Synchronization of Nonlinear Circuits in Dynamic Electrical Networks

Florian Dörfler, Sairaj V. Dhople, Brian B. Johnson, and Abdullah Hamadeh

***Abstract*— Synchronization of coupled oscillators is a pervasive theme of multi-disciplinary research. Focused on circuit-theoretic applications, in this paper, we derive sufficient conditions for global asymptotic synchronization in a system of identical nonlinear circuits coupled through linear time-invariant (LTI) electrical networks. The nonlinear circuits are composed of a parallel combination of passive LTI circuit elements and a nonlinear voltage-dependent current source with finite gain. The terminals of the nonlinear circuits are coupled through LTI networks characterized by either identical per-unit-length impedances or identical effective impedances between any two terminals. This setup is motivated by a recently proposed control strategy for inverters in microgrids. We analyze synchronization by means of an input-output analysis of a coordinate-transformed system that emphasizes signal differences. To apply the synchronization analysis to a broad class of networks, we leverage recent results on Kron reduction—a circuit reduction and transformation procedure that reveals the interactions of the nonlinear circuits. We illustrate our results with simulations in networks of coupled Chua's circuits.**

I. Introduction

Synchronization of nonlinear electrical circuits coupled through complex networks is integral to modeling, analysis, and control in application areas such as the ac electrical grid, solid-state circuit oscillators, semiconductor laser arrays, secure communications, and microwave oscillator arrays [1], [2]. This paper focuses on the global asymptotic synchronization of terminal voltages in a class of nonlinear circuits coupled through passive LTI electrical networks.

We assume that the circuits are composed of a parallel combination of passive LTI circuit elements and a nonlinear voltage-dependent current source with finite gain. A variety of chaotic and hyperchaotic circuits as well as nonlinear oscillators [3]–[9] admit this general model. A collection of such identical circuits are coupled through passive LTI electrical networks. We consider two broad classes of LTI networks: i) *Uniform networks*, that have arbitrary topologies and arbitrary (but identical) per-unit-length line impedances. These include purely resistive and lossless networks as special cases. ii) *Homogeneous networks*, that are characterized by identical effective impedances between any pair of terminals (the effective impedance is the equivalent (complex) impedance between any two nodes in the network with the others open circuited). They are encountered in engineered setups (such as power grid monitoring and electrical impedance tomography), in large-scale random networks or regular lattices, and in idealized settings where all terminals are electrically uniformly distributed.

The motivation for this work stems from developing control paradigms for parallel-connected power electronics inverters in low-inertia microgrids. The key idea pertains to controlling inverters to emulate the dynamics of nonlinear Liénard-type limit-cycle oscillators [10]–[13]. The oscillators (inverters) are coupled (connected) through the existing microgrid electrical network, and synchrony emerges in this system with no external forcing in the form of a utility grid or any communication beyond the physical electrical network. This paper generalizes the previous results by establishing synchronization conditions for a much wider class of nonlinear electrical circuits and networks.

The approach adopted in this paper builds on previous work in [12]–[16], where incremental $\mathcal{L}_2$ methods were used to analyze synchronization in feedback systems thereby offering an alternate perspective compared to the rich body of literature that has examined synchronization problems with (incremental) Lyapunov- and passivity-based methods [10], [11], [17]–[23]. To investigate synchronization, the linear and nonlinear subsystems are compartmentalized, and a coordinate transformation is applied to recover a *differential system* emphasizing signal differences. Synchronization is guaranteed by ensuring the stability of the differential system with a small-gain argument. The suite of synchronization conditions presented in this paper generalize our previous efforts in [12] from parallel networks with star topologies to arbitrary topologies. Integral to this generalization is a model-reduction and circuit-transformation procedure called *Kron reduction* [24]–[26] that uncovers the interactions between the nonlinear circuits. In this regard, a related contribution of this work pertains to leveraging recent results on structural and spectral properties of Kron reduction [25] in deriving synchronization conditions. In particular, we extend some key lemmas from [25] from the real-symmetric to the complex-symmetric (and not necessarily Hermitian) case. We also offer some converse results to statements in [25].

This manuscript is organized as follows. Section II introduces some notation. Section III describes the nonlinear circuits and their interactions. Section IV formulates the synchronization problem. Section V establishes sufficient synchronization conditions. Section VI offers simulation case studies to validate our approach. Finally, Section VII concludes the paper. In the interest of brevity, we do not provide proofs for the matrix-theoretic results pertaining to Kron reduction; the complete set of proofs are in [25], [27].

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II. Notation and Preliminaries

Given a complex-valued N-tuple $\{u_1, \ldots, u_N\}$, the associated column vector is $u = [u_1, \ldots, u_N]^{\mathrm{T}}$, where $(\cdot)^{\mathrm{T}}$ denotes transposition (without conjugation). Let I be the $N \times N$ identity matrix, and let $\mathbf{1}$ and $\mathbf{0}$ be the N-dimensional vectors of ones and zeros. The Moore-Penrose pseudo inverse of a matrix U is denoted by $U^{\dagger}$. Let $\mathrm{j} = \sqrt{-1}$ be the imaginary unit. The cardinality of a finite set $\mathcal{N}$ is $|\mathcal{N}|$.

