Distributed multi-hop reactive power compensation in smart micro-grids subject to saturation constraints

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Abstract— **In this paper we address the problem of exploiting the distributed energy resources (DER) available in a smart micro-grid to minimize the power distribution losses via optimal reactive power compensation. Due to their typically small size, the amount of reactive power provided by each micro-generator is subject to tight saturation constraints. As a consequence, it might be impossible to achieve convergence to the global optimum based on algorithms that rely on short-range, gossip-type communication. We therefore propose a randomized multi-hop protocol that guarantees convergence of the distributed optimization algorithm also when only short-range communications are possible, at the expense of some additional communication overhead.**

I. Introduction

A prominent part of the modernization of the electric power distribution network consists in the development of *smart micro-grids*. A micro-grid is a portion of the power distribution network which is populated by a large number of micro-generators, and that can be managed independently from the rest of the grid. In particular, the power inverters that equip each micro-generator can be controlled in a synergistic and coordinated way in order to make the micro-grid more reliable, to improve energy efficiency, to guarantee higher quality of the service, and to allow the exploitation of larger shares of renewable energy sources.

For example, the power inverters can be exploited in order to provide distributed reactive power compensation, one among the most important ancillary services. In order to provide this service, each inverter can be controlled locally, according to its own measurements (as in [1]), or can be controlled by a central supervising unit (as in [2], [3]). Both approaches have some drawbacks: the *local* approach exhibits suboptimal performance due to the lack of coordination between agents; the *centralized* approach scales badly with the number of devices in the micro-grid, yields large-scale optimization problems, is hardly reconfigurable when a new device enters or leaves the micro-grid, and requires a very reliable communication channel.

More recently, *distributed* approaches have been proposed, for example in [4] and [5]. In these works, the coordination between different inverters is achieved via short-range communication (possibly based on the technology of power line communication) and by exploiting the on-board intelligence available at each device. This way, optimal behavior of the micro-grid can be achieved without relying on any central unit, thus improving robustness, enabling plug-and-play procedures, and requiring limited communication resources.

This work stems from the solution proposed in [5]. The large-scale optimization problem corresponding to the minimization of reactive power flows (and thus distribution losses) is decomposed into small tractable subproblems, each one involving only a pair of inverters. Iteratively, a pair of inverters is activated, and they optimally adjust the amount of reactive power they inject into the grid in order to minimize losses on the power lines. The proposed algorithm can be considered as a randomized and asynchronous version of the block-coordinate and Gauss-Seidel descent algorithms available in the literature on parallel optimization [6]. The convergence and the performance of this algorithm has been analyzed in [5] for the unconstrained case, i.e. when there are no bounds limiting the amount of reactive power that each inverter can inject in the grid. In practice, however, these bounds are quite tight because of the small size of the single devices. Extending the class of block-coordinate descent algorithms to the constrained case is not straightforward. In [7], convergence has been proved when the feasible set is the Cartesian product of convex sets, but without any linear equality constraint (which instead is present in the application that we are considering). In [8], the authors consider a general framework which includes the problem that we are considering here. However, the choice of which coordinates to update at each iteration is not randomized; on the contrary, they are chosen in order to guarantee a sufficient predicted descent, *a la* Gauss-Southwell. In [9], components are updated pair-wise (as in this work) but at least some of the components must be uniformly bounded away from their limits, which is not the case in our application. Finally, in [10], the author proves convergence of the algorithm when the choice of the set of components that are updated at each iteration is given by the solution of another optimization problem (similar to what is proposed in [11]). This procedure cannot be applied in this context, where there is no supervisor capable of triggering the correct sequence of devices.

We propose instead an asynchronous and randomized distributed algorithm which relies only on local communications between devices. In order to guarantee convergence, a multi-hop communication strategy is implemented. We show that, at the cost of a little communication overhead, the algorithm is globally convergent to the optimum of the constrained optimization problem.

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II. Micro-grid model

For the purpose of this paper, we model a micro-grid as a radial graph $\mathcal{G}$, in which edges represent the power lines, and the n nodes (whose set is denoted by $\mathcal{V}$) represent both loads and generators that are connected to the micro-grid. These include the residential and industrial consumers, micro-generators, and also the point of connection of the micro-grid to the transmission grid (called point of common coupling, or PCC).

