{23b}$$

We emphasize that the systems (23) and (19) are equivalent representations of the microgrid with quadratic droop control at the inverters. □

Equation (23a) has the same structure of the original load reactive power flow equation (6), and in Theorem II.2 we introduced an approximate solution to equation (6). It is natural to follow a similar path and apply the same approximation to equation (23a) to find an approximate solution for the voltages E_L and E_I, while E_R is considered to be fixed. Using the expression of the approximate solution it is possible to study the stability of the unique equilibrium $\tilde{Q}_I = \mathbf{0}$ of (23b). This strategy leads to the Theorem III.2. In order to state it, we define the *reference baseline voltage* $\tilde{E}_N$ by

$$\tilde{E}_N := \min_{i\in\mathcal{V}_R} E_i^R,$$

and the *reference voltage spread* $\tilde{\eta}$ such that

$$E_R = \tilde{E}_N(\mathbb{1} + \tilde{\eta}).$$

Note that $\tilde{\eta} \geq \mathbf{0}$ is a vector with nonnegative entries. We define the inverse X of the truncated Laplacian matrix by

$$X = \begin{bmatrix} X_{LL} & X_{LI} \\ X_{IL} & X_{II} \end{bmatrix} := \begin{bmatrix} L_{LL} & L_{LI} \\ L_{IL} & L_{II}+C \end{bmatrix}^{-1},$$

and the hybrid matrix

$$M = \begin{bmatrix} M_L \\ M_I \end{bmatrix} := -\begin{bmatrix} L_{LL} & L_{LI} \\ L_{IL} & L_{II}+C \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} \\ -C \end{bmatrix} = \begin{bmatrix} X_{LI} \\ X_{II} \end{bmatrix} C\,.$$

We are now ready to state the main result of this section.

Theorem III.2 (Existence and stability of the fixed point). *There exists a minimum reference baseline voltage $\tilde{E}_N^{\min}$ such that for all $\tilde{E}_N > \tilde{E}_N^{\min}$ the differential-algebraic system (19) has a locally exponentially stable high-voltage fixed point given by*

$$\begin{bmatrix} E_L^{eq} \\ E_I^{eq} \end{bmatrix} = \tilde{E}_N(\mathbb{1} + M\tilde{\eta}) + \\ + \frac{1}{\tilde{E}_N}\begin{bmatrix} X_{LL} \\ X_{LI} \end{bmatrix} \operatorname{diag}^{-1}(\mathbb{1} + M_L\tilde{\eta})Q_L + \frac{1}{\tilde{E}_N^3}k^{\text{eq}}\,, \tag{24}$$

where the norm of the term k^{eq} can be bounded as

$$\|k^{\text{eq}}\|_2 \leq \tilde{\gamma}\,, \tag{25}$$

with $\tilde{\gamma}$ only depending on the network parameters $\tilde{L}$, Q_L and $\tilde{\eta}$.

Theorem III.2 takes inspiration from Theorem II.2 but addresses a different problem. While Theorem II.2 gives an approximate solution of the algebraic equation (6), Theorem III.2 provides the approximate expression of a fixed point of a differential-algebraic system, and studies its stability. The proof of Theorem III.2 is more involved than that of Theorem II.2, and we do not report it here. Note that setting $k^{\text{eq}} = \mathbf{0}$ in equation (24) gives an approximate expression for the solution of (23a). The approximate expression is the same we introduced in Theorem II.2, except that here it is applied to an extended network with $\tilde{Q}_I = \mathbf{0}$. The close relationship between the two theorems allows us to verify the accuracy of the approximation (14) of Theorem II.2 by performing a numerical analysis only on the system (23) (see Section IV).

IV. Numerical study

In this section we test the results obtained in Theorem III.2 on an islanded version of the standard IEEE 37 distribution network, which we report in Figure 2. The nominal operating voltage of the network is 4.8kV, the line susceptances vary in the range $[-0.5\text{ S}, -10\text{ S}]$ with R/X ratios of approximately one, while the reactive power demands vary for each load in the interval $[-30\text{ kvar}, -70\text{ kvar}]$. The sources in this

