Amidst centralized and distributed frequency control in power systems

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Abstract—We propose a novel frequency control approach in between centralized and distributed architectures, that is a generalized continuous-time feedback control version of the dual decomposition optimization method. Specifically, a convex combination of the frequency measurements is centrally aggregated and followed by an integral control, which is broadcast as the control signal, and then by the optimal local allocations. We show that our controller comprises many previously proposed strategies for specific parameter sets. Under mild parametric assumptions, we prove local asymptotic stability of the closed-loop equilibria of the power system, which is modeled as a nonlinear, differential-algebraic, dynamical system that includes traditional generators, frequency-responsive as well as passive loads, where the sources are already equipped with primary droop control. Our feedback control is designed such that the closed-loop equilibria of the power system solve the optimal economic dispatch problem.

I. Introduction

The quintessential task of power system operation is to match electrical load and generation. The power balance in an AC grid can be directly accessed via the system frequency, making frequency regulation the fundamental mechanism to ensure the load-generation balance. This task is subject to operational constraints, system stability, and economic interests, and it is traditionally accomplished by adjusting generation in a hierarchical structure consisting of three layers: primary (droop control), secondary (automatic generation control) and tertiary (economic dispatch) - from fast to slow timescales, and from decentralized to centralized architectures [1], [2].

With the increasing integration of distributed renewable energy generation, such as solar and wind power, the grid is subject to larger and faster fluctuations in power supply. Therefore, frequency control requires more fast-ramping generators to act as spinning reserves, which is expensive, inefficient, and the resulting emissions defeat the purpose of renewables [3]. As a complement, distributed frequency control through inverter-based sources [4] or loads [5] has a high potential due to the fast ramping capabilities of these devices. In any case, the task of frequency regulation will have to be shouldered by more and more small-scale and distributed devices.

From a control-theoretic perspective, the main objective of frequency control is to stabilize the system frequency to the nominal value, subject to operational constraints and economic interests such as load sharing, optimal economic

generation dispatch, or according to the outcome of reserve markets. Possible further constraints include a partial information structure accounting for a distributed generation environment, liberalized markets, and limited system knowledge. A plethora of strategies has been developed to address these tasks ranging from fully decentralized to centralized architectures, partially relying on time-scale separation and hierarchical control, and being dependent on the detailed system model, load and generation forecasts. While robustness to failures and uncertainties is an issue for centralized architectures relying on detailed models, this is not the case for distributed frequency control approaches. The major drawback of the latter, in terms of practical implementation, is that a massive (often bidirectional) communication architecture is required. We postpone a detailed literature review to Section II-D, after introducing the problem setup.

In this paper, we consider a nonlinear, differential-algebraic, and heterogeneous power system model including traditional generation, power electronic sources, and frequency-responsive as well as passive loads. We assume that the sources are already equipped with primary droop control, and we focus on designing the secondary control strategy while simultaneously solving a tertiary economic dispatch problem.

Our control approach falls square in between centralized and distributed architectures, and is motivated and developed by exploiting parallels in dual decomposition methods in optimization [6], auctions in markets [7], mean field control [8], as well as classic Automatic Generation Control (AGC) [1]. Interestingly, our controller includes many previous frequency control strategies for specific parameter sets.

Specifically, we first develop an online optimization routine for the steady-state dynamics that evaluates the price of frequency violation in feedback with the optimal generation response of each generator. Next, we propose a continuous-time feedback control version of this optimization scheme as a centralized aggregation of a convex combination of measurements, followed by integral control and optimal local allocations of the broadcast control signal. Our feedback control law is such that the closed-loop equilibria of the power system solve the optimal economic dispatch problem, and in addition is transiently optimal, in the sense that identical marginal costs are achieved during transients. Under mild assumptions on the parameter design, we prove local asymptotic stability of the closed-loop equilibria of the nonlinear differential-algebraic power system.

We emphasize that our frequency control scheme does not require any model knowledge, it relies only on unidirectional communication, and it is privacy preserving, that is, no partici-

pant needs to communicate its internal model or cost function.

The paper is organized as follows. In Section II we formally introduce the frequency control problem in power systems, that includes both the frequency regulation and the optimal economic dispatch. In Section III we propose our novel frequency control method, and in Section IV we show local asymptotic stability of the set of closed-loop equilibria, under proper choice of some parameters. In Section V, we illustrate the performance of our controller with a simulation case study on the IEEE39 New England grid, and we also compare to other controllers. Section VI concludes the paper and raises some open questions.

Notation

$\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{R}_{\geq0}$ respectively denote the set of real, positive real, non-negative real numbers; $A^{\top} \in \mathbb{R}^{m \times n}$ denotes the transpose of $A \in \mathbb{R}^{n \times m}$. For given matrices $A_1, \ldots, A_M$, $\mathrm{diag}\,(A_1, \ldots, A_M)$ denotes the block diagonal matrix with $A_1, \ldots, A_M$ in block diagonal positions. $\mathbb{1}$ ($\mathbb{0}$) denotes a matrix/vector with elements all equal to 1 (0).

II. THE FREQUENCY CONTROL PROBLEM IN POWER SYSTEMS

A. Power system model

Consider a power system modeled as a graph $G = (\mathcal{V}, \mathcal{E})$ with nodes (or buses) $\mathcal{V} = \{1, \ldots, N\}$ and edges (or branches) $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. With each bus $i \in \mathcal{V}$, we associate the complex voltage phasor $V_i \exp(\mathrm{i}\theta_i)$ corresponding to a harmonic voltage waveform $V_i \cos(\omega^* t + \theta_i)$, where $\omega^* = 2\pi \cdot 50\,\mathrm{Hz}$ is the nominal network frequency. We consider a high-voltage transmission network with lossless lines. The network topology is then induced by the sparse susceptance matrix $\tilde{B} \in \mathbb{R}^{N \times N}$. We partition the set of buses according to the power sources and sinks connected to it as $\mathcal{V} = \mathcal{G} \cup \mathcal{F} \cup \mathcal{P}$ corresponding to synchronous generators $\mathcal{G}$, buses with frequency-responsive devices $\mathcal{G}$ (e.g., frequency-sensitive loads or inverter sources performing droop control), and passive buses $\mathcal{P}$ (e.g., static loads or inverters performing maximum power-point tracking). The associated dynamic model reads as follows [9]:

$$\forall i \in \mathcal{G} : M_i \ddot{\theta}_i + D_i \dot{\theta}_i = P_i + u_i - \sum_{j \in \mathcal{V}} B_{i,j} \sin(\theta_i - \theta_j) \tag{1a}$$