We limit our study to the steady state behavior of the system, when all voltages and currents are sinusoidal signals at the same frequency ω_0. Each signal can therefore be represented via a complex number whose absolute value corresponds to the signal root-mean-square value, and whose phase corresponds to the phase of the signal at $t = 0$. Therefore the complex number $y = |y|e^{j\angle y}$ represents the signal $\mathbf{y}(t) = |y|\sqrt{2}\sin(\omega_0 t + \angle y)$.

In this notation, the steady state of a micro-grid is described by the following system variables:

- $u_v \in \mathbb{C},\ v \in \mathcal{V}$ are the grid voltages at the points where the nodes are connected to the grid;
- $i_v \in \mathbb{C},\ v \in \mathcal{V}$ are the currents injected by the nodes into the grid.

Each node v of the micro-grid is characterized by a law relating its injected current i_v with its voltage u_v. We model the node corresponding to the PCC (which we assume to be node 0) as a constant voltage generator at the nominal voltage U_N with arbitrary, but fixed, angle:

$$u_0 = U_N e^{j\angle u_0}.$$

We assume instead that the voltage u_v and the current i_v of every node v, but the PCC, satisfy the following law

$$u_v \bar{i}_v = s_v \left|\frac{u_v}{U_N}\right|^{\eta_v}, \quad \forall v \in \mathcal{V}\backslash\{0\}, \tag{1}$$

where $\bar{i}_v$ is the complex conjugate of i_v, $s_v = p_v + jq_v$ is the *nominal complex power* (p_v and q_v being the *active* and *reactive* nominal power, respectively), and η_v is a characteristic parameter of the node v. The model (1) is called *exponential model* [12] and is widely adopted in the literature on power flow analysis [13]. Notice that s_v is the complex power that the node would inject into the grid if the voltage at its point of connection was the nominal voltage U_N. Micro-generators fit in this model with $\eta_v = 0$, as they generally are commanded via a complex power reference and they are capable of injecting it independently from the voltage at their point of connection [14], [15].

We assume that all the micro-grid power line impedances have the same inductance/resistance ratio, i.e. for each edge e of the graph $\mathcal{G}$ we have

$$z_e = |z_e|e^{j\theta}.$$

We denote by Z^{eff}_{hk} the effective impedance of the electric network between node h and k. As the network is radial, this corresponds to the impedance of the path connecting node h to node k.

We adopt here the approximated model proposed in [16] for the analysis of the micro-grid power flows. The approximation is based on the fact that the micro-grid operating point, in its regular regime, is characterized by a relatively high nominal voltage compared to the voltage drops across the power lines, and by relatively small power distribution losses, compared to the power delivered to the loads. According to this analysis, node voltages are approximated by an affine expression of the injected complex powers:

$$u_v \approx e^{j\angle u_0}\left(U_N + \frac{1}{U_N}e^{j\theta}\mathbf{1}_v X\bar{s} + \frac{1}{U_N}\lambda_v(U_N)\right), \tag{2}$$

where $\mathbf{1}_v$ is the vector with 1 in position v and 0 elsewhere, $\lambda_v(U_N)$ is infinitesimal for large nominal voltages U_N, s is the vector whose elements are the nominal complex node powers $s_v, v \in \mathcal{V}\backslash\{0\}$, augmented with $s_0 := -\sum_{v\in\mathcal{V}\backslash\{0\}} s_v$, and where the matrix X depends on the electrical topology of the micro-grid and satisfies

$$(\mathbf{1}_h - \mathbf{1}_k)^T X(\mathbf{1}_h - \mathbf{1}_k) = \left|Z^{\text{eff}}_{hk}\right|, \quad h,k \in \mathcal{C}. \tag{3}$$

III. Optimal reactive power flow problem

Similarly to what has been done for example in [17], we choose active power distribution losses on the power lines as a metric for optimality of reactive power flows.

In the scenario that we are considering, we are allowed to command only a subset $\mathcal{C} \subset \mathcal{V}$ of cardinality $|\mathcal{C}|$ of the nodes of the micro-grid. The nodes in $\mathcal{C}$ are the micro-generators participating to the reactive power compensation, hence they will be denoted as *compensators*.

