Optimal Consumption Policies for Power-Constrained Flexible Loads Under Dynamic Pricing

Donatello Materassi, Saverio Bolognani, Mardavij Roozbehani, and Munther A. Dahleh

Abstract—**This paper analyzes the response of an individual consumer with a flexible demand for electricity to exogenous and stochastic electricity prices. The contributions of this paper are twofold. First, we propose a comprehensive model for which the optimal policy can be explicitly computed. The proposed consumer model features a time varying bound on power consumption, the presence of a nondeferrable load profile, the presence of a load profile that can be deferred up to a deadline, and the possibility of curtailment. Second, we describe the algorithm that, via explicit backward iterations of the Bellman equation, returns the optimal response of such consumer in a very general energy market scenario featuring a correlated/nonstationary price process and multiple energy procurement sources.**

Index Terms—**Power demand, power generation economics, power load control.**

I. Introduction

MECHANISMS for real-time demand response are likely to be one of the most significant technologies that will emerge in future power grids. Consumers or smart devices with communication and computation capabilities will be able to adjust their consumption in real-time in order to mitigate the effects of exogenous uncertainties and intermittencies of renewable generation. Developing methodologies to determine the stability properties and quantify the reliability of the system resulting from the interaction between consumers with reconfigurable energy needs and the electricity market is a fundamental problem that still has to be fully addressed [1].

Success in the development of such methodologies relies on progress in three directions.

1) Development of tractable models from basic principles at the individual device/consumer level: these models should be expressive enough to capture the most important characteristics of power consumption and, at the same time, abstract enough to be tractable for analysis and design.
2) Development of techniques to obtain aggregate models with foundations in the individual models in order to reliably characterize the aggregate behavior of a large number of consumers.
3) Identification and validation of such models with actual data.

This paper contributes to the first goal by studying the optimal response of an individual consumer to stochastically varying electricity prices. The proposed consumer model allows for a very general energy demand, featuring: 1) a time varying bound on the instantaneous power consumption; 2) the presence of a nondeferrable load profile; 3) the presence of a load profile that can be deferred up to a given deadline; and 4) the possibility of curtailment at the deadline, at a cost. For such demand, it is possible to compute the exact optimal behavior of the consumer that wants to optimize her energy consumption in a real time market with time correlated and nonstationary prices, and multiple energy procurement options.

The optimal response of similar classes of consumers to an exogenous price has been investigated since the seminal contribution in [2] and its characterization has been shown to be closely related to variations of the inventory control problem (see [3]–[5] and the references therein). While these tools have been applied, for example, to extend the solution of the optimal energy storage problem from a deterministic setting [6] to a stochastic one [7], [8], the existing literature on the implications for individual flexible consumers in more practical scenarios is relatively narrow.

A similar problem has been formulated under the terminology of multistage energy procurement. In this scenario, large consumers have the option of purchasing energy from different sources in the day-ahead auction and in the real-time market, and want to minimize the expected energy cost, possibly while mitigating risk [9]. Stochastic optimization methods [10], [11] and nonprobabilistic decision strategies [12] have been specialized for this problem, obtaining complex algorithms whose computational burden can partially be mitigated via approximation or reduction techniques. If only two stages are assumed (one day-ahead and one real-time transaction), then the resulting optimal policy [13] is a special case of the scenario that, we propose in this paper.

Several papers, including [14]–[19], have considered certain applications of deferrable power demand, for which they derived specialized strategies, often based on heuristics or certain features of the specific application. In other cases,

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computationally intensive algorithms have been proposed for more general and comprehensive formulations of the deferrable load problem, based on approximate solutions of the corresponding stochastic dynamic program [20], heuristic strategies [21], Monte Carlo methods [22], or robust discrete optimization [23]. Because of their complexity, these methods do not usually provide analytical solutions and are not well suited for the derivation of aggregate models.

There have been previous contributions toward the mathematical understanding of the optimal strategy of a deferrable load responding to an exogenous price, based on dynamic programming (DP), such as [24]–[28]. However, results in [24] are limited to the case of independent prices only proving that the "value function" is piecewise linear in the backlogged demand, with no explicit expressions for the computation of the value functions. In [25], a detailed model that takes into account bounds on consumption and correlated prices has been proposed, showing that the value function is convex with respect to the backlogged demand. In [26], optimal threshold policies have been derived for the case of both interruptible and noninterruptible tasks. However, the authors assumed that consumption is discrete (on/off) in nature, and prices are independent in each period. Extension of these results to more general settings is not immediate. In [27], a quasi-analytic solution to the problem is derived, but in the case of no bounds of consumptions and only for an independent and identically distributed process. In [28], a Markovian price model is considered, and the optimal threshold policy for a simpler model of a delay-averse load is computed, discussing an explicit iterative derivation of the thresholds and the computational complexity of this derivation.

Compared to these works, in this paper, we model flexible loads taking into account multiple features simultaneously (i.e., a time-varying bound on power consumption, a firm demand profile, a shiftable demand profile with a hard deadline). The optimal policy for these loads is fully described by a set of thresholds: the consumer will consume only when the price falls below a certain level which depends on the time left to his deadline, on the future consumption needs, and on the information about the price process. The key technical achievement is the development of an exact procedure to compute such thresholds, without relying on any approximation technique, in several energy market scenarios of practical relevance (i.e., with nonstationary and correlated prices, with possible curtailment, and with multiple procurement sources). Because the proposed algorithm is amenable to implementation in relative small computational units (i.e., in the computational core of smart appliances), it is reasonable to adopt this policy as an atomic model of the response of flexible loads to real-time energy prices, which is the ultimate goal of this paper.

This paper is organized as follows. We first formulate the problem in terms of DP (Section II), and we describe a general approach for its solution via the stochastic Bellman equation. In Section III, we show how to explicitly perform the iterations of the Bellman equation in the simplified case of an independent identically distributed (i.i.d). price process, in order to provide the necessary basic intuition. This allows us to use a simpler notation since the value functions associated with the stochastic Bellman equation are independent of the previous history. The extension of the i.i.d. hypothesis to the more general case of correlated prices is then discussed in Section IV. In Section V, we show how the same approach can also be used in the case of multiple energy procurement options, at different prices. The procedure to compute the optimal policy, and the resulting consumer response to a price signal, is illustrated numerically in Section VI.