$$\forall i \in \mathcal{F} : D_i \dot{\theta}_i = P_i + u_i - \sum_{j \in \mathcal{V}} B_{i,j} \sin(\theta_i - \theta_j) \tag{1b}$$

$$\forall i \in \mathcal{P} : 0 = P_i + u_i - \sum_{j \in \mathcal{V}} B_{i,j} \sin(\theta_i - \theta_j), \tag{1c}$$

where $P_i \in \mathbb{R}$ is a constant power injection or demand (positive for sources and negative for loads), $u_i \in \mathcal{U}_i = [\underline{u}_i, \overline{u}_i] \subset \mathbb{R}$ is a controllable injection or demand, and $B_{i,j} := \tilde{B}_{i,j} V_i V_j$ is the effective susceptance for all $i, j \in \mathcal{V}$. A generator $i \in \mathcal{G}$ is characterized by its rotational inertia $M_i > 0$ and primary droop control coefficient $D_i > 0$. A frequency-responsive device $i \in \mathcal{F}$ is characterized by its frequency-sensitivity $D_i > 0$ (e.g, the droop coefficient for inverters or actively controlled loads, or the damping of a frequency-dependent load). Finally, passive buses (inverters performing maximum power-point tracking and static loads)

have no dynamics. We do not model reactive power and voltage dynamics, since they do not affect the forthcoming analyses – though everything can be extended.

B. Frequency regulation

We note that, if there is a stable synchronized solution to (1) satisfying $\dot{\theta}_i = \omega_{\mathrm{sync}} > 0$ for all $i \in \mathcal{V}$, then by summing up all steady-state equations (1), the *synchronous frequency* is

$$\omega_{\mathrm{sync}} = \frac{\sum_{i \in \mathcal{V}} P_i + u_i}{\sum_{i \in \mathcal{G} \cup \mathcal{F}} D_i} \tag{2}$$

If transmission losses are integrated in the power system model (1), there would be another strictly negative term on the right-hand side of (2) that depends on the steady-state flow pattern.

Note that in absence of controllable injections $\{u_i\}_{i \in \mathcal{V}}$ the synchronous frequency ω_{sync} is determined by the constant power injections $\{P_i\}_{i \in \mathcal{V}}$ of possibly slow-ramping generation units, fluctuating renewable sources, and unknown loads. We are interested in regulating the frequency deviation (2) to its nominal (zero) value by scheduling the controllable injections:

Problem 1 (Frequency regulation): Schedule the controllable injections $\{u_i \in \mathcal{U}_i\}_{i \in \mathcal{V}}$ to balance load and generation, that is, so that the frequency deviation ω_{sync} in (2) is zero. $\square$

C. Centralized and competitive resource allocation

A basic *feasibility condition* to solve Problem 1 is that the total power imbalance can be met by the controllable and constrained injections $\{u_i \in \mathcal{U}_i = [\underline{u}_i, \overline{u}_i]\}_{i \in \mathcal{V}}$.

Assumption 1 (Feasibility): $-\sum_{i \in \mathcal{V}} P_i \in \sum_{i \in \mathcal{V}} [\underline{u}_i, \overline{u}_i]$. $\square$

If this feasibility condition is met, then there can be many options to schedule the controllable injections $\{u_i\}_{i \in \mathcal{V}}$ to (asymptotically) regulate ω_{sync} in (2) to zero.

Since we are also interested in solving a resource allocation problem, we associate to every controllable injection a cost function to trade-off operating costs, emissions, capacities, and other levels of preference.

Problem 2 (Optimal economic power dispatch): Schedule the controllable injections $\{u_i \in \mathcal{U}_i\}_{i \in \mathcal{V}}$ to balance load and generation, while minimizing the aggregate operational cost

$$\begin{aligned} \min_{u \in \mathbb{R}^N} \quad & \sum_{i \in \mathcal{V}} J_i(u_i) \\ \mathrm{s.t.} \quad & \sum_{i \in \mathcal{V}} P_i + u_i = 0, \end{aligned} \tag{3}$$

where, for all $i \in \mathcal{V}$, $J_i : \mathcal{U}_i \to \mathbb{R}$ is strictly convex and differentiable on its domain. $\square$

Note that the constraints $\{u_i \in \mathcal{U}_i = [\underline{u}_i, \overline{u}_i]\}_{i \in \mathcal{V}}$ can be directly incorporated in the domain of the cost functions $\{J_i\}_{i \in \mathcal{V}}$, e.g, via barrier functions. The economic dispatch problem in (3) is typically solved on different time scales and, in a longer planning horizon, and it often includes binary unit-commitment constraints and inequality constraints penalizing power flows violating the thermal constraints. Here we focus on the reserve scheduling problem, where fast ramping generation and controllable loads are dispatched to meet the

real-time net demand indicated by the frequency deviation in (2).

Let us now consider the Lagrangian function associated with the economic dispatch optimization problem in (3), that is,

$$\ell(u, \lambda) := \sum_{i \in \mathcal{V}} J_i(u_i) - \lambda (u_i + P_i), \qquad (4)$$

where the scalar $\lambda \in \mathbb{R}$ is the Lagrange multiplier associated with the constraint $\sum_{i \in \mathcal{V}} u_i + P_i = 0$ in (3). The necessary KKT optimality conditions [10] require that

$$\frac{\partial \ell(u, \lambda)}{\partial u} = 0 \implies J_i'(u_i^\star) = \lambda^\star \ \forall i \in \mathcal{V}, \qquad (5)$$

where J_i' is the derivative of J_i. A basic insight from condition (5) is the *economic dispatch criterion* [2] stating that all marginal utilities must be identical in the unconstrained case:

$$J_i'(u_i^\star) = J_j'(u_j^\star) \quad \forall i, j \in \mathcal{V}. \qquad (6)$$

So far we took the perspective of centralized social welfare optimization. From a competitive market perspective (in particular, a spot market), consider the *utility maximization* (cost minus benefit minimization) of each market participant $i \in \mathcal{V}$

$$\min_{u_i \in \mathcal{U}_i} \ell(u, \lambda) = \min_{u_i \in \mathcal{U}_i} \{ J_i(u_i) - \lambda u_i \}. \qquad (7)$$

Here, λ is the *nodal price* which is identical for every participant (in this setup neglecting network congestion). The optimal generation as a function of the price is then obtained by $u_i(\lambda^\star) = J_i'^{-1}(\lambda^\star)$. Accordingly, the constraint in (3) can then be formulated as the intersection of the aggregated supply bid and demand curves:

$$0 = \sum_{i \in \mathcal{V}} P_i + u_i = \sum_{i \in \mathcal{V}} P_i + J_i'^{-1}(\lambda^\star). \qquad (8)$$

Based upon (8) the *market clearing price* λ^* is determined.