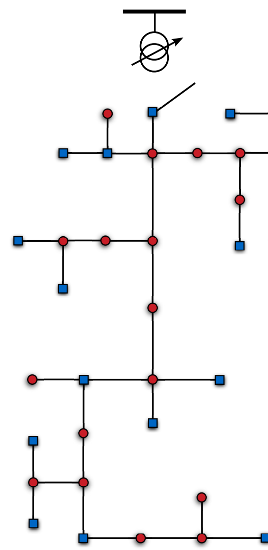

Fig. 2. Islanded IEEE 37 bus distribution network containing loads ● and inverters ■

network are DC/AC inverters, whose voltage magnitudes are governed by the quadratic droop controller (18). The reference voltage magnitudes are fixed values in the interval $[\tilde{E}_N, \tilde{E}_N + 0.1\tilde{E}_N]$. We simulate the resulting differential-algebraic system (23) for different values of $\tilde{E}_N$ and study:

a) the threshold $\tilde{E}_N^{\min}$ above which the fixed point (24) exists and is stable, and
b) the accuracy of the approximated fixed point expression resulting from (24); we consider two variations: in the *approximation* 1 (**A1**) we set $k^{\text{eq}} = \mathbf{0}$ in (24); the *approximation* 2 (**A2**) is instead

$$\begin{bmatrix} E_L^{eq} \\ E_I^{eq} \end{bmatrix} = \tilde{E}_N(\mathbb{1} + M\tilde{\eta}) + \frac{1}{\tilde{E}_N}\begin{bmatrix} X_{LL} \\ X_{LI} \end{bmatrix} Q_L\,. \tag{26}$$

Equation (26) is the "incomplete" approximation (12) formulated for the equations (23a); comparison with this simpler approximation will illustrate that the intuitive analysis used to arrive at (12) is improved upon with the rigorous results of Theorem II.2 and Theorem III.2. To quantify the error between the true fixed point E_{nonlin}^{eq} of the nonlinear system and the approximations given by **A1** and **A2**, we introduce the *relative approximation errors*

$$\delta_i := \frac{\|E_{\text{nonlin}}^{eq} - E_{\text{approx},\mathbf{A}i}^{eq}\|_\infty}{E_N}\,, \quad i \in \{1,2\}\,.$$

In studying the accuracy of Theorem III.2, we implicitly also study the accuracy of Theorem II.2, as we noted at the end of Section III.

The threshold $\tilde{E}_N^{\min}$ above which a stable high-voltage fixed point exists was found by simulation to be roughly 860V, well below the operating voltage of the system (4.8kV). Figure 3 reports the relative approximation errors δ_1 and δ_2 for the approximations **A1** and **A2** in the IEEE 37 network, for different values of $\tilde{E}_N$.

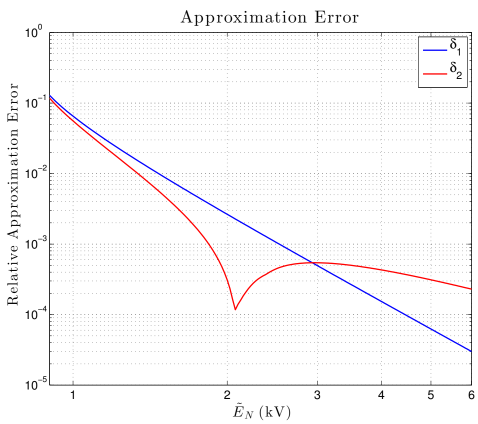

Fig. 3. Relative approximation error for simulation of the IEEE 37 bus distribution network as a function of the nominal network voltage. The scales on both axes are logarithmic.

Note first that both relative approximation errors decrease rapidly as $\tilde{E}_N$ grows. In particular at the 4.8kV nominal operating voltage of the network, the relative error using both approximations is below 0.1%, with the accuracy of **A1** being below 0.01%. For large values of $\tilde{E}_N$ the approximation **A1** is more accurate, so the exact characterization of the coefficient of the term $\frac{1}{\tilde{E}_N}$ leads to a better approximation in the practical operational regime. The curious and smooth behavior of the relative approximation error — and the relation of this behavior to the bounds in (25) — is a subject of future research.

V. Conclusions

In this work we have presented novel analytic expressions for the approximate solution of the decoupled reactive power flow equations. In addition to being readily applicable in transmission networks, we have demonstrated the flexibility of our result by using it to study the behavior of droop-controlled inverters in an islanded microgrid. Through simulation, we have demonstrated that our results are practical and very accurate. Future work in this direction seeks to quantify the threshold $\tilde{E}_N^{\min}$, examine analytically the results of Figure 3, and further relax the assumption of small angular differences. We further envision an extensive set of case studies, with the goal of demonstrating conclusively the usefulness of this approximation in power system planning and operation.

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