II. Model of Individual Consumer With Deferrable Demand

The consumer has a predetermined energy demand profile defined over a known time horizon consisting of n intervals, indexed by $k = 0, \dots, n-1$. Such demand profile is made of two components: 1) the firm demand $d_k^{(f)},\ k = 0, \dots, n-1$; and 2) the shiftable demand $d_k^{(s)},\ k = 0, \dots, n-1$. The firm demand $d_k^{(f)}$ needs to be satisfied at time k, and cannot be deferred in the future. Instead, the shiftable demand $d_k^{(s)}$ describes energy needs that can be satisfied either at time k or at a any later time up to the last time interval $n-1$. In a way, the demand profile $d_k^{(s)}$ defines how early the energy needs of the consumer can be served, opposed to the time deadline n that poses a limit to the deferral of consumption. It is assumed that both the profiles $d^{(f)}$ and $d^{(s)}$ are known to the consumer at the beginning of the optimization horizon.

The consumer's backlogged demand at the kth interval is denoted by $x_k \in (-\infty, 0]$, where $x_k = 0$ means that there is no backlogged demand. The state variable x_k is updated according to $x_{k+1} = x_k + u_k - d_k^{(f)} - d_k^{(s)}$, where, $u_k \in [0, u_k^{\max}]$ is the amount of energy that the consumer decides to withdraw from the grid during the kth period, and $u_k^{\max}$ is a possibly time varying limit on the instantaneous power consumption.

It is also assumed, for now, that the energy cost is linear in the consumption u_k, namely it is equal to $\lambda_k u_k$. The case with different energy procurement options available to the consumer is analyzed in Section V. The price λ_k is a stochastic process with known probability distribution. In particular, in this paper, we first consider the case where $\{\lambda_j\}_{j=-\infty}^{+\infty}$ is an independent and identically distributed process, and then, we extend the results to the case where $\{\lambda_j\}_{j=-\infty}^{+\infty}$ is a Markov process of order m, namely that the conditional distribution of λ_k given the whole price history depends only on the past m prices $\{\lambda_j\}_{j=k-m}^{k-1}$.

Finally, we assume that the consumer suffers a penalty $C(x_n) \geq 0$ determined by any residual backlogged demand x_n at the end of the optimization horizon. The function $C(x_n)$ models the curtailment that the consumer would adopt if her energy needs were to be partially satisfied. We assume that the function $C(x_n)$, defined on negative values of x_n, is nonincreasing and convex, since a rational consumer will curtail consumption at the expense of less essential needs first. We also assume that $C(x_n)$ is piecewise linear. This assumption is only marginally restrictive, since any continuous curtailment function $C(x_n)$ can be approximated with a piecewise linear function.

Remark 1 (Multiple Time Deadlines): The proposed model assumes that the entirety of the consumer demand shares the same time deadline n. Further investigation shows that there is indeed a qualitative difference in the optimal consumption problem with a single or multiple deadlines. Materassi *et al.* [29] showed how this latter problem is in fact a scheduling problem with a rich structure of the feasibility set, and different tools are need to be employed to aggregate the individual policies of the consumers.

A. Optimization-Based Model

The consumer's energy management problem is formulated as a finite-horizon DP problem

$$\begin{aligned} \min \quad & \mathbb{E}\left[C(x_n) + \sum_{k=0}^{n-1} \lambda_k u_k\right] \\ \text{subject to} \quad & x_{k+1} = x_k + u_k - d_k^{(f)} - d_k^{(s)} \\ & x_0 = 0 \\ & x_k \leq 0 \\ & d_k^{(f)} \leq u_k \leq u_k^{\max}. \end{aligned} \tag{1}$$

The situation where no curtailment is allowed can be taken into account by considering the limit case $C(x_n) = -\gamma x_n$, with γ arbitrarily large. This condition is equivalent to including the additional terminal constraint $x_n = 0$.

In this paper, we will always assume that the problem is feasible.

We observe that, we can reformulate the problem in an equivalent way by "redefining" u_k as $u_k - d_k^{(f)}$ and removing the explicit presence of a firm demand. Also, the constraint $x_k \leq 0$ can be removed, if we introduce the constraint

$$u_k \leq \bar{u}_k(x_k) := \min\left\{u_k^{\max} - d_k^{(f)}, d_k^{(s)} - x_k\right\}. \tag{2}$$

Thus, the consumer optimization problem becomes

$$\begin{aligned} \min \quad & \mathbb{E}\left[C(x_n) + \sum_{k=0}^{n-1} \lambda_k u_k\right] \\ \text{subject to} \quad & x_{k+1} = x_k - d_k + u_k \\ & x_0 = 0 \\ & 0 \leq u_k \leq \bar{u}_k(x_k) \end{aligned} \tag{3}$$

where now the input u_k has a bound that depends on the state x_k and the demand profile d_k, $k = 1, \ldots, n-1$, is equal to the deferrable demand $d^{(s)}$ only. Thus, we have no loss of generality, if we consider the optimization problem (3), in which the firm demand and the constraint $x_k \leq 0$ do not appear explicitly.

B. Solution via Bellman Stochastic Equation

The optimization problem (3) is a (stochastic) dynamic program. The solution of a dynamic program is immediately obtained if it is possible to solve the (stochastic) Bellman equation associated with it [30]. The Bellman equation is an equation in which the unknown is a function. Usually the function solving the Bellman equation is referred to as the value function. In general, an explicit form for the value function is difficult to obtain. In the following, we show that for (3) the value function can be obtained using explicit formulas.

We consider the general case in which the stochastic price λ_k is a Markov process of order m. Thus, the conditional distribution of λ_k given the past history $\{\lambda_j\}_{j=-\infty}^{k-1}$ depends only on a limited number of past prices $\lambda_{k-1}, \ldots, \lambda_{k-m}$. In this case, the approach via DP is effective since it is necessary to keep track of only the last m price values in order to obtain the optimal solution.

The Bellman stochastic equation [30], associated with program (3) and defined in terms of the value function $V(x_k, \{\lambda_j\}_{j=k-m+1}^{k})$ for $k = 0, \ldots, n-1$, is

$$\begin{aligned} V_k\left(x_k, \{\lambda_j\}_{j=k-m+1}^{k}\right) &= \min_{u_k \in [0, \bar{u}_k]} \left\{\lambda_k u_k + \mathbb{E}\left[V_{k+1}\left(x_{k+1}, \{\lambda_j\}_{j=k-m+2}^{k+1}\right) \middle| x_k, \{\lambda_j\}_{j=k-m+1}^{k}\right]\right\} \end{aligned} \tag{4}$$

with final condition given by the function

$$V_n\left(x_n, \{\lambda_j\}_{j=n-m+1}^{n}\right) = C(x_n) \tag{5}$$

for every history $\{\lambda_j\}_{j=n-m+1}^{n}$.