We assume that for these agents we are only allowed to command the amount q_v of reactive power injected into the grid, as the decision on the amount of active power p_v follows imperative economic criteria (for example, in the case of renewable energy sources, any available active power is typically injected into the grid to replace generation from traditional plants, which are more expensive and exhibit a worse environmental impact – see [18]).

Because the power electronics (inverter) of each micro-generator is capable of processing only a maximum rated amount $S_v^{\max}$ of *apparent power* $|s_v|^2 = p_v^2 + q_v^2$, for each of them we introduce a bound on the maximum reactive power that can be injected by that agent into the grid:

$$|q_v| \leq \sqrt{\left(S_v^{\max}\right)^2 - p_v^2} := q_v^{\max}. \tag{4}$$

Following the approach proposed in [16] (that we do not recall here in its details), we can express the problem of optimal reactive power injection at the compensators as a convex, quadratic, linearly constrained problem, in the form

$$\begin{aligned} \min_{q_v, v\in\mathcal{C}} \quad & J(q), \qquad \text{where} \quad J(q) = \frac{\cos\theta}{2} q^T X q, \\ \text{subject to} \quad & \sum_{v\in\mathcal{V}} q_v = 0 \\ & q \in \mathcal{B}, \end{aligned} \tag{5}$$

where X is positive definite on the feasible subspace, while $\mathcal{B}$ is a *box constraint set* defined as

$$\mathcal{B} = \{q \,|\, |q_v| \le q_v^{\max},\ \forall v \in \mathcal{C}\}\,.$$

Notice that the cost function in (5) is not *separable* into individual terms, and therefore most of the decomposition methods available in the recent literature on distributed optimization (e.g.[19], [20], [21]) cannot be directly applied.

IV. Proposed Distributed Algorithm

In this section, we propose a fully distributed and randomized algorithm for the optimization problem (5).

A. Available measurements and communication graph

In the scenario that we are considering, micro-generators are provided with measurement, processing, and communication capabilities. In particular, they are equipped with synchronized phasor measurement units that allow them to obtain voltage measurements at their point of connection with respect to a common time reference.

Micro-generators can also communicate, by exploiting the electrical grid as a communication channel, via powerline communication (PLC). Each of them can communicate only with a subset of neighbors which are sufficiently close in the electrical topology. We can therefore construct a connected *communication graph* $\mathcal{H}$ whose nodes are the elements of $\mathcal{C}$ and whose edges connect compensators that can exchange data via PLC links (see Figure 1).

We assume that each compensator h knows the electric distance $\left|Z_{hk}^{\text{eff}}\right|$ between itself and any neighbor k in the communication graph $\mathcal{H}$. This is a reasonable hypothesis, since the mutual effective impedances can be obtained from either *a priori* knowledge of the local grid topology, via online estimation as in [22], or via ranging technologies over power line communications as suggested in [23].

Notice that such architecture can be constructed either via automatic discovery (plug and play) or via manual configuration at the time of deployment. In any case, the insertion or removal of compensators require limited reconfiguration of a confined neighborhood of the micro-grid.

B. Gossip multi-hop algorithm

Let each compensator be provided with a timer, each one governed by an independent, identical, Poisson process, which triggers the compensator after exponentially distributed waiting times. By assuming that the time required for the execution of the algorithm is negligible with respect to the typical waiting time of the Poisson processes, we do not consider in this analysis the event of concurrent activation of different clusters. With this assumption, the algorithm execution can be described by a sequence $\sigma(t) \in \mathcal{C}$, $t \in \mathbb{Z}_{\ge 0}$, where $\sigma(t)$ is the only compensator triggered at the iteration t. As the timers are identical, and because of the Markovianity of the Poisson processes, the sequence $\sigma(t)$ results to be a sequence of independent symbols uniformly distributed in $\mathcal{C}$, with probability

$$\mathbb{P}[\sigma(t) = h] = \frac{1}{|\mathcal{C}|}, \quad \forall h \in \mathcal{C}.$$

When a compensator h is triggered, it initiates a **Send** procedure, as described hereafter.