Denote the Laplace transform of a continuous-time function $f(t)$ by f. The Euclidean norm of a complex vector, u, is denoted by $\|u\|_2$ and is defined as $\|u\|_2 := \sqrt{u^* u}$, where $(\cdot)^*$ denotes the conjugate transpose. The space of all piecewise continuous functions such that

$$\|u\|_{\mathcal{L}_2}^2 := \int_0^{\infty} u(t)^{\mathrm{T}} u(t) \, \mathrm{d}t < \infty, \tag{1}$$

is denoted as $\mathcal{L}_2$, where $\|u\|_{\mathcal{L}_2}$ is referred to as the $\mathcal{L}_2$ *norm* of u. If $u \in \mathcal{L}_2$, then u is said to be *bounded*. A causal system $\mathcal{H}$ with input u and output y is *finite-gain $\mathcal{L}_2$ stable* if there are finite and non-negative constants γ and η such that

$$\|y\|_{\mathcal{L}_2} =: \|\mathcal{H}(u)\|_{\mathcal{L}_2} \leq \gamma \|u\|_{\mathcal{L}_2} + \eta, \quad \forall u \in \mathcal{L}_2. \tag{2}$$

The smallest value of γ for which there exists a η such that (2) is satisfied is called the $\mathcal{L}_2$ *gain* of the system. If $\mathcal{H}$ is linear and represented by the transfer matrix $H : \mathbb{C} \to \mathbb{C}^{N \times N}$, it can be shown that the $\mathcal{L}_2$ gain of $\mathcal{H}$, denoted $\gamma(\mathcal{H})$, is equal to its H-*infinity norm* $\|\mathcal{H}\|_\infty$ defined as

$$\gamma(\mathcal{H}) = \|\mathcal{H}\|_\infty := \sup_{\omega \in \mathbb{R}} \frac{\|H(\mathrm{j}\omega) u(\mathrm{j}\omega)\|_2}{\|u(\mathrm{j}\omega)\|_2}, \tag{3}$$

where $\|u(\mathrm{j}\omega)\|_2 = 1$, provided that all poles of H have strictly negative real parts [28]. For a SISO transfer function, $h : \mathbb{C} \to \mathbb{C}$, $\gamma(\mathcal{H}) = \|\mathcal{H}\|_\infty = \sup_{\omega \in \mathbb{R}} \|h(\mathrm{j}\omega)\|_2$.

A construct we will find particularly useful in assessing signal differences is the $N \times N$ *projector matrix* [22]

$$\Pi := I - \frac{1}{N} \mathbf{1}\mathbf{1}^{\mathrm{T}}. \tag{4}$$

For a vector u, let $\widetilde{u} := \Pi u$ be the corresponding *differential vector* measuring the deviation of each entry u_i from the average value $\left(\sum_{j=1}^N u_j\right)/N$ [14]–[16], [22].

A causal system with input u and output y is said to be *differentially finite $\mathcal{L}_2$ gain stable* [14] if there exist finite, non-negative constants, $\widetilde{\gamma}$ and $\widetilde{\eta}$, such that

$$\|\widetilde{y}\|_{\mathcal{L}_2} \leq \widetilde{\gamma} \|\widetilde{u}\|_{\mathcal{L}_2} + \widetilde{\eta}, \quad \forall \widetilde{u} \in \mathcal{L}_2, \tag{5}$$

where $\widetilde{y} = \Pi y$. The smallest value of $\widetilde{\gamma}$ for which there exists a non-negative value of $\widetilde{\eta}$ such that (5) is satisfied is referred to as the *differential $\mathcal{L}_2$ gain* of H. The differential $\mathcal{L}_2$ gain of a system provides a measure of the largest amplification imparted to input signal differences.

Given a symmetric and nonnegative matrix $A \in \mathbb{R}^{N \times N}$, we define its *Laplacian matrix* L component-wise by $l_{nm} = -a_{nm}$ for off-diagonal elements and $l_{nn} = \sum_{m=1}^N a_{nm}$ for diagonal elements. The Laplacian has zero row and column sums, it is symmetric and positive semidefinite, and its zero eigenvalue is simple if and only if the graph is connected.

III. Coupled Nonlinear Electrical Circuits

A. Nonlinear Circuit Model

An electrical schematic of the nonlinear circuits studied in this work is depicted in Fig. 1. Each circuit has a linear subsystem composed of an arbitrary connection of passive circuit elements described by the impedance, $z_{\mathrm{ckt}} \in \mathbb{C}$, and a nonlinear voltage-dependent current source $i_{\mathrm{g}} = -g(v)$. We require that the function $g(\cdot)$ be globally Lipschitz, that is, there is a finite constant $\sigma > 0$ so that

$$|g(x) - g(y)| \leq \sigma |x - y| \quad \forall x, y \in \mathbb{R}. \tag{6}$$

A wide class of electrical circuits can be described within this framework. One example is Chua's circuit [5], [9], for which the impedance z_{ckt} and nonlinear function $g(\cdot)$ are illustrated in Fig. 2(a). In related work on voltage synchronization of voltage source inverters in small-scale power systems [10]–[13], Liénard-type dead-zone oscillators are considered with z_{ckt} and $g(\cdot)$ illustrated in Fig. 2(b). Some families of hyperchaotic circuits and negative-resistance oscillators can also be described with the model above, see, e.g., [4]–[9].