Observe that the dynamics of the state x_k depends deterministically on the decision variable u_k. Assuming that the state x_k does not affect the future prices (the consumer is a price taker), then (4) becomes

$$\begin{aligned} V_k\left(x_k, \{\lambda_j\}_{j=k-m+1}^{k}\right) &= \min_{u_k \in [0, \bar{u}_k]} \left\{\lambda_k u_k + \mathbb{E}\left[V_{k+1}\left(x_k - d_k + u_k, \{\lambda_j\}_{j=k-m+2}^{k+1}\right) \middle| \{\lambda_j\}_{j=k-m+1}^{k}\right]\right\}. \end{aligned}$$

Observe that, given the deterministic dependence of x_{k+1} on x_k, u_k and d_k, we can define the average value function as

$$\begin{aligned} W_k\left(x_{k+1}, \{\lambda_j\}_{j=k-m+1}^{k}\right) &:= \mathbb{E}\left[V_{k+1}\left(x_{k+1}, \{\lambda_j\}_{j=k-m+2}^{k+1}\right) \middle| \{\lambda_j\}_{j=k-m+1}^{k}\right] \end{aligned} \tag{6}$$

for $k = 0, \ldots, n-1$.

We then rewrite (4) as

$$\begin{aligned} V_k\left(x_k, \{\lambda_j\}_{j=k-m+1}^{k}\right) &= \min_{u_k \in [0, \bar{u}_k]} \left\{\lambda_k u_k + W_k\left(x_k - d_k + u_k, \{\lambda_j\}_{j=k-m+1}^{k}\right)\right\}. \end{aligned}$$

In order to minimize the quantity $\lambda_k u_k + W_k(x_k - d_k + u_k, \{\lambda_j\}_{j=k-m+1}^{k})$, let us differentiate with respect to the decision variable u_k and let us equate to zero. This leads to

$$-\omega_k\left(x_k - d_k + u_k, \{\lambda_j\}_{j=k-m+1}^{k}\right) = \lambda_k \tag{7}$$

where, we have defined

$$\omega_k\left(x_{k+1}, \{\lambda_j\}_{j=k-m+1}^{k}\right) := \frac{\partial}{\partial x_{k+1}} W_k\left(x_{k+1}, \{\lambda_j\}_{j=k-m+1}^{k}\right). \tag{8}$$

A graphical representation of (7) is given in Fig. 1. Equation (7) has a straightforward economic interpretation. The curve $-\omega_k(x_k - d_k + u_k, \{\lambda_j\}_{j=k-m+1}^{k})$ represents the marginal utility associated with the consumption of energy given the price history, while the line at λ_k represents the associated marginal cost. In the presence of no other constraints, a consumer would consume the quantity $\hat{u}_k$ that corresponds to

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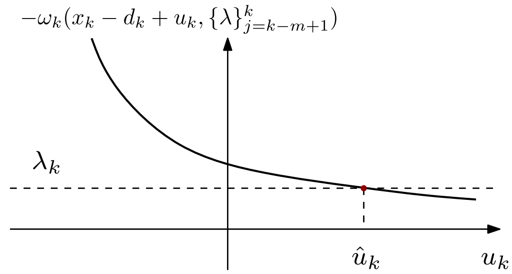

Fig. 1. Graphical interpretation of (7). The function $-\omega_k(x_k - d_k + u_k, \{\lambda_j\}_{j=k-m+1}^{k})$ represents the utility associated to the unit $x_k - d_k + u_k$ of backlogged energy for the next time interval. The line at λ_k represents the marginal price for consumption. The value $\hat{u}_k$ where the two functions intersect is the optimal consumption.

the point of intersection between the two curves. Such intersection, if it exists, is unique when $-\omega_k(x_k - d_k + u_k, \{\lambda_j\}_{j=k-m+1}^{k})$ is continuous and monotonically decreasing. In the case of $-\omega_k(x_k - d_k + u_k, \{\lambda_j\}_{j=k-m+1}^{k})$ monotonically decreasing, but with no intersections between the two curves (i.e., because of discontinuities), the optimal consumption $\hat{u}_k$ would be

$$\hat{u}_k := \sup\left\{u_k \middle| \omega_k\left(x_k - d_k + u_k, \left\{\lambda_j\right\}_{j=k-m+1}^{k}\right) + \lambda_k \le 0\right\} \quad (9)$$

with the convention that $\sup\{\emptyset\} = -\infty$. Observe that if the marginal price curve lies completely below the marginal value curve, we have $\hat{u}_k = +\infty$. Instead, if the marginal price curve lies above the marginal value curve, we have $\hat{u}_k = -\infty$.

It is immediate to verify that $\hat{u}_k$ in (9) is the optimal consumption when there are no constraints on u_k, even in the case where $-\omega_k(x_k - d_k + u_k, \{\lambda_j\}_{j=k-m+1}^{k})$ is monotonically nonincreasing (e.g., piecewise constant).

When u_k is constrained in $[0, \overline{u}_k(x_k)]$, (9) is still useful to determine the optimal consumption. If $-\omega_k(x_k - d_k + u_k, \{\lambda_j\}_{j=k-m+1}^{k})$ is monotonically nonincreasing in u_k, the optimal consumption u_k^* is determined as follows:

$$u_k^* = \begin{cases} 0 & \text{if } \hat{u}_k < 0 \\ \overline{u}_k(x) & \text{if } \hat{u}_k > \overline{u}_k(x) \\ \hat{u}_k & \text{otherwise.} \end{cases} \quad (10)$$

The policy provided by (10) has again a straightforward economic interpretation. The optimal consumption $\hat{u}_k$, that would happen in the presence of no constraints, is determined by intersecting the averaged marginal value function $-\omega_k(x_k - d_k + u_k, \{\lambda_j\}_{j=k-m+1}^{k})$ and the current marginal price λ_k. In the presence of the constraints, the optimal level of consumption is forced to lie in the interval $[0, \overline{u}_k(x_k)]$, motivating the final form of (10).