Independent of a centralized optimization or competitive market setup, solving Problem 2 also amounts to asymptotically regulating the frequency, that is, solving Problem 1. Frequency regulation is often referred to as secondary control, whereas offline optimization is referred to as tertiary control. As there are no clear boundaries between these two control objectives, there are many solutions available in the literature to solve the optimal economic power dispatch in (3) via online frequency regulation algorithms. These solutions range from the classic centralized *automatic generation control* (AGC) [1], [2] to distributed optimal frequency regulation based on continuous-time or discrete-time averaging algorithms. Let us provide a brief review in the following paragraph.

D. Literature review on (de)centralized frequency regulation

To regulate steady-state deviations in the synchronous frequency, one may consider simple *decentralized secondary integral controllers* at every source, that is,

$$k \dot{p}_i = \dot{\theta}_i, \quad u_i = -p_i \quad \forall i \in \mathcal{V},$$

where $k > 0$ is a constant gain. Such decentralized integral controllers regulate the frequency, but they induce additional closed-loop equilibria resulting in undesired injection profiles violating load sharing and economic dispatch objectives [11]. Indeed, it is well known in power systems [9], as well as in control theory [12], that multiple decentralized integral controllers generally fail to achieve frequency regulation while maintaining a desired injection profile among generating units, and they may induce internal instabilities [13].

The current industrial standard is the centralized AGC [1], [2] where a single frequency measurement is integrated (typically together with the area control error) at a site $i^\star \in \mathcal{V}$, and the required generation mismatch is then allocated to individual generating units according to their *participation factors* $\{1/A_i > 0\}_{i \in \mathcal{V}}$, often selected as inverse ratings of the sources, which define their individual contribution:

$$k \dot{p} = \dot{\theta}_{i^\star}, \quad u_i = -\frac{1}{A_i} p \quad \forall i \in \mathcal{V}. \qquad (9)$$

Note that the AGC signal (9) may be written as

$$k \dot{p} = \dot{\theta}_{i^\star}, \quad u_i = -J_i'^{-1}(p) \quad \forall i \in \mathcal{V}, \qquad (10)$$

if the cost function J_i is defined as $J_i(u_i) = \frac{1}{2} A_i u_i^2$. Hence, the AGC strategy (10) achieves identical marginal costs as in (6) and is thus implicitly optimal for a quadratic cost function. On the other hand, the above strategy is centralized; in a distributed generation environment, such a setup is not robust; additionally, a single node $i^\star$ may not have the authority to command the secondary control strategies of all other nodes.

As a remedy to the above problems, *distributed secondary integral controllers* have been proposed that average the integral actions among the generation units through a communication network between the controllers. Different distributed secondary integral approaches have been proposed on the basis of *continuous-time consensus averaging* with all-to-all [14]–[16] or nearest-neighbor [17]–[20] communication. These distributed secondary control approaches can be merged with the tertiary optimization layer, based on the economic dispatch criterion (6) that all marginal utilities must be identical, so the integral control gains can be then adjusted to this aim. Different approaches realize this objective based on *continuous-time optimization* approaches [11], [13], [21]–[27], game-theoretic ideas [28], nodal pricing algorithms [29], or discrete-time algorithms [30]–[32]. All of these algorithms essentially rely on the fact that frequencies and marginal costs should be identical in an optimal steady state. Accordingly, for all $i \in \mathcal{V}$, distributed integral controllers of the form

$$k \dot{p}_i = \dot{\theta}_i - \sum_{j \in \mathcal{V}} w_{i,j} \left(J_i'(u_i) - J_j'(u_j) \right), \quad u_i = -p_i, \qquad (11)$$

are added, where $W = W^\top \in \mathbb{R}_{\geq 0}^{N \times N}$ induces an undirected and connected communication network. One major drawback of these distributed strategies in a real-world implementation is that the existing controllable load and generation units must be retrofit with massive bidirectional communication architecture to execute the distributed algorithms. In comparison with the centralized price-based coordination in (7)-(8), the strategy in (11) relies on bilateral agreements, and one can imagine scenarios where individual agents aim to maximize their benefit by reporting biased marginal costs to their neighbors. Finally, aside from the above concerns on operational cost and market power, other issues include vulnerabilities to cyber-physical security breaches and the utilities' concern that they give the power system control out of their hands.

III. PRICE-BROADCASTING FREQUENCY CONTROL

A. Market-based insight: discrete-time dual decomposition

In the following, we provide an alternative frequency control algorithm based on a central price update for violating the power balance, and inspired by the market-based insights to economic dispatch optimization presented in Subsection II-C.

Specifically, we exploit the fact that the Lagrangian function $\ell(u, \lambda)$ in (4) associated with the optimization problem in (3) is separable. Therefore, as the individual costs $\{J_i\}_{i \in \mathcal{V}}$ are strictly convex and bounded, the optimization problem in (3) can be solved iteratively via the gather-and-broadcast dual decomposition method [6, Sections 2.1–2.2]. This reads as the following iterative primal-dual algorithm, where $k \in \mathbb{N}$ denotes a discrete time step, and $(\alpha_k)_{k \in \mathbb{N}}$ is a sequence of sufficiently small positive scalars:

$$u_i(k+1) := \arg\min_{u \in \mathbb{R}} J_i(u) - \lambda(k)\, u, \quad \forall i \in \mathcal{V}, \quad (12)$$

$$\lambda(k+1) := \lambda(k) - \alpha_k \left(\sum_{i \in \mathcal{V}} P_i + u_i(k+1) \right) \quad (13)$$

At every discrete time step k, each node $i \in \mathcal{V}$ computes its optimal injection according to (12) as a function of the current price $\lambda(k)$. At the same time, the price $\lambda(k)$ for the power imbalance is updated in (13) via a discrete-time integral type control of the power balancing error, that is directly measurable through the frequency signal

$$\omega(k) := \sum_{i \in \mathcal{V}} P_i + u_i(k+1), \quad (14)$$

thus, the dual update in (13) can drive the frequency error to zero. The iteration in (12)–(13) can be implemented in a *semi-decentralized* fashion. The dual update (or integral control) in (13) determining the current price $\lambda(k)$ is performed at a central site[1] using the steady-state frequency error (14), and the primal update (12) can then be carried out locally as a function of the current price $\lambda(k)$ and the generator cost function $J_i(\cdot)$. In this regard, the gather-and-broadcast update in (12)-(13) is conceptually similar to AGC in (9), with the advantage of guaranteed global convergence [33, Chapter 6], [6, Chapter 2], even for non-quadratic costs and local injection constraints.