B. Electrical Network Model

The nonlinear circuits are coupled through a passive, connected, LTI electrical network. The nodes of the network are collected in the set $\mathcal{A}$, and branches (or edges) of the network are represented by the set $\mathcal{E} := \{(m, n)\} \subset \mathcal{A} \times \mathcal{A}$. Let $\mathcal{N} := \{1, \ldots, N\} \subseteq \mathcal{A}$ be the set of *boundary nodes* to which the nonlinear circuits are connected, and let $\mathcal{I} = \mathcal{A} \backslash \mathcal{N}$ be the set of *interior nodes*. Let $y_{mn} \in \mathbb{C}$ be the series admittance corresponding to the branch $(m, n) \in \mathcal{E}$.

Denote the vectors that collect the nodal current injections and node voltages in the network by $i_{\mathcal{A}}$ and $v_{\mathcal{A}}$, respectively. The coupling between the circuits can be described by Kirchhoff's and Ohm's laws, which reads in vector form as

$$i_{\mathcal{A}} = Y_{\mathcal{A}} v_{\mathcal{A}}, \tag{7}$$

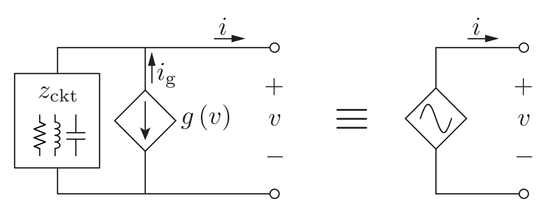


Fig. 1. Electrical schematic of nonlinear circuit studied in this work. Each circuit is composed of a linear subsystem modeled by a passive impedance, z_{ckt}, and a nonlinear voltage-dependent current source, $g(\cdot)$. The circuit symbol used to represent the nonlinear circuit is depicted on the right.

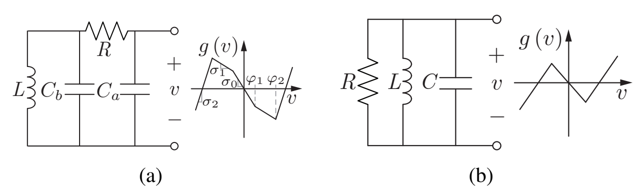


Fig. 2. The linear impedance z_{ckt} and the nonlinear current source $g(\cdot)$ illustrated for (a) Chua's circuit, and (b) the dead-zone oscillator.

where $Y_{\mathcal{A}} \in \mathbb{C}^{|\mathcal{A}|\times|\mathcal{A}|}$ is the admittance matrix defined as

$$[Y_{\mathcal{A}}]_{mn} := \begin{cases} \sum_{(k,m)\in\mathcal{E}} y_{mk}, & \text{if } m = n, \\ -y_{mn}, & \text{if } (m,n) \in \mathcal{E}, \\ 0, & \text{otherwise}. \end{cases} \tag{8}$$

Since the network has no shunt elements (which would appear as additional diagonal terms in $Y_{\mathcal{A}}$ and render $Y_{\mathcal{A}}$ diagonally dominant), it follows that $Y_{\mathcal{A}}$ is a singular complex-valued Laplacian matrix with zero row and column sums.

In subsequent analysis, we will find it useful to consider the effective impedance between nodes in the network. The *effective impedance*, z_{nm}, between nodes n and m is the potential difference between nodes n and m, when a unit current is injected in node n and extracted from node m. In this case, the current-balance equations are $e_n - e_m = Y_{\mathcal{A}} v$, where e_n is the canonical vector of all zeros except with a 1 in the n^{th} position. The effective impedance is then

$$z_{nm} = (e_n - e_m)^{\mathrm{T}} v = (e_n - e_m)^{\mathrm{T}} Y_{\mathcal{A}}^{\dagger} (e_n - e_m) \,. \tag{9}$$

The effective impedance is an electric and graph-theoretic distance measure, see [25] for details and further references.

Let $i = [i_1, \ldots, i_N]^{\mathrm{T}}$ and $v = [v_1, \ldots, v_N]^{\mathrm{T}}$ be the vectors collecting the current injections and terminal voltages of the nonlinear circuits, and let $i_{\mathcal{I}}$ and $v_{\mathcal{I}}$ be the vectors collecting the current injections and nodal voltages for the interior nodes. With this notation in place, we can rewrite (7) as

$$\begin{bmatrix} i \\ i_{\mathcal{I}} \end{bmatrix} = \begin{bmatrix} Y_{\mathcal{N}\mathcal{N}} & Y_{\mathcal{N}\mathcal{I}} \\ Y_{\mathcal{N}\mathcal{I}}^{\mathrm{T}} & Y_{\mathcal{I}\mathcal{I}} \end{bmatrix} \begin{bmatrix} v \\ v_{\mathcal{I}} \end{bmatrix}. \tag{10}$$

Since the internal nodes are only connected to passive LTI circuit elements, all entries of $i_{\mathcal{I}}$ are equal to zero in (10).