In the following, we will first show how to compute the quantity $\hat{u}_k$ defined in (9), and thus the optimal consumption defined in (10), when the price process $\{\lambda_j\}_{j=-\infty}^{+\infty}$ where the variables λ_j are assumed independent (but not necessarily identically distributed). This is a scenario similar to [27], with the main difference given by the fact that the constraint $u_k \in [0, \overline{u}_k(x_k)]$ is taken into account.

In Section IV, we will then extend the results to the case of generic Markov prices, and in Section V, we will show how to allow different energy procurement options, and thus more complex structures of the energy price.

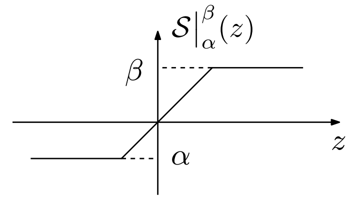

Fig. 2. Graphical representation of $\mathcal{S}\big|_{\alpha(z)}^{\beta(z)}(z)$.

III. Independent and Identically Distributed Price Process

The main goal of this section is to provide an explicit formula for the computation of the averaged marginal value function $\omega_k(x_{k+1}, \{\lambda_j\}_{j=k-m+1}^{k})$, defined in (8), in the case of a price process $\{\lambda_j\}_{j=-\infty}^{+\infty}$ independent and identically distributed.

The knowledge of the averaged marginal function ω_k at each time $k = 0, \ldots, n-1$ determines the optimal consumption policy as indicated in (9) and (10).

We start again from the Bellman equation (4), and observe that, the situation of i.i.d prices would correspond to the case $m = 0$. For $m = 0$, (4) can still be applied with a small modification. Indeed, we notice that for $m = 0$, the function W_k does not depend on λ_k. Instead, the function V_k still keeps its dependence on λ_k since such a variable is contained in the argument of the min operator yielding

$$V_k(x_k, \lambda_k) = \min_{u_k}\{\lambda_k u_k + \mathbb{E}[V_{k+1}(x_{k+1}, \lambda_{k+1})]\} \quad (11)$$

for $k = 0, \ldots, n-1$, and

$$V_n(x_n, \lambda_n) = C(x_n). \quad (12)$$

We have the following result.

Theorem 1: Consider the optimal consumption problem in the case of an i.i.d price process, with curtailment penalty $C_n(x_n)$ that is convex and piecewise linear. The optimal policy is given by (9) and (10), where the averaged marginal value functions $\omega_k(x_{k+1})$, for $k = 0, \ldots, n-1$, are piecewise constant and monotonically nonincreasing, and are given by the backward iterations

$$-\omega_{k-1}(x_k) = \int \mathcal{S}\big|_{-\omega_k(x_k - d_k + \overline{u}_k(x_k))}^{-\omega_k(x_k - d_k)}(\lambda_k)\, dP(\lambda_k) \quad (13)$$

$$-\omega_{n-1}(x_n) = -\frac{dC(x_n)}{dx_n} \quad (14)$$

where the saturation function $\mathcal{S}\big|_{\alpha}^{\beta}(\cdot)$ is defined as

$$\mathcal{S}\big|_{\alpha(z)}^{\beta(z)}(z) := \begin{cases} \alpha & \text{if } z < \alpha \\ \beta & \text{if } z > \beta \\ z & \text{otherwise} \end{cases} \quad (15)$$

for $\alpha(z) \le \beta(z)\ \forall z$, and represented in Fig. 2.

The property that $-\omega_k$, for $k = 0, \ldots, n-1$, are nonincreasing and piecewise constant in x_{k+1} has practical relevance, since piecewise linear functions are extremely memory-efficient to store and manipulate.

We prove this results constructively, by using the backward iterations of the Bellman equation. Defining for $k = 0, \ldots, n-1$, we find

$$V_k(x_k, \lambda_k) = \min_{u_k \in [0, \overline{u}_k]}\{\lambda_k u_k + W_k(x_k - d_k + u_k)\}.$$

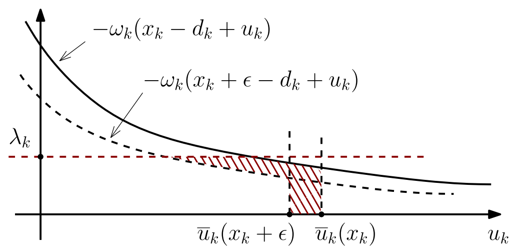

Fig. 3. Interpretation of the difference of the two integrals in (17), for $\overline{u}_k(x_k) \neq \overline{u}_k(x_k + \varepsilon)$.

The averaged marginal value function ω_k is therefore simply defined as

$$\omega_k(x_{k+1}) = \frac{\partial}{\partial x_{k+1}} \mathbb{E}[V_{k+1}(x_{k+1}, \lambda_{k+1})]. \quad (16)$$

Recall that, we are considering a curtailment penalty $C(x_n)$ which is convex and piecewise linear. Then, according to (12) and (16), $-\omega_{n-1}(x_n) = -dC/dx_n$ is monotonically nonincreasing and piecewise constant. Adopting this as a base case, we assume by induction, that $-\omega_k(x_{k+1})$ is monotonically nonincreasing in x_{k+1}.

In order to find the minimum of the function $\lambda_k u_k + W_k(x_k - d_k + u_k)$ for u_k in the interval $[0, \overline{u}_k(x_k)]$ and for a fixed λ, we can simply integrate its derivative with respect to u_k as long as it is nonpositive in the interval $[0, \overline{u}_k(x_k)]$. We thus have

$$V_k(x_k, \lambda_k) = W_k(x_k - d_k) + \int_0^{\overline{u}_k(x_k)} \min\left\{\frac{dW_k}{du_k}(x_k - d_k + u_k) + \lambda_k, 0\right\} du_k.$$

Equivalently, we find that

$$V_k(x_k, \lambda_k) = W_k\,(x_k - d_k + \bar{u}_k(x_k)) + \int_0^{\bar{u}_k(x_k)} \min\{\lambda_k, -\omega_k(x_k - d_k + u_k)\}\, du_k.$$