Proposition 1 (Discrete-time global convergence): The sequence $(u_1(k), \ldots, u_N(k); \lambda(k))_{k \in \mathbb{N}}$ defined iteratively in (12)–(13) asymptotically converges, from any initial condition, to the unique primal-dual optimal solution to (3). □

As a final remark, from a market perspective, the updates (12) and (13) correspond to an iterative local utility maximization (12), communication of bids $u_i^\star(k+1)$, subsequent price announcement (13), which is again followed by the optimal generation response (12), and so on. Such a scheme is referred as an *auction* [7]. Auctions are known to be decentralized yet robust market mechanisms compared to the *exchange trade* based on a (central) price (7)-(8) and the bilateral *over the*

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[1] In principle, the update in (13) can also be carried out locally using $\dot\theta_i$ instead of (14), since the steady-state frequency error (2) in (14) is identical throughout the network. On the other hand, if the step sizes $\{\alpha_k\}_k$ are not identical or if the frequencies are not perfectly synchronized in steady state, then such decentralized integral updates may fail to solve the economic dispatch problem in (3), see [11] for a continuous-time analysis.

counter trading scheme (11) [34] - all of which lead to a Pareto-optimal solution to (7) or the optimal solution to (3).

B. Continuous-time price-broadcasting frequency control

Motivated by the dual decomposition algorithm in (12)-(13), we derive a *continuous-time* version that acts as a feedback control law stabilizing the frequency deviations of the nonlinear differential-algebraic model in (1). The following setup complements the different centralized and distributed frequency regulation approaches reviewed in Section II-D.

We assume that a central aggregator collects a set of frequency measurements in the network and integrates these measurements to form the overall area frequency error as

$$k\,\dot p = \sum_{i \in \mathcal{V}} C_i \dot\theta_i, \quad (15)$$

where $k > 0$ is a scalar gain, and $C_i \in [0, 1]$ is a set of convex coefficients. Next the signal p from (15) is broadcast to the individual nodes, where it is dispatched according to

$$u_i := -J_i'^{-1}(p) \quad \forall i \in \mathcal{V}. \quad (16)$$

This feedback control scheme relies on the following mean-field-type loop [8]: construction of the measurement average $\sum_{i \in \mathcal{V}} C_i \dot\theta_i$ as a global variable, that is centrally processed via an integrator, and then broadcast back to the individual nodes. Note that the broadcast-topology is "one-to-all", whereas in principle the measurement aggregation can include either only one measurement or possibly all measurements.

We observe that generation allocation in (16) is transiently optimal, that is, it achieves identical marginal costs $J_i'(u_i^\star(t)) = J_j'(u_j^\star(t))$ as in (6) for all $t \geq 0$, that is, *even during transients*.

C. Comparison with methods proposed in the literature

For specific parameter choices, the price-broadcasting frequency control in (15)–(16) reduces to different control architectures proposed in the literature, as summarized next.

Automatic Generation Control [1], [2]: If only a single measurement coefficient C_i is non-zero and each cost is J_i is quadratic, then the control scheme in (15)-(16) reduces exactly to the conventional AGC in (9), where the area control error is allocated to the individual generators according to their participation factors corresponding to inverse marginal costs.

All-to-all averaging control [11], [14]–[16]: The gains $C_i = D_i$ for all $i \in \mathcal{V}$ have been employed for the analysis of centralized averaging-based PI controllers in [11], [14] whose experimental implementations can be found in [15], [16].

Mean field control [8]: If all frequencies are measured and weighted equally, i.e., $C_i = 1$ for all $i \in \mathcal{V}$, then we have a true mean-field setup where all nodes are treated equally.

Market mechanism [7]: The control scheme in (15)–(16) corresponds to a continuos auction mechanism, where the accumulated frequency error in (15) serves as pricing signal.

IV. CLOSED-LOOP STABILITY ANALYSIS

A. Closed-loop equilibria

The overall closed-loop system in (1), (15)–(16) has the property that if an equilibrium exists, then, by summing all steady-state equations (1), (15)–(16), it satisfies

$$\sum_{i \in \mathcal{V}} P_i = \sum_{i \in \mathcal{V}} J_i'^{-1}(p^*). \tag{17}$$

Due to the strict convexity of the cost functions $\{J_i\}_{i \in \mathcal{V}}$, it holds that $\{J_i'^{-1}\}_{i \in \mathcal{V}}$ are strictly increasing functions, and therefore (17) admits a unique solution $p^* \in \mathbb{R}$.

In vector form the closed-loop system in (1), (15)–(16) reads as

$$\dot{\theta}_\mathcal{G} = \omega_\mathcal{G} \tag{18a}$$

$$M \dot{\omega}_\mathcal{G} = -D_\mathcal{G} \omega_\mathcal{G} - \left(\nabla_\mathcal{G} U(\theta) - P_\mathcal{G} \right) - J_\mathcal{G}'^{-1}(p) \tag{18b}$$

$$D_\mathcal{F} \dot{\theta}_\mathcal{F} = -\left(\nabla_\mathcal{F} U(\theta) - P_\mathcal{F} \right) - J_\mathcal{F}'^{-1}(p) \tag{18c}$$

$$\mathbb{0}_{|\mathcal{P}|} = -\left(\nabla_\mathcal{P} U(\theta) - P_\mathcal{P} \right) - J_\mathcal{P}'^{-1}(p) \tag{18d}$$

$$k \dot{p} = \mathbb{1}_N^\top C \dot{\theta}, \tag{18e}$$

where we used the shorthand notations $M_\mathcal{G} := \operatorname{diag}(\{M_i\}_{i \in \mathcal{G}})$, $D_\mathcal{G} := \operatorname{diag}(\{D_i\}_{i \in \mathcal{G}})$, $D_\mathcal{F} := \operatorname{diag}(\{D_i\}_{i \in \mathcal{F}})$, $C := \operatorname{diag}(\{C_i\}_{i \in \mathcal{V}})$, and we introduced the network potential function

$$U : \mathbb{T}^n \to \mathbb{R}, \quad U(\theta) := \sum_{(i,j) \in \mathcal{E}} B_{i,j} \left(1 - \cos(\theta_i - \theta_j) \right)$$

satisfying $\mathbb{1}_N^\top \nabla U(\theta) = 0$ due to the symmetry of the flow.

In steady state, with $\dot{\theta} = \mathbb{0}_N$, $\dot{\omega} = \mathbb{0}_N$, and $\dot{p} = 0$, the equilibria of the closed-loop system in (18) can be characterized as follows.