For RLC networks without shunt elements (as considered here), the submatrix $Y_{\mathcal{I}\mathcal{I}}$ is irreducibly block diagonally dominant (due to connectivity) and hence nonsingular [29, Corollary 6.2.27]. It follows that $v_{\mathcal{I}} = -Y_{\mathcal{I}\mathcal{I}}^{-1} Y_{\mathcal{N}\mathcal{I}}^{\mathrm{T}} v$, which when substituted in (10) yields the following relation among the nonlinear-circuit current injections and terminal voltages:

$$i = \left(Y_{\mathcal{N}\mathcal{N}} - Y_{\mathcal{N}\mathcal{I}} Y_{\mathcal{I}\mathcal{I}}^{-1} Y_{\mathcal{N}\mathcal{I}}^{\mathrm{T}}\right) v =: Y v. \tag{11}$$

This model reduction through a *Schur complement* of the admittance matrix is known as *Kron reduction* [25]. We refer to the matrix Y in (11) as the *Kron-reduced admittance matrix*. From a control-theoretic perspective, (11) is a minimal realization of the circuit (10). Even though the Kron-reduced admittance matrix Y is well defined, it is not necessarily the admittance matrix of a passive circuit. Figure 3 illustrates an electrical network and its Kron-reduced counterpart. Notice that in this case, Kron reduction is equivalent to the celebrated star-delta transformation.

Similar to the original network, the Kron-reduced network has no shunt elements, and its admittance matrix Y commutes with the projector matrix Π; see [25], [27] for a proof.

Lemma 1 *The following statements are equivalent:*

(i) *The original electrical network has no shunt elements.*

(ii) *The Kron-reduced network has no shunt elements.*

If statements (i) and (ii) are true, then Y commutes with Π.

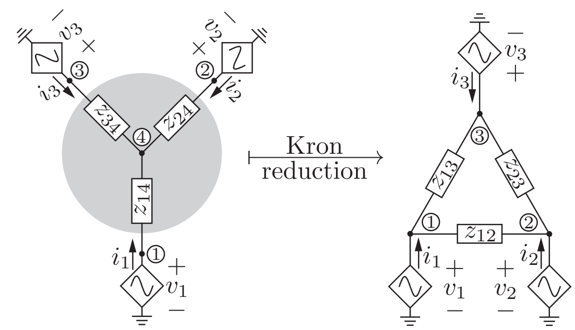


Fig. 3. Kron reduction illustrated for a representative network where $N = 3$, $\mathcal{A} = \{1,2,3,4\}$, $\mathcal{N} = \{1,2,3\}$, and $\mathcal{I} = \{4\}$.

IV. Problem Statement & Differential System

A. Global Asymptotic Synchronization

We are interested in global asymptotic synchronization of the terminal voltages of the nonlinear circuits coupled through the electrical LTI network described in Section III. In particular, we seek sufficient conditions that ensure

$$\lim_{t\to\infty} (v_j(t) - v_k(t)) = 0 \quad \forall j,k = 1,\ldots,N. \tag{12}$$

For ease of analysis, we find it useful to implement a coordinate transformation by employing the projector matrix, Π, to obtain the corresponding *differential system* emphasizing signal differences. To highlight the analytical advantages afforded by this coordinate transformation, note that:

$$\widetilde{v}(t)^{\mathrm{T}}\widetilde{v}(t) = (\Pi v(t))^{\mathrm{T}}(\Pi v(t)) = \frac{1}{2N}\sum_{j,k=1}^{N}(v_j(t) - v_k(t))^2 \,.$$

Hence, condition (12) equivalently reads as $\lim_{t\to\infty}\widetilde{v}(t) = \lim_{t\to\infty}\Pi v(t) = 0$. The coordinate transformation with the projector matrix allows us to cast the synchronization problem as a stability problem in the differential coordinates.

B. Compartmentalization of Linear and Nonlinear Systems

We seek a system description where the linear and nonlinear subsystems in the network of coupled nonlinear circuits are clearly compartmentalized. In light of the importance of differential signals in facilitating the derivation of synchronization conditions, the compartmentalization is sought in the coordinates of the corresponding differential system.

Let $i_{\mathrm{g}} := [i_{\mathrm{g}1}, \ldots, i_{\mathrm{g}N}]^{\mathrm{T}}$ collect the currents sourced by the nonlinear voltage-dependent current sources. From Fig. 1, we see that the terminal voltage of the j^{th} nonlinear circuit, v_j, can be expressed as

$$v_j = z_{\mathrm{ckt}}\left(i_{\mathrm{g}j} - i_j\right), \forall j = 1,\ldots,N.$$