In order to compute $\omega_{k-1}(x_k)$ according to (16), we take $\varepsilon > 0$ sufficiently small and compute

$$\begin{aligned} &V_k(x_k + \varepsilon, \lambda_k) - V_k(x_k, \lambda_k) \\ &\quad = W_k(x_k + \varepsilon - d_k + \bar{u}_k(x_k + \varepsilon)) \\ &\qquad + \int_0^{\bar{u}_k(x_k+\varepsilon)} \min\{\lambda_k, -\omega_k(x_k + \varepsilon - d_k + u_k)\}\, du_k \\ &\qquad - W_k\,(x_k - d_k + \bar{u}_k(x_k)) \\ &\qquad - \int_0^{\bar{u}_k(x_k)} \min\{\lambda_k, -\omega_k(x_k - d_k + u_k)\}\, du_k. \quad (17) \end{aligned}$$

The graphical interpretation of the difference between the two integrals in the last equation is given in Fig. 3. The value of $V_k(x_k + \varepsilon, \lambda_k) - V_k(x_k, \lambda_k)$ is given by the difference of the two integrals as represented in the figure summed to the difference of two terms depending on the function $W_k(x_{k+1})$. While it is not possible to provide an explicit solution for the general case, this quantity can be computed in the case where $W_k(x_{k+1})$ is a piecewise linear functions (and thus ω_k is piecewise constant). We therefore have two cases.

A. Case $\overline{u}_k(x_k) = u_k^{max}$—$d_k^{(f)}$

Given the definition (2) for $\overline{u}_k(x_k)$, in this case, we also have that, almost everywhere and for ε sufficiently small, $\overline{u}_k(x_k + \varepsilon) = u_k^{\max} - d_k^{(f)}$. Since $-\omega_k(x_{k+1})$ is piecewise constant, the difference of the two integrals is given, by

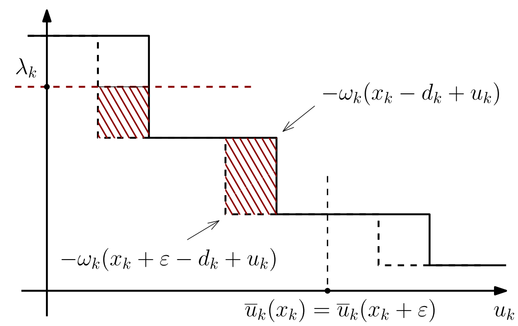

Fig. 4. Interpretation of the difference of the two integrals in (17) in the case of piecewise marginal value function, for $\overline{u}_k(x_k) = \overline{u}_k(x_k + \varepsilon) = u_k^{\max}$—$d_k^{(f)}$.

$$\Delta A = \begin{cases} 0 & \text{if } \lambda_k < -\omega_k(x_k - d_k + \overline{u}_k(x_k)) \\ -\varepsilon\,[-\omega_k(x_k - d_k) + \omega_k(x_k - d_k + \overline{u}_k(x_k))] & \\ \qquad \text{if } \lambda_k > -\omega_k(x_k - d_k) & \\ -\varepsilon[\lambda_k + \omega_k(x_k - d_k + \overline{u}_k(x_k))] & \text{otherwise.} \end{cases} \quad (18)$$

This is exemplified by the plot in Fig. 4. At the same time, it holds almost everywhere and for sufficiently small $\varepsilon > 0$ that

$$\begin{aligned} W_k(x_k + \varepsilon - d_k + \overline{u}_k(x_k + \varepsilon)) - W_k(x_k - d_k + \overline{u}_k(x_k)) \\ = \varepsilon\omega_k(x_k - d_k + \overline{u}_k(x_k)). \quad (19) \end{aligned}$$

Thus, using (18) and (19) in (17), we find that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\, [V_k(x_k + \varepsilon, \lambda_k) - V_k(x_k, \lambda_k)]/\varepsilon \\ &\qquad = -\mathcal{S}\big|_{-\omega_k(x_k - d_k + \overline{u}_k(x_k))}^{-\omega_k(x_k - d_k)}(\lambda_k). \quad (20) \end{aligned}$$

By averaging (20) over λ_k, we find a formula for the update of the marginal expected value function ω_k

$$-\,\omega_{k-1}(x_k) = \int \mathcal{S}\big|_{-\omega_k(x_k - d_k + \overline{u}_k(x_k))}^{-\omega_k(x_k - d_k)}(\lambda_k)\; dP(\lambda_k) \quad (21)$$

where $dP(\lambda_k)$ is the probability density associated with the random variable λ_k.

B. Case $\overline{u}_k(x_k) < u_k^{max}$—$d_k^{(f)}$

Given the definition (2) for $\overline{u}_k(x_k)$, in this case, we also have that, almost everywhere and for ε sufficiently small, $\overline{u}_k(x_k)$—$\overline{u}_k(x_k + \varepsilon) = \varepsilon$. In this case, we therefore have that

$$\begin{aligned} W_{k+1}\,(x_k + \varepsilon - d_k + \bar{u}_k(x_k + \varepsilon)) \\ -\; W_{k+1}(x_k - d_k + \bar{u}_k(x_k)) = 0. \quad (22) \end{aligned}$$

Instead, the area between the two integrals is given by

$$\Delta A = \begin{cases} -\varepsilon\,[-\omega_k(x_k - d_k)] & \text{if } \lambda_k > -\omega_k(x_k - d_k) \\ -\varepsilon\lambda_k & \text{otherwise.} \end{cases} \quad (23)$$

Thus, using (22) and (23) in (17), we find that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\, [V_k(x_k + \varepsilon, \lambda_k) - V_k(x_k, \lambda_k)]/\varepsilon \\ &\qquad = \; -\mathcal{S}\big|_{-\infty}^{-\omega_k(x_k - d_k)}(\lambda_k). \end{aligned}$$

Averaging the last expression with respect to λ_k, we find that

$$-\omega_{k-1}(x_k) = \int \mathcal{S}\big|_{-\infty}^{-\omega_k(x_k-d_k)}(\lambda_k)dP(\lambda_k) \qquad (24)$$

where $P(\lambda_k)$ is the probability measure associated with the random variable λ_k.

The function $-\omega_k(x_{k+1})$ is defined for $x_{k+1} \leq 0$. By formally extending it to $-\omega_k(x_{k+1}) = -\infty$ for $x_{k+1} > 0$, (21) and (24) can be expressed in the unified form (13). Observe that, since the function $-\omega_k(x_{k+1})$ is piecewise constant and monotonically nonincreasing, also $-\omega_{k-1}(x_k)$ is piecewise constant and monotonically nonincreasing, as stated in Theorem 2.