Lemma 1: The equilibria $(\theta^*, \omega^*, p^*)$ of the closed-loop system (18) are such that $\omega_\mathcal{G}^* = \mathbb{0}_{|\mathcal{G}|}$, $p^* \in \mathbb{R}$ is the unique solution to (17), and the steady-state injections are $\nabla U(\theta^*) = P - J'^{-1}(p^*)$. Moreover, each equilibrium is an optimal solution to the economic dispatch problem in (3). □

Proof: From (18a) with $\dot{\theta}_\mathcal{G} = \mathbb{0}_{|\mathcal{G}|}$, it immediately follows that $\omega_\mathcal{G}^* = \mathbb{0}_{|\mathcal{G}|}$. Equations (18b)–(18d) read in steady state as

$$\begin{bmatrix} I & I \end{bmatrix} \begin{bmatrix} \nabla U(\theta^*) - P \\ J'^{-1}(p^*) \end{bmatrix} = \mathbb{0}_N.$$

If we multiply these equations by $\mathbb{1}_N^\top$, since $\mathbb{1}_N^\top \nabla U(\theta^*) = 0$, we obtain equation (17), which admits a unique solution $p^* \in \mathbb{R}$. Finally, optimality of the steady-state injections follows by construction of the control law in (16). ∎

B. Local asymptotic stability

In the following we perform a stability analysis of the equilibria of the differential-algebraic closed-loop system in (18), characterized in Lemma 1. For simplicity and to aim at a compact presentation, we make the following assumptions.

Assumption 2 (Quadratic cost functions): For all $i \in \mathcal{V}$, the cost function J_i in (3) is defined as $J_i(u_i) := \frac{1}{2} A_i u_i^2$, for some $A_i > 0$. □

Assumption 3 (Coefficients): For all $i \in \mathcal{V}$, $C_i A_i = 1$. □

Assumptions 2 and 3 restrict the design of parameters, yet they are meaningful in a large-scale decentralized generation setups with simple quadratic cost functions with identical coefficients (up to a constant factor) and identically weighted measurements. We remark that extensive numerical tests indicate that the Assumptions 2 and 3 are not necessary for closed-loop local asymptotic stability.

We are now ready to state the main technical result of the paper, that is, the local asymptotic stability of the equilibria of the differential-algebraic nonlinear closed-loop system (18), transient optimality and asymptotic solution to the frequency regulation (Problem 1) and the optimal economic power dispatch (Problem 2).

Theorem 1 (Local asymptotic stability): If Assumptions 2 and 3 hold, then any equilibrium of the closed-loop system in (18) satisfying $|\theta_i^* - \theta_j^*| < \pi/2$ for all $(i, j) \in \mathcal{E}$ is locally asymptotically stable. The control inputs $\{u_i(\cdot)\}_{i \in \mathcal{V}}$ defined as in (16) satisfy (6) for all $t \geq 0$, and asymptotically solve Problems 1, 2. □

Proof: Motivated by Lemma 1, we consider the error coordinates $\tilde{p} := p - p^*$ and, for all $i \in \mathcal{V}$, the parameter change $\tilde{P}_i := P_i - J_i'^{-1}(p^*) = P_i - A_i^{-1} p^*$. This change of coordinates and Assumptions 2, 3 render the closed-loop system in (18) to

$$\dot{\theta}_\mathcal{G} = \omega_\mathcal{G} \tag{19a}$$

$$M \dot{\omega}_\mathcal{G} = -D_\mathcal{G} \omega_\mathcal{G} - \left(\nabla_\mathcal{G} U(\theta) - \nabla_\mathcal{G} U(\theta^*) \right)$$
$$\quad - A_\mathcal{G}^{-1} \mathbb{1}_{|\mathcal{G}|} \tilde{p} \tag{19b}$$

$$D_\mathcal{F} \dot{\theta}_\mathcal{F} = -\left(\nabla_\mathcal{F} U(\theta) - \nabla_\mathcal{F} U(\theta^*) \right) - A_\mathcal{F}^{-1} \mathbb{1}_{|\mathcal{F}|} \tilde{p} \tag{19c}$$

$$\mathbb{0}_{|\mathcal{P}|} = -\left(\nabla_\mathcal{P} U(\theta) - \nabla_\mathcal{P} U(\theta^*) \right) - A_\mathcal{P}^{-1} \mathbb{1}_{|\mathcal{P}|} \tilde{p} \tag{19d}$$

$$k \dot{p} = \mathbb{1}_N^\top A^{-1} \dot{\theta}, \tag{19e}$$

where we have defined the shorthand notation $A = \operatorname{diag}(\{A_i\}_{i \in \mathcal{V}})$, and $\nabla U(\theta^*) := \tilde{P}$, which satisfies $\mathbb{1}_N^\top \nabla U(\theta^*) = \mathbb{1}_N^\top \tilde{P} = 0$. To analyze the local asymptotic stability of the closed-loop equilibria, we consider the incremental Lyapunov function

$$V(\omega, \theta, \tilde{p}) := \frac{1}{2} \omega_\mathcal{G}^\top M \omega_\mathcal{G} +$$
$$\quad U(\theta) - U(\theta^*) - \nabla U(\theta^*) \cdot (\theta - \theta^*) + \frac{k}{2} \tilde{p}^2, \tag{20}$$

whose critical points correspond to the equilibria of the closed-loop system in (18). The derivative of V in (20) along