By collecting all v_j's, we can write

$$v = Z_{\mathrm{ckt}}\left(i_{\mathrm{g}} - i\right) = Z_{\mathrm{ckt}} i_{\mathrm{g}} - Z_{\mathrm{ckt}} Y v, \tag{13}$$

where $Z_{\mathrm{ckt}} := z_{\mathrm{ckt}} I \in \mathbb{C}^{N\times N}$, and we substituted $i = Yv$ from (11). A multiplication of both sides of (13) by the projector matrix Π yields the differential terminal-voltage vector

$$\widetilde{v} = \Pi v = \Pi\left(Z_{\mathrm{ckt}}\left(i_{\mathrm{g}} - Yv\right)\right) = Z_{\mathrm{ckt}}\left(\widetilde{i}_{\mathrm{g}} - Y\widetilde{v}\right), \tag{14}$$

where we leveraged the commutativity property $\Pi Y = Y\Pi$. We can now isolate $\widetilde{v}$ in (14) as follows:

$$\widetilde{v} = (I + Z_{\text{ckt}}Y)^{-1} Z_{\text{ckt}}\widetilde{i}_{\text{g}} = \mathcal{F}(Z_{\text{ckt}}, Y)\,\widetilde{i}_{\text{g}}, \tag{15}$$

where $\mathcal{F}(Z_{\text{ckt}}, Y)$ is the *linear fractional transformation*

$$\mathcal{F}(Z_{\text{ckt}}(s), Y(s)) := (I + Z_{\text{ckt}}(s)Y(s))^{-1} Z_{\text{ckt}}(s), \tag{16}$$

that captures the negative feedback interconnection of Z_{ckt} and Y. The differential system (15) admits the compact block-diagram representation in Fig. 4. The linear and nonlinear portions of the system are clearly compartmentalized by $\mathcal{F}(Z_{\text{ckt}}, Y)$ and the map $\widetilde{g}: \widetilde{v} \to -\widetilde{i}_{\text{g}}$, respectively.

V. Conditions for Synchronization

The results in this paper apply to Kron-reduced admittance matrices that satisfy the following *normality property*:

(P) The Kron-reduced admittance matrix Y is normal, that is, $YY^* = Y^*Y$. Consequently, Y can be diagonalized by a unitary matrix Q: $Y = Q\Lambda Q^*$, where $QQ^* = I$ and Λ is the diagonal matrix of the eigenvalues of Y.

We begin this section by describing classes of electrical networks that satisfy the normality property, and then present sufficient conditions for global asymptotic synchronization.

A. Identifying Electrical Networks that Satisfy Property (P)

We consider the following two broad classes of networks:

(i) *Networks with uniform line characteristics* [26], in which the branch admittances satisfy $y_{nm} = y_{\text{series}} a_{nm}$ for all $(n, m) \in \mathcal{E}$, where $a_{nm} \in \mathbb{R}$ is real-valued and $y_{\text{series}} \in \mathbb{C} \setminus \{0\}$ is identical for every branch.

(ii) *Homogeneous networks* [25], in which the effective impedances are identical for all boundary nodes, that is, $z_{nm} =: z_{\text{eff}} = r + \mathrm{j}x, r, x \in \mathbb{R}, \forall n, m \in \mathcal{N}$;

We begin our analysis with networks having uniform line characteristics. Essentially, all branches in these networks are made of the same material, and the admittance of each branch (n, m) depends on its constant per-unit-length admittance $y_{\text{series}} \in \mathbb{C}$ and its length $a_{nm} > 0$. These networks include as special cases resistive and lossless networks for which y_{series} is real-valued or purely imaginary, respectively. For these networks one can express $Y_{\mathcal{A}} = y_{\text{series}} \cdot L_{\mathcal{A}}$, where $L_{\mathcal{A}}$ is a symmetric, positive semidefinite, and real-valued Laplacian.

Lemma 2 *([27, Lemma 1]) Consider a network with uniform line characteristics, that is, $Y_{\mathcal{A}} = y_{\text{series}} \cdot L_{\mathcal{A}}$, where $L_{\mathcal{A}} \in \mathbb{R}^{|\mathcal{A}|\times|\mathcal{A}|}$ is a real-valued Laplacian and $y_{\text{series}} \in \mathbb{C}$. Then, the Kron-reduced network has uniform line characteristics with the Kron-reduced admittance matrix given by*

$$Y = y_{\text{series}} L, \quad L = L_{\mathcal{N}\mathcal{N}} - L_{\mathcal{N}\mathcal{I}} L_{\mathcal{I}\mathcal{I}}^{-1} L_{\mathcal{N}\mathcal{I}}^{\mathrm{T}}. \tag{17}$$

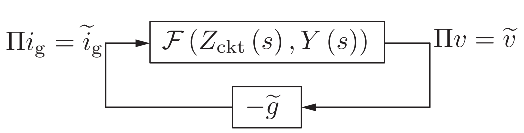


Fig. 4. Block-diagram of the differential system. The linear and nonlinear portions of the system are compartmentalized in $\mathcal{F}(\cdot,\cdot)$ and $\widetilde{g}$, respectively.

Due to the special form of the Kron-reduced matrix (17), Y is diagonalizable with a unitary matrix, and (P) is satisfied.

To address homogeneous networks, we recall from [25, Theorem III.4] that (in the purely resistive case) a sparse electrical network becomes denser under Kron reduction and even complete under mild connectivity assumptions. However, the branch admittances in the reduced network are still heterogeneous and reflect the topology and electrical properties of the original network. It turns out that for a homogeneous original network, the associated Kron-reduced network is characterized by identical branch admittances.