IV. Markov Price Process

The previous derivation can be extended to the case where the price process is not a sequence of independent random variables. This can be achieved by simply considering the proper conditional distributions in (13), instead of the static probability measure $P(\lambda_k)$.

The following result follows.

Theorem 2: Consider the optimal consumption problem with curtailment penalty $C_n(x_n)$ that is convex and piecewise linear. The optimal policy is given by (9) and (10), where the averaged marginal value functions $-\omega_k(x_{k+1}, \{\lambda_j\}_{j=k-m+1}^{k-1})$, for $k = 0, \ldots, n-1$, are piecewise constant in x_{k+1}, monotonically nonincreasing in x_{k+1}, and are given by the backward iterations

$$-\omega_{k-1}\left(x_k, \left\{\lambda_j\right\}_{j=k-m}^{k-1}\right)\right) = \int -\mathcal{S}\big|_{-\omega_k\left(x_k-d+\bar{u}_k(x_k), \{\lambda_j\}_{j=k-m+1}^{k}\right)}^{-\omega_k\left(x_k-d, \{\lambda_j\}_{j=k-m+1}^{k}\right)}(\lambda_k)\, dP\left(\lambda_k \big| \left\{\lambda_j\right\}_{j=k-m}^{k-1}\right) \qquad (25)$$

where $P(\lambda_k|\{\lambda_j\}_{j=k-m}^{k-1})$ is the conditional probability measure of λ_k given $\{\lambda_j\}_{j=k-m}^{k-1}$, and

$$-\omega_{n-1}\left(x_n, \{\lambda_j\}_{j=n-m}^{n-1}\right) = -\frac{dC(x_n)}{dx_n}. \qquad (26)$$

Therefore, also in the case of correlated prices, the optimal strategy of consumption is in the form of thresholds that can be explicitly computed from the price history.

Also in this case, in order to prove the optimality of the policy, we construct the explicit iterations for the averaged marginal value function. Starting from the stochastic Bellman equation (4) for each price λ_{k+1} fix the price history $\{\lambda_j\}_{j=k-m+1}^{k}$. As in the previous scenario, by assuming that $-\omega_k$ is monotonically nondecreasing in x_{k+1}, for each fixed price history, we find that

$$V_k(x_k, \left\{\lambda_j\right\}_{j=k-m+1}^{k}) = W_k(x_k - d_k + \bar{u}_k(x_k), \left\{\lambda_j\right\}_{j=k-m+1}^{k}) + \int_0^{\bar{u}_k(x_k)} \min\left\{\lambda_k, -\omega_k(x_k - d_k + u_k, \left\{\lambda_j\right\}_{j=k-m+1}^{k})\right\} du_k.$$

Proceeding exactly as in the previous scenario, we also find that, with the formal extension $-\omega_k(x_{k+1}, \{\lambda_j\}_{j=k-m+1}^{k}) = -\infty$ for $x_k > 0$ and by considering $\varepsilon > 0$ sufficiently small, we can write

$$\left[V_k\left(x_k + \varepsilon, \left\{\lambda_j\right\}_{j=k-m+1}^{k}\right) - V_k\left(x_k, \left\{\lambda_j\right\}_{j=k-m+1}^{k}\right)\right]/\varepsilon = -\mathcal{S}\big|_{-\omega_k\left(x_k-d+\bar{u}_k(x_k), \{\lambda_j\}_{j=k-m+1}^{k}\right)}^{-\omega_k\left(x_k-d, \{\lambda_j\}_{j=k-m+1}^{k}\right)}(\lambda_k) \qquad (27)$$

for every price history $\{\lambda_j\}_{j=k-m}^{k-1}$. Taking the limit for $\varepsilon \to 0$ and averaging with respect to λ_k, given the history $\{\lambda_j\}_{j=k-m+1}^{k-1}$, we obtain (25).

Since the final condition of the backward iterations is given by (26) and $C(x_n)$ is a piecewise linear and convex function, then it is guaranteed by iterations (25) that each $-\omega_k$ is piecewise constant and monotonically decreasing in x_{k+1} for a fixed history $\{\lambda_j\}_{j=k-m+1}^{k-1}$.

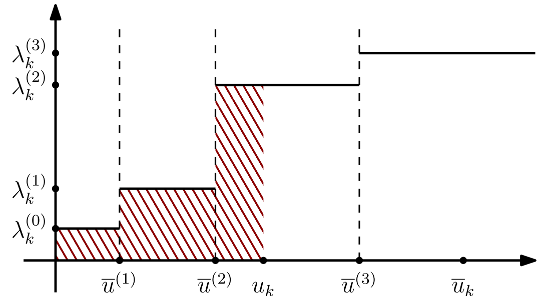

Fig. 5. Schematic representation of the piece-wise constant price curve that approximates the procurement cost in the presence of multiple sources. The shaded area corresponds to the total price paid to consume u_k at time k.

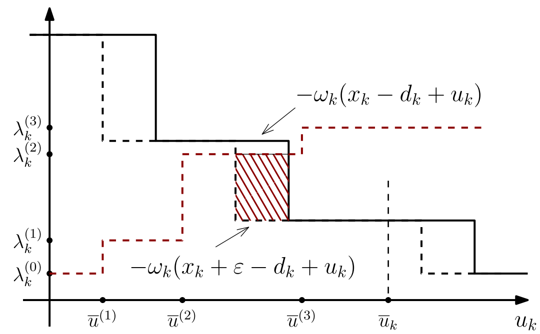

Fig. 6. Interpretation of the difference of the two integrals in (17), in the case of a piecewise constant energy price (energy procurement scenario).

V. Energy Procurement Problem

In the derivation of the optimal policy, we assumed that the energy cost is linear in the consumption, i.e., the individual consumer faces the marginal real-time energy price λ_k. In many practical cases, however, the consumer is allowed to purchase energy from multiple sources, possibly including its own generation facilities. If this is the case, then the resulting marginal energy price is a convex, nondecreasing function $\lambda_k(u_k)$ of the power consumption, and the total energy cost for the consumer becomes $\int_0^{u_k} \lambda_k(z)dz$.