trajectories of the closed-loop system in (19) reads as

$$
\begin{aligned}
\dot{V}&(\omega, \theta, \tilde{p}) \\
&= \omega_{\mathcal{G}}^{\top} M \dot{\omega}_{\mathcal{G}} + \left(\nabla_{\mathcal{G}} U(\theta) - \nabla_{\mathcal{G}} U(\theta^*)\right)^{\top} \dot{\theta}_{\mathcal{G}} \\
&\quad + \left(\nabla_{\mathcal{F}} U(\theta) - \nabla_{\mathcal{F}} U(\theta^*)\right)^{\top} \dot{\theta}_{\mathcal{F}} \\
&\quad + \left(\nabla_{\mathcal{P}} U(\theta) - \nabla_{\mathcal{P}} U(\theta^*)\right)^{\top} \dot{\theta}_{\mathcal{P}} + k\, \tilde{p}\, \dot{\tilde{p}} \\
&= -\omega_{\mathcal{G}}^{\top} D_{\mathcal{G}} \omega_{\mathcal{G}} - \omega_{\mathcal{G}}^{\top}\left(\nabla_{\mathcal{G}} U(\theta) - \nabla_{\mathcal{G}} U(\theta^*)\right) \\
&\quad - \omega_{\mathcal{G}}^{\top} A_{\mathcal{G}}^{-1} \mathbb{1}_{|\mathcal{G}|} \tilde{p} + \left(\nabla_{\mathcal{G}} U(\theta) - \nabla_{\mathcal{G}} U(\theta^*)\right)^{\top} \omega_{\mathcal{G}} \\
&\quad - \left(\nabla_{\mathcal{F}} U(\theta) - \nabla_{\mathcal{F}} U(\theta^*)\right)^{\top} D_{\mathcal{F}}^{-1} \left(\nabla_{\mathcal{F}} U(\theta) - \nabla_{\mathcal{F}} U(\theta^*)\right) \\
&\quad - \left(\nabla_{\mathcal{F}} U(\theta) - \nabla_{\mathcal{F}} U(\theta^*)\right)^{\top} D_{\mathcal{F}}^{-1} A_{\mathcal{F}}^{-1} \mathbb{1}_{|\mathcal{F}|} \tilde{p} \\
&\quad - \left(A_{\mathcal{P}}^{-1} \mathbb{1}_{|\mathcal{P}|} \tilde{p}\right)^{\top} \omega_{\mathcal{P}} + \mathbb{1}_N^{\top} A^{-1} \omega \tilde{p} \\
&= -\omega_{\mathcal{G}}^{\top} D_{\mathcal{G}} \omega_{\mathcal{G}} - \omega_{\mathcal{G}}^{\top} A_{\mathcal{G}}^{-1} \mathbb{1}_{|\mathcal{G}|} \tilde{p} \\
&\quad - \left(\nabla_{\mathcal{F}} U(\theta) - \nabla_{\mathcal{F}} U(\theta^*)\right)^{\top} D_{\mathcal{F}}^{-1} \left(\nabla_{\mathcal{F}} U(\theta) - \nabla_{\mathcal{F}} U(\theta^*)\right) \\
&\quad - \left(\nabla_{\mathcal{F}} U(\theta) - \nabla_{\mathcal{F}} U(\theta^*)\right)^{\top} D_{\mathcal{F}}^{-1} A_{\mathcal{F}}^{-1} \mathbb{1}_{|\mathcal{F}|} \tilde{p} \\
&\quad - \left(A_{\mathcal{P}}^{-1} \mathbb{1}_{|\mathcal{P}|} \tilde{p}\right)^{\top} \omega_{\mathcal{P}} + \mathbb{1}_N^{\top} A^{-1} \omega \tilde{p} \\
&= -\omega_{\mathcal{G}}^{\top} D_{\mathcal{G}} \omega_{\mathcal{G}} - \omega_{\mathcal{G}}^{\top} A_{\mathcal{G}}^{-1} \mathbb{1}_{|\mathcal{G}|} \tilde{p} \\
&\quad - \left(\nabla_{\mathcal{F}} U(\theta) - \nabla_{\mathcal{F}} U(\theta^*)\right)^{\top} D_{\mathcal{F}}^{-1} \left(\nabla_{\mathcal{F}} U(\theta) - \nabla_{\mathcal{F}} U(\theta^*)\right) \\
&\quad - \left(\nabla_{\mathcal{F}} U(\theta) - \nabla_{\mathcal{F}} U(\theta^*)\right)^{\top} D_{\mathcal{F}}^{-1} A_{\mathcal{F}}^{-1} \mathbb{1}_{|\mathcal{F}|} \tilde{p} \\
&\quad - \left(A_{\mathcal{P}}^{-1} \mathbb{1}_{|\mathcal{P}|} \tilde{p}\right)^{\top} \omega_{\mathcal{P}} + \mathbb{1}_{|\mathcal{G}|}^{\top} A_{\mathcal{G}}^{-1} \omega_{\mathcal{G}} \tilde{p} \\
&\quad - \mathbb{1}_{|\mathcal{F}|}^{\top} A_{\mathcal{F}}^{-1} D_{\mathcal{F}}^{-1} \left(\nabla_{\mathcal{F}} U(\theta) - \nabla_{\mathcal{F}} U(\theta^*) + A_{\mathcal{F}}^{-1} \mathbb{1}_{|\mathcal{F}|} \tilde{p}\right) \\
&\quad + \mathbb{1}_{|\mathcal{P}|}^{\top} A_{\mathcal{P}}^{-1} \omega_{\mathcal{P}} \tilde{p} \\
&= -\omega_{\mathcal{G}}^{\top} D_{\mathcal{G}} \omega_{\mathcal{G}} \\
&\quad - \left(\nabla_{\mathcal{F}} U(\theta) - \nabla_{\mathcal{F}} U(\theta^*)\right)^{\top} D_{\mathcal{F}}^{-1} \left(\nabla_{\mathcal{F}} U(\theta) - \nabla_{\mathcal{F}} U(\theta^*)\right) \tilde{p} \\
&\quad - \left(\nabla_{\mathcal{F}} U(\theta) - \nabla_{\mathcal{F}} U(\theta^*)\right)^{\top} D_{\mathcal{F}}^{-1} A_{\mathcal{F}}^{-1} \mathbb{1}_{|\mathcal{F}|} \tilde{p} \\
&\quad - \mathbb{1}_{|\mathcal{F}|}^{\top} A_{\mathcal{F}}^{-1} D_{\mathcal{F}}^{-1} \left(\nabla_{\mathcal{F}} U(\theta) - \nabla_{\mathcal{F}} U(\theta^*)\right) \tilde{p} \\
&\quad - \mathbb{1}_{|\mathcal{F}|}^{\top} A_{\mathcal{F}}^{-1} D_{\mathcal{F}}^{-1} A_{\mathcal{F}}^{-1} \mathbb{1}_{|\mathcal{F}|} \tilde{p}^2 \,,
\end{aligned}
$$

where we introduced the notation $\omega := (\omega_{\mathcal{G}}, \omega_{\mathcal{F}}, \omega_{\mathcal{P}}) = (\dot{\theta}_{\mathcal{G}}, \dot{\theta}_{\mathcal{F}}, \dot{\theta}_{\mathcal{P}})$ and made use of the algebraic constraint in (19d). Thus we obtain the compact form in (21), as displayed on page 7. The matrix Q in (21) is positive semi-definite, and thus the Lyapunov function $V(\omega, \theta, \tilde{p})$ is non-increasing along the trajectories of the closed-loop system.