Lemma 3 *([27, Lemma 1]) The following are equivalent:*

(i) *The original network is homogeneous: for all boundary nodes $n, m \in \{1, \ldots, N\}$, the pairwise effective impedances take the uniform value $z_{nm} = z_{\text{eff}} \in \mathbb{C}\setminus\{0\}$.*

(ii) *The Kron-reduced network is complete and the branch admittances take the uniform value $y_{\text{series}} \in \mathbb{C} \setminus \{0\}$. Equivalently, the Kron-reduced admittance matrix is*

$$Y = y_{\text{series}}\Gamma\,, \tag{18}$$

where $\Gamma = NI - \mathbf{1}\mathbf{1}^{\mathrm{T}} = N\Pi$ is the Laplacian matrix of the complete graph.

If statements (i) and (ii) are true, then $z_{\text{eff}} = 2/(Ny_{\text{series}})$.

Since the Kron-reduced admittance matrix Y in the homogeneous case (18) is a special case of the Kron-reduced admittance matrix for the network with uniform line characteristics, it naturally satisfies (P).

B. Sufficient Condition for Asymptotic Synchronization

In this subsection, we derive sufficient conditions to ensure global asymptotic synchronization in the network of coupled nonlinear circuits. The following lemma establishes the global Lipschitz constant σ from (6) as an upper bound on the differential $\mathcal{L}_2$ gain of the function $g(\cdot)$.

Lemma 4 *The differential $\mathcal{L}_2$ gain of $g(\cdot)$ is upper bounded by the Lipschitz constant σ: $\widetilde{\gamma}(g) := \|\widetilde{i}_{\text{g}}\|_{\mathcal{L}_2} / \|\widetilde{v}\|_{\mathcal{L}_2} \leq \sigma$.*

Lemma 4 can be proved analogous to [12, Lemma 1]. We now provide a sufficient synchronization condition for the case where the nonlinear circuits are connected through networks with admittance matrices of the form (17) or (18).

Theorem 1 *Assume that the network that couples the system of N identical nonlinear circuits has no shunt elements and a Kron-reduced admittance matrix of the form $Y = y_{\text{series}}L$ as in (17) (or $Y = y_{\text{series}}\Gamma$ as in (18) as a particular case). The terminal voltages of the nonlinear circuits synchronize in the sense of (12) if for all $j \in \{2, \ldots, N\}$*

$$\|\mathcal{F}(z_{\text{ckt}}(\mathrm{j}\omega), y_{\text{series}}(\mathrm{j}\omega)\lambda_j)\|_{\infty}\sigma < 1\,, \tag{19}$$

where λ_j, $j \in \{2, \ldots, N\}$, are the nonzero eigenvalues of the Laplacian matrix L.

Proof: Consider the block-diagram of the differential system in Fig. 4. From Lemma 4, we have

$$\|\widetilde{i}_{\text{g}}\|_{\mathcal{L}_2} \leq \sigma\|\widetilde{v}\|_{\mathcal{L}_2}. \tag{20}$$

For the linear fractional transformation, we can write

$$\|\widetilde{v}\|_{\mathcal{L}_2} \leq \widetilde{\gamma}\left(\mathcal{F}\left(Z_{\mathrm{ckt}}, Y\right)\right)\|\widetilde{i}_{\mathrm{g}}\|_{\mathcal{L}_2} + \eta, \tag{21}$$

for some non-negative η, where $\widetilde{\gamma}\left(\mathcal{F}\left(Z_{\mathrm{ckt}}, Y\right)\right)$ denotes the differential $\mathcal{L}_2$ gain of the linear fractional transformation. By combining (20) and (21), we arrive at

$$\|\widetilde{v}\|_{\mathcal{L}_2} \leq \widetilde{\gamma}\left(\mathcal{F}\left(Z_{\mathrm{ckt}}, Y\right)\right)\sigma\|\widetilde{v}\|_{\mathcal{L}_2} + \eta. \tag{22}$$

By isolating $\|\widetilde{v}\|_{\mathcal{L}_2}$ from (22), we can write

$$\|\widetilde{v}\|_{\mathcal{L}_2} \leq \eta/(1 - \widetilde{\gamma}\left(\mathcal{F}\left(Z_{\mathrm{ckt}}, Y\right)\right)\sigma), \tag{23}$$

provided that the following condition holds

$$\widetilde{\gamma}\left(\mathcal{F}\left(Z_{\mathrm{ckt}}, Y\right)\right)\sigma < 1. \tag{24}$$

If (24) holds true, then $\widetilde{v} \in \mathcal{L}_2$. It follows from Barbalat's lemma [28] that $\lim_{t\to\infty} \widetilde{v}(t) = \mathbf{0}$. Hence, if the network of nonlinear circuits satisfies the condition (24), global asymptotic synchronization can be guaranteed.