The methodology proposed in this paper can be extended to this scenario, as long as we can approximate the real-time price curve $\lambda_k(u_k)$ with a piecewise constant function. Let $\{\bar{u}^{(1)}, \bar{u}^{(2)}, \ldots, \bar{u}^{(M)}\}$ be a set of fixed consumption thresholds. Then, at every time k, the energy price curve is described

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by a vector $\boldsymbol{\lambda_k} = [\lambda_k^{(0)}, \lambda_k^{(1)}, \lambda_k^{(2)}, \ldots, \lambda_k^{(M)}]$, describing the marginal costs in the corresponding energy consumption intervals (see Fig. 5).

The derivation of the optimal policy described in Section III can be repeated in this case. Similarly as before, the difference $V_k(x_k+\epsilon, \boldsymbol{\lambda}_k) - V_k(x_k, \boldsymbol{\lambda}_k)$ can be computed explicitly, by evaluating the area between the two curves $\min\{\lambda_k(u_k), -\omega_k(x_k - d_k + u_k)\}$ and $\min\{\lambda_k, -\omega_k(x_k + \epsilon - d_k + u_k)\}$. In this case, however, this area needs to be computed separately for each consumption interval (see how Fig. 6 replaces Fig. 4).

The backward iteration (13) of the Bellman equation (for the i.i.d. case, and similarly for the case of correlated prices) is finally replaced by

$$- \omega_{k-1}(x_k) = \int \sum_{i=0}^{M_k-1} \mathcal{S}\Big|_{-\bar{\omega}_k^{(i+1)}}^{-\bar{\omega}_k^{(i)}} \left(\lambda_k^{(i)} + \bar{\omega}_k^{(i+1)}\right) + \mathcal{S}\Big|_{-\bar{\omega}_k}^{-\bar{\omega}_k^{(M_k)}} \left(\lambda_k^{(M_k)}\right) \, dP(\boldsymbol{\lambda}_k)$$

where M_k is such that $\bar{u}^{(M_k)}$ is the largest consumption threshold smaller than $\bar{u}_k$, and we introduced the compact notations

$$\bar{\omega}_k^{(i)} = \omega_k\left(x_k - d_k + \bar{u}^{(i)}\right), \qquad \bar{\omega}_k = \omega_k\left(x_k - d_k + \bar{u}_k\right).$$

While the computational complexity of the iteration surely increased (depending on the number of consumption thresholds), it is again an explicit iteration that preserves the piece-wise nature of the averaged marginal value functions $-\omega_k(x_{k+1})$, and ultimately returns a threshold policy.

VI. Numerical Example

A. Example for i.i.d. Price Process

We first illustrate, via a simple numerical example, the computation of the derivatives of the expected value functions (W_k) for $k = 0, \ldots, n-1$ and how these functions define the optimal consumption for a price-responsive energy user. Consider an i.i.d price process $\{\lambda_k\}_{k \in \mathbf{Z}}$ with probability distributions that is uniform in the interval $[0, 1]$. Consider a price responsive user with demand profile $(d_0, d_1, d_2) = (3, 0, 0)$ over an optimization horizon $n = 3$ and bound on consumption $\overline{u}_k = 2$, for $k = 0, 1, 2$. Assume that the task is "critical," thus the backlog at time $n = 3$ has to be zero, $x_3 = 0$. We want to compute the derivatives of the value functions. Since the task is critical, we initialize the backward iterations as

$$-\omega_{n-1}(x_n) = -\omega_2(x_3) = \begin{cases} +\infty & \text{for } -2 \le x_3 < 0 \\ -\infty & \text{for } 0 \le x_3 \end{cases}$$

as shown in Fig. 7(a). Applying the backward iteration (13), we find

$$\begin{aligned} -\omega_1(x_2) &= \int \mathcal{S}\Big|_{-\omega_2(x_2+2)}^{-\omega_2(x_2)} (\lambda_2) dP(\lambda_2) \\ &= \begin{cases} +\infty & \text{for } -4 \le x_2 < -2 \\ \frac{1}{2} & \text{for } -2 \le x_2 < 0 \\ -\infty & \text{for } 0 \le x_2 \end{cases} \end{aligned}$$

that is shown in Fig. 7(b). Notice that, in order to compute $-\omega_1(x_2)$, we only have to compute the integral in (13) for a finite number of values of x_2, given the piecewise constant nature of the functions ω_k.

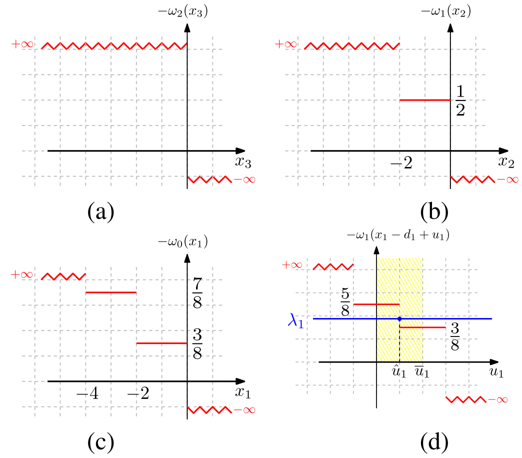

Fig. 7. Derivatives of the marginal value functions $-\omega_3$, $-\omega_2$, and $-\omega_1$ for the numerical example of Section VI in (a)–(c), respectively. (d) Determination of the optimal consumption u_0^* at time $k = 0$. If the price λ_0 is above $7/8$, then the optimal policies provides $u_0^* = 0$; if the price is between $3/8$ and $7/8$ (as in the figure), the optimal policy provides $u_0^* = 1$; if the price is below $3/8$, the optimal policy provides $u_0^* = \bar{u}_0 = 2$.

Applying again (13), we find

$$\begin{aligned} -\omega_0(x_1) &= \int \mathcal{S}\Big|_{-\omega_1(x_1+2)}^{-\omega_1(x_1)} (\lambda_1) dP(\lambda_1) \\ &= \begin{cases} +\infty & \text{for } -6 \le x_1 < -4 \\ \frac{5}{8} & \text{for } -4 \le x_1 < -2 \\ \frac{3}{8} & \text{for } -2 \le x_1 < 0 \\ -\infty & \text{for } 0 \le x_1 \end{cases} \end{aligned}$$

that is shown in Fig. 7(c). Now that $-\omega_0(x_1)$ is computed, we can plot $-\omega_0(x_0 - d_0 + u_0)$ as a function of u_0 and intersect this curve with the horizontal line λ_0 in order to determine the optimal consumption, using the expressions (9) and (10). As Fig. 7(d) suggests, if the price is above $7/8$, then $\hat{u}_0 = -1$ and the optimal policy returns $u_0^* = 0$; if the price is between $3/8$ and $7/8$ [as in Fig. 7(d)] the optimal policy provides $u_0^* = \hat{u}_0 = 1$; if the price is below $3/8$, then $\hat{u}_0 = 3$ and then the optimal policy provides $u_0^* = 2$.