In the following, we apply the LaSalle invariance principle for DAE systems [35, Theorem 3]. To do so, we need to construct a compact set (i) in which the vector field in (19) is twice continuously differentiable, (ii) that is forward invariant for the dynamics in (19), and (iii) in which the Jacobian with respect to $\theta_{\mathcal{P}}$ of the algebraic equations in (18d) is nonsingular (so that solvability of the algebraic equations (18d) with respect to $\theta_{\mathcal{P}}$ is guaranteed by the implicit function theorem). Therefore, we show that, the sublevel set

$$
\Omega_c := \big\{ (\omega, \theta, \tilde{p}) \in \mathbb{R}^{2N+1} \,\big|\, V(\omega, \theta, \tilde{p}) \le c, \\
|\theta_i - \theta_j| < \pi/2 \;\; \forall (i,j) \in \mathcal{E} \big\} \quad (22)
$$

satisfies all of the above conditions, for sufficiently small $c > 0$. First, observe that the differential-algebraic vector field (19)

is twice continuously differentiable in Ω_c. Next, we show that the dynamics are bounded in Ω_c. Note that the Hessian of $U(\cdot)$ has (i,j) element $\left[\nabla^2 U(\theta)\right]_{i,j}$ equal to

$$
\frac{\partial^2 U(\theta)}{\partial \theta_j \partial \theta_i} = \left\{ \begin{array}{ll} -B_{i,j} \cos\left(\theta_i - \theta_j\right) & \text{if } j \ne i \\ \sum_{k=1, k \ne i}^N B_{i,k} \cos\left(\theta_i - \theta_k\right) & \text{if } j = i. \end{array} \right. \quad (23)
$$

Therefore, $\nabla^2 U(\theta^*)$ is a positive semidefinite and irreducible Laplacian matrix with nullspace corresponding to the rotational symmetry, i.e., the dynamics in (19) are invariant under a rigid rotation of all angles θ. Hence, within Ω_c in (22), $U(\theta)$ is locally positive definite (modulo symmetry). Thus, the Lyapunov function V is locally positive definite and its sublevel sets are compact (modulo rotational symmetry). The above reasoning guarantees boundedness of the frequencies ω, the integral variable $\tilde{p}$, as well as the relative angles $\theta_i - \theta_j$, that is, $v^{\top} \theta$ is bounded for any $v \in \mathbb{R}^N$ such that $v \perp \mathbb{1}_N$. To show boundedness of the remaining coordinate, the sum of all angles $\mathbb{1}_N^{\top} \theta$, we first note from (21) that $\tilde{p} = \sum_{i \mathcal{V}} C_i \theta_i$ is bounded. Thus, both $\mathbb{1}_N^{\top} C \theta$ and $v^{\top} \theta$ are bounded for any $v \perp \mathbb{1}_n$. It follows that $\mathbb{1}_N^{\top} \theta$ is bounded, and the dynamics are bounded in the compact set Ω_c in (22).

Finally, within Ω_c, the Jacobian matrix associated to the algebraic equation (18d) is a principal submatrix of the irreducible Laplacian matrix in (23). Since submatrices of irreducible Laplacians are nonsingular [36, Lemma 2.1], it follows that the algebraic equations (18d) are solvable with respect to $\theta_{\mathcal{P}}$. Since all the conditions of the LaSalle invariance principle for DAE systems [35, Theorem 3] are met, we conclude that the closed-loop dynamics asymptotically converge to largest invariant set in Ω_c satisfying $\dot{V}(\omega, \theta, \tilde{p}) = 0$, that is, to the set of vectors $(\omega, \theta, \tilde{p})$ such that

$$
\begin{bmatrix} \omega_{\mathcal{G}} \\ \nabla_{\mathcal{F}} U(\theta) - \nabla_{\mathcal{F}} U(\theta^*) \\ \tilde{p} \end{bmatrix} \in \ker(Q).
$$

Due to the block-diagonal structure of Q, from the $(1,1)$−block we conclude that $\lim_{t \to \infty} \omega_{\mathcal{G}}(t) = 0$, that is, the generator states converge to the set of equilibria. Thus, as $\dot{\omega}_{\mathcal{G}} = \mathbb{0}_{|\mathcal{G}|}$ and $\omega_{\mathcal{G}}(t) \to \mathbb{0}_{|\mathcal{G}|}$, by (19b), we get that $\lim_{t \to \infty} \nabla_{\mathcal{G}} U(\theta(t)) - \nabla_{\mathcal{G}} U(\theta^*) = -A_{\mathcal{G}}^{-1} \mathbb{1}_{|\mathcal{G}|} \tilde{p}(t)$. Then, due to the second block row of Q, we have that $\lim_{t \to \infty} \nabla_{\mathcal{F}} U(\theta(t)) - \nabla_{\mathcal{F}} U(\theta^*) = -A_{\mathcal{F}}^{-1} \mathbb{1}_{|\mathcal{F}|} \tilde{p}(t)$. In addition, by the algebraic constraint in (19d), we have that $\nabla_{\mathcal{P}} U(\theta) - \nabla_{\mathcal{P}} U(\theta^*) = -A_{\mathcal{P}}^{-1} \mathbb{1}_{|\mathcal{P}|} \tilde{p}(t)$.

Finally, since $\mathbb{1}_N^{\top} \left(\nabla U(\theta(t)) - \nabla U(\theta^*)\right) = 0$ for all $t \ge 0$, and $\lim_{t \to \infty} \mathbb{1}_N^{\top} \left(\nabla U(\theta(t)) - \nabla U(\theta^*)\right) = \lim_{t \to \infty} -\left(A_{\mathcal{G}}^{-1} \mathbb{1}_{|\mathcal{G}|} + A_{\mathcal{F}}^{-1} \mathbb{1}_{|\mathcal{F}|} + A_{\mathcal{P}}^{-1} \mathbb{1}_{|\mathcal{P}|}\right) \tilde{p}(t) = 0$, we conclude that $\lim_{t \to \infty} \tilde{p}(t) = 0$. Hence, asymptotically we have that $\omega(t) \to \mathbb{0}_{|\mathcal{G}|}$, $\tilde{p}(t) \to 0$, and $\nabla U(\theta(t)) \to \nabla U(\theta^*)$, therefore the set of equilibria inside Ω_c, that correspond to the critical points of the Lyapunov function V, are locally asymptotically stable.

Condition (6) on the identical marginal costs follows directly by (16), as $J_i(u_i) = -p$ for all $i \in \mathcal{V}$, which implies that the control inputs $\lim_{t \to \infty}\{u_i(t)\}_{i \in \mathcal{V}}$ solve Problem 2. ∎