In the remainder of the proof, we establish an equivalent condition for (24). By definition of the differential $\mathcal{L}_2$ gain of the linear fractional transformation, we can express

$$\begin{aligned}\widetilde{\gamma}\left(\mathcal{F}\left(Z_{\mathrm{ckt}}, Y\right)\right) &= \sup_{\omega\in\mathbb{R}} \frac{\left\|\mathcal{F}\left(z_{\mathrm{ckt}}\left(\mathrm{j}\omega\right)I, Y\left(\mathrm{j}\omega\right)\right)\widetilde{i}_{\mathrm{g}}\left(\mathrm{j}\omega\right)\right\|_2}{\left\|\widetilde{i}_{\mathrm{g}}(\mathrm{j}\omega)\right\|_2} \\ &= \sup_{\omega\in\mathbb{R}} \frac{\left\|\left(I + z_{\mathrm{ckt}}(\mathrm{j}\omega)Y\left(\mathrm{j}\omega\right)\right)^{-1} z_{\mathrm{ckt}}(\mathrm{j}\omega)\widetilde{i}_{\mathrm{g}}\left(\mathrm{j}\omega\right)\right\|_2}{\left\|\widetilde{i}_{\mathrm{g}}(\mathrm{j}\omega)\right\|_2} \\ &= \sup_{\omega\in\mathbb{R}} \frac{\left\|Q\left(I + z_{\mathrm{ckt}}(\mathrm{j}\omega)y_{\mathrm{series}}(\mathrm{j}\omega)\Lambda\right)^{-1} z_{\mathrm{ckt}}(\mathrm{j}\omega)Q^{\mathrm{T}}\widetilde{i}_{\mathrm{g}}(\mathrm{j}\omega)\right\|_2}{\left\|Q^{\mathrm{T}}\widetilde{i}_{\mathrm{g}}(\mathrm{j}\omega)\right\|_2},\end{aligned}$$

where we made use of property (P) to diagonalize the admittance matrix as $Y = y_{\mathrm{series}}L = y_{\mathrm{series}} \cdot Q\Lambda Q^{\mathrm{T}}$, where Q is unitary and Λ is a diagonal matrix containing the real-valued and nonnegative eigenvalues of the Laplacian L.

Given that the Kron-reduced network is connected, Y and L have simple zero eigenvalues with associated eigenvectors $q_1 = (1/\sqrt{N})\mathbf{1}$. Since $\mathbf{1}^{\mathrm{T}}\Pi = \mathbf{0}^{\mathrm{T}}$, we obtain $Q^{\mathrm{T}}\widetilde{i}_{\mathrm{g}} = Q^{\mathrm{T}}\Pi i_{\mathrm{g}} = [0,\, p]^{\mathrm{T}}$, where $p \in \mathbb{C}^{N-1}$ is made of the nonzero elements of $Q^{\mathrm{T}}\Pi i_{\mathrm{g}}$. If we denote the diagonal matrix of non-zero eigenvalues of Y by Λ_{N-1}, we get:

$$\begin{aligned}&\widetilde{\gamma}\left(\mathcal{F}\left(z_{\mathrm{ckt}}I, Y\right)\right) \\ &= \sup_{\omega\in\mathbb{R}} \frac{\left\|\left(I_{N-1} + z_{\mathrm{ckt}}(\mathrm{j}\omega)y_{\mathrm{series}}(\mathrm{j}\omega)\Lambda_{N-1}\right)^{-1} z_{\mathrm{ckt}}(\mathrm{j}\omega)p(\mathrm{j}\omega)\right\|_2}{\|p(\mathrm{j}\omega)\|_2} \\ &= \max_{j=2,\dots,N} \sup_{\omega\in\mathbb{R}} \left|\frac{z_{\mathrm{ckt}}(\mathrm{j}\omega)}{1 + z_{\mathrm{ckt}}(\mathrm{j}\omega)y_{\mathrm{series}}(\mathrm{j}\omega)\lambda_j}\right|. \end{aligned} \tag{25}$$

By combining (25) and (24), we arrive at condition (19). □

For a homogeneous network, the Kron-reduced matrix is $Y = y_{\mathrm{series}}\Gamma$. Since the eigenvalues of Γ, are $\lambda_1 = 0$ and $\lambda_2 = \cdots = \lambda_N = N$, condition (19) reduces to

$$\|\mathcal{F}(z_{\mathrm{ckt}}(\mathrm{j}\omega), y_{\mathrm{series}}(\mathrm{j}\omega)N)\|_\infty\sigma < 1.$$

For purely resistive systems, the synchronization condition (19) has to be evaluated only for the second-smallest eigenvalue λ_2 of the Laplacian, which is known as the *algebraic connectivity*. It is known that the algebraic connectivity in a resistive Kron-reduced network upper-bounds the algebraic connectivity in the original network [25, Theorem III.5]. Hence, condition (19) implies that the nonlinear circuits should be sufficiently strongly connected, which is aligned with synchronization results in complex oscillator networks with a static interconnection topology (i.e., with only resistive elements) and without unstable internal oscillator dynamics (e.g., passive oscillator subsystems) [2]. On the other hand, if the interconnecting network is *dynamic*, for example, if it contains capacitive or inductive storage elements, then the synchronization condition (19) needs to be evaluated for all nonzero network modes λ_j, $j \in \{2, \dots, N\}$.