This simple numerical example provides also a qualitative assessment of the complexity of the proposed algorithm. In fact, the number of thresholds that describe the value function increases at every backward step in the iterative procedures, and for each threshold, we need to compute an expectation operation (see also the discussion in [28] for a similar algorithm). The complexity of this latter operation depends on the pricing model: in the case of discretized prices, this operation reduces to a sum over the possible price levels (and their probabilities).

B. Correlated Prices

Let us consider the following scenario. There is a contract in place between a utility company and a household/facility where energy is delivered according to a mechanism of real-time pricing that considers only five possible prices: "high," "medium/high," "medium," "medium/low," and "low." Prices are updated at a given frequency (for example every 15 min)

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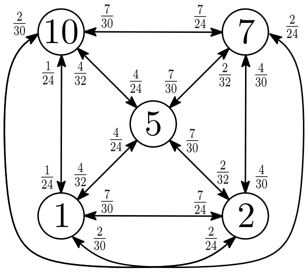

Fig. 8. Graph representation of the price process modeled as Markov process. The labels on the arrows are the corresponding transition probabilities from one price to another (transitions from one state to itself have been omitted).

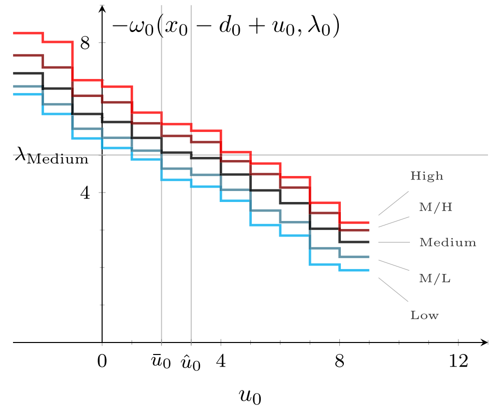

Fig. 9. Computed function $-\omega_0(x_1, \lambda_0) = \lambda_0$ for the example in Section VI-B. The price λ_0 can assume only five discrete values, thus there are five piecewise constant functions of $x_1 = x_0 - d_0 + u_0$, each one associated to a value of λ_0. Once λ_0 is known, the appropriate piecewise function is selected, and used to determine the optimal consumption.

and kept constant in between updates. A contract of this kind (with only a few discretized prices) might provide some of the advantages of real-time pricing without exposing the consumer to the high variability of the market. The consumer (or better a smart appliance) monitors the price and determines the appropriate consumption trying to minimize the expected cost. Let us assume that the price process is well modeled by a Markov process, so that the probability of having a certain price during the next time interval depends only on the price at the current interval. Let the five price levels be [10, 7, 5, 2, 1] and the transition probabilities be as represented in Fig. 8. Observe that a higher prices a time k is positively correlated with a higher price at time $k+1$ and vice versa. Positive correlation in the price process has already been observed in power systems [31], thus determining an optimal consumption policy in the case of a correlated price process has great practical relevance.

Assume that a task requires nine energy units and has to be completed within six time steps ($k = 0, 1, 2, 3, 4, 5$) with no curtailment allowed. We model the task assuming a shiftable demand profile $d^{(s)} = [9, 0, 0, 0, 0, 0]$. Assume also that the household has a constraint on the energy that it can withdraw from the grid, namely $u_k^{\max} = 3$, for all k. However, suppose that there are other processes already scheduled that can not be either postponed or anticipated. They are modeled as a firm component of the demand $d^{(f)} = [1, 1, 0, 0, 2, 2]$. Applying the backward iterations proposed in Theorem 2, it is possible to compute the function $-\omega_k(x_{k+1}, \lambda_k)$, for $k = 0, 1, 2, 3, 4, 5$. Notice that, by considering discretized prices, the functions $-\omega_k(x_{k+1}, \lambda_k)$ can be represented (and stored) as five piecewise constant functions of x_{k+1}, each one associated to a value of λ_k. For $k = 0$ the result of these computations is reported in Fig. 9. Observe that higher value functions corresponds to higher values of λ_0. This is a direct consequence of the positive correlation of the price process: if the λ_0 is high it is more likely that future prices will be high as well.

The optimal consumption is, therefore determined immediately from the five value functions computed at $k = 0$, as shown before in Fig. 7(d). Once the price λ_0 is known, the appropriate value function is selected. The intersection of the selected curve with the price level λ_0 determines the value $\hat{u}_0$ for which $-\omega_0(x_0 - d_0 + \hat{u}_0, \lambda_0) = \lambda_0$, as in (9). The optimal consumption u_0^* is then computed according to (10), i.e., by limiting the consumption to the interval $[0, \bar{u}_0]$, if it falls outside. In the example of Fig. 9, we assumed that the price λ_0 is medium, and we obtained $\hat{u}_0 = 3$. As $\bar{u}_0 = u_0^{\max} - d_0^{(f)} = 2$, then $u_0^* = 2$.

This numerical example also shows how, as expected, the number of thresholds increases with the length of the optimization horizon n. It is indeed possible to construct examples where the number of thresholds grows exponentially in n. However, in the specific case where the bounds $\bar{u}_k$ are all commensurable and uniformly bounded, the number of thresholds grows linearly with the time horizon. Regardless of these asymptotic behaviors, in practice only the specific application (in particular the price update frequency and the price discretization) is going to dictate the computational requirements of this procedure. However, the resulting complexity is expected to be typically manageable by the computational core of a smart appliance, at least for a wide variety of scenarios.

VII. Conclusion

In this paper, we have considered the problem of optimal consumption for an energy user with flexible loads, responding to correlated real time energy prices from multiple procurement sources. The model that we introduced for the individual user allows to describe a very general scenario, including power consumption constraints, firm and shiftable demand profiles, and possible curtailment. In this model, loads are aggregated by deadline. This choice proves to be effective. In this case, we show how the optimal consumption strategy can be explicitly computed, via backward iterations of a Bellman stochastic equation.

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