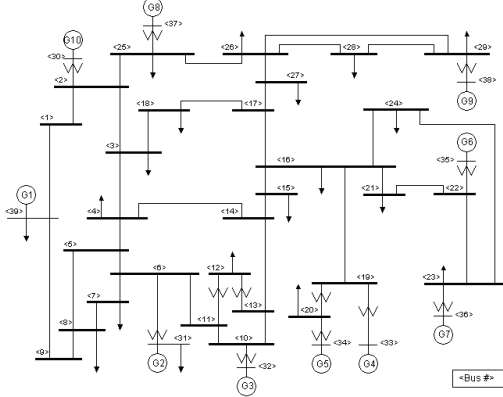$$\dot{V}(\omega,\theta,\tilde{p}) = - \begin{bmatrix} \omega_{\mathcal{G}} \\ \nabla_{\mathcal{F}}U(\theta) - \nabla_{\mathcal{F}}U(\theta^*) \\ \tilde{p} \end{bmatrix}^{\top} \underbrace{\begin{bmatrix} D_{\mathcal{G}} & \mathbb{0} & \mathbb{0}_{|\mathcal{G}|} \\ \hline \star & D_{\mathcal{F}}^{-1} & D_{\mathcal{F}}^{-1}A_{\mathcal{F}}^{-1}\mathbb{1}_{|\mathcal{F}|} \\ \star & \star & \mathbb{1}_{|\mathcal{F}|}^{\top}A_{\mathcal{F}}^{-1}D_{\mathcal{F}}^{-1}A_{\mathcal{F}}^{-1}\mathbb{1}_{|\mathcal{F}|} \end{bmatrix}}_{=:Q \succcurlyeq 0} \begin{bmatrix} \omega_{\mathcal{G}} \\ \nabla_{\mathcal{F}}U(\theta) - \nabla_{\mathcal{F}}U(\theta^*) \\ \tilde{p} \end{bmatrix} \qquad (21)$$

Fig. 1: IEEE New England test power system.

V. NUMERICAL SIMULATIONS

We evaluate the performance of our proposed controller on the IEEE New England test power system shown in Figure 1. The system has 10 generators and 39 buses, serving a total load of about 6 GW. The generator inertia coefficients $\{M_i\}_i$ and line susceptances $\{B_{i,j}\}_{(i,j)}$ are obtained from the Power System Toolbox. The droop coefficients are chosen uniformly, as $D_i = 1$ for all the buses i. The cost coefficients $\{A_i\}_i$ are randomly generated, uniformly in $(0,1)$.

We simulate a scenario in which at time $t = 1$ s, the energy demand changes by 33 MW at the buses 4, 12 and 20, creating a power imbalance and causing the bus frequencies to drop below the nominal value of 60 Hz. We compare our control strategy with the fully decentralized integral control [20, Section III] and with the recently introduced distributed averaging integral (DAI) control [20, Section IV], where we use the same communication graph as in [20, Section V]. Figure 2 shows the frequencies and the marginal costs of five generators for the three control schemes.

We note that all controllers drive the system frequency to its nominal value. As expected, the decentralised integral controller does not achieve convergence of the marginal cost; on the other hand, both the DAI controller and the proposed one asymptotically solve the optimal economic dispatch problem. In addition, our controller guarantees identical marginal costs even during the transient. The main benefit of our controller is that, as Figure 2 shows, the closed-loop frequency responses induced are qualitatively at least as good as the ones due to the DAI controller, despite the communication requirements are significantly lower.

VI. CONCLUSIONS

We have proposed a novel frequency control approach that achieves both local asymptotic stability of the closed-loop equilibria of power systems, modeled as a nonlinear, differential-algebraic, dynamical system, and economic-dispatch optimality. The control architecture is semi-decentralized, hence the communication requirements are significantly lower than those of distributed architectures.

Open problems

Extensive numeric simulations show that the closed-loop system in (18) has locally asymptotically stable equilibria for *any* non-negative convex-combination selection of the measurement coefficients $\{C_i\}_i$, and also for convex non-quadratic cost functions $\{J_i\}_i$, including the conventional (non-smooth) dead-zones and saturation nonlinearities for $J_i'^{-1}$. Formal proofs of these claims are currently not available.

An important extension would be the inclusion of forecasts and inter-temporal constraints into our frequency control architecture, with the aim of designing predictive control actions, while maintaining minimal communication requirements.

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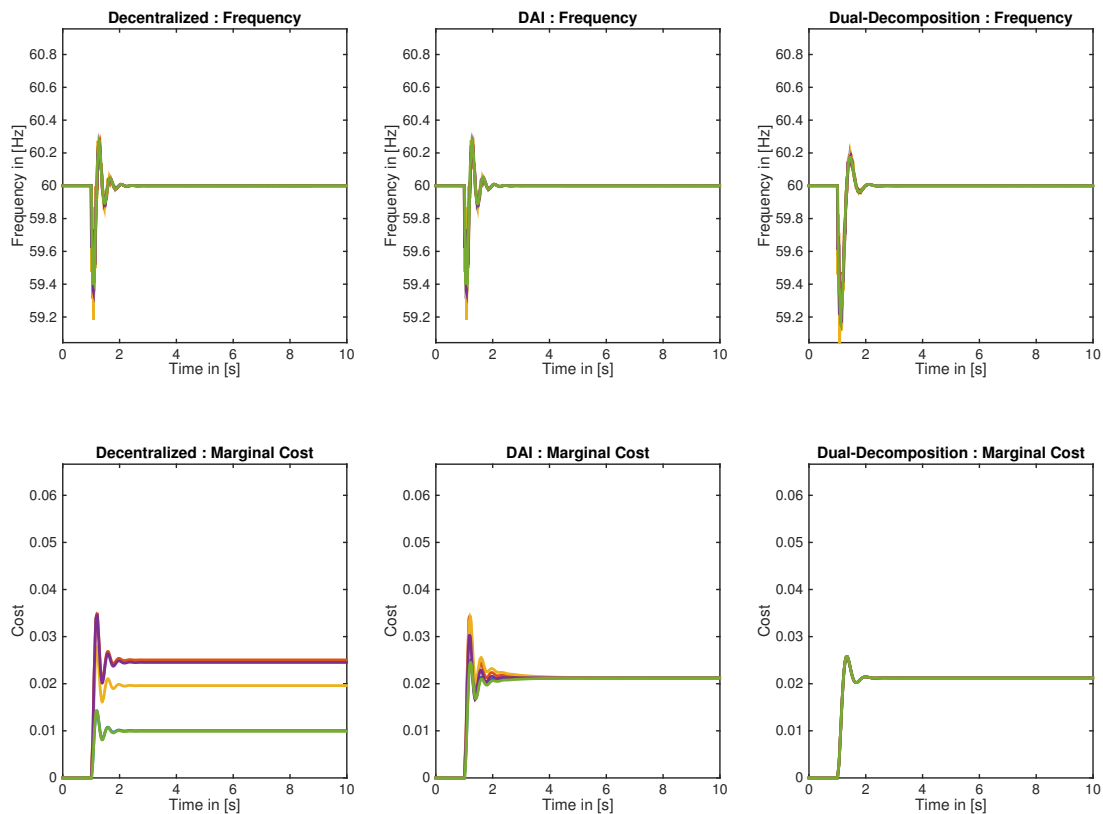
Fig. 2: Frequency and marginal costs of generators 2, 4, 8, 10, for the different control schemes.

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