VI. Case Studies

We now present simulation case studies to validate the synchronization conditions in an illustrative LTI electrical network that interconnects *Chua's circuits* [9]. The voltage-dependent current source in Chua's circuit is illustrated in Fig. 2(a). Since the function $g(\cdot)$ in Fig. 2(a) is piecewise linear, it satisfies (6). All parameters can be found in [27].

Consider a connected lossless network with uniform (inductive) line characteristics, as illustrated in Fig. 5. For the set of network parameters in [27] the sufficient synchronization condition (19) is met. Figure 6(a) illustrates the terminal voltages and the voltage synchronization error with non-identical initial conditions as the voltages begin to pull into phase. Figure 6(b) depicts a three-dimensional view of the internal states of the Chua's circuits as a function of time, and clearly demonstrates the chaotic double-scroll attractor [9] in the asymptotic limit. For another set of network parameters found in [27], the sufficient synchronization condition (19) is violated (in particular, $||\mathcal{F}(\cdot,\cdot)||_\infty\sigma = 1.07 > 1$). While this is not an indication that the terminal voltages cannot synchronize (since the condition (19) is only sufficient), it turns out that, in this case, the terminal voltages indeed do not synchronize, as illustrated in the simulation in Fig. 7 (with the same initial conditions as before in Fig. 6).

VII. Conclusions

We derived a synchronization condition for a class of nonlinear electrical circuits coupled through dynamic LTI electrical networks. We considered particular classes of networks,

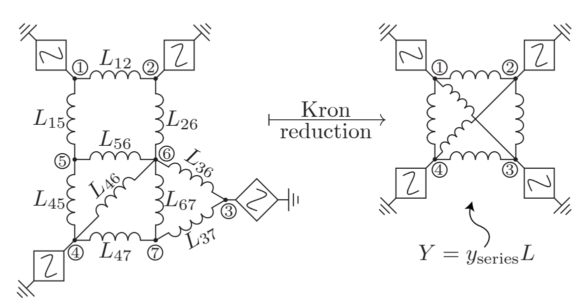
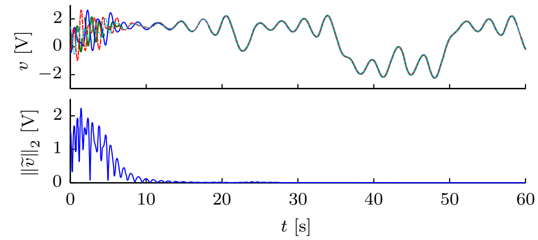
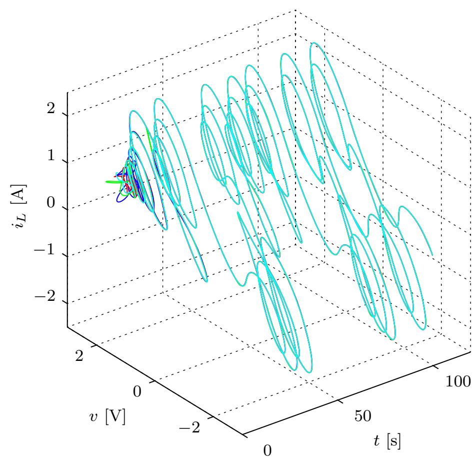


Fig. 5. Schematic of lossless network and its Kron-reduced counterpart.



(a) Terminal voltages $v(t)$ and synchronization error $||\widetilde{v}(t)||_2$.



(b) Chaotic double-scroll attractor in the asymptotic limit.

Fig. 6. Synchronization of terminal voltages in Chua's circuits.

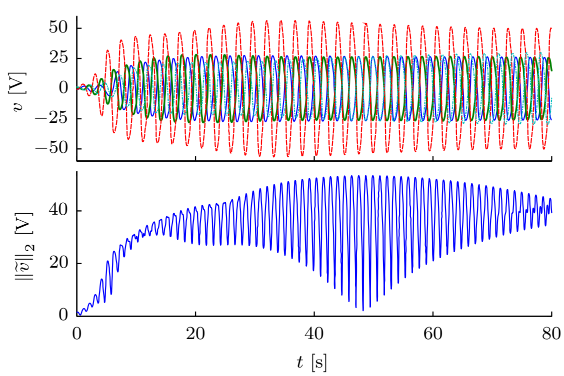


Fig. 7. Terminal voltages $v(t)$ and synchronization error $||\widetilde{v}(t)||_2$ when condition (19) fails: $||\mathcal{F}(z_{\mathrm{ckt}}(\mathrm{j}\omega), y_{\mathrm{series}}^{-1}(\mathrm{j}\omega)\lambda_j||_\infty \sigma \nless 1, j = 2, 3, 4$.

where perfect synchronization of the terminal voltages can be achieved. These classes included homogeneous networks and networks with uniform line characteristics. In ongoing and future work, we aim at extending the present analysis to more general circuits with heterogeneous shunt elements and without normal Kron-reduced matrices. Finally, we want to alleviate the potentially restrictive global Lipschitz condition (6) on the nonlinear current sources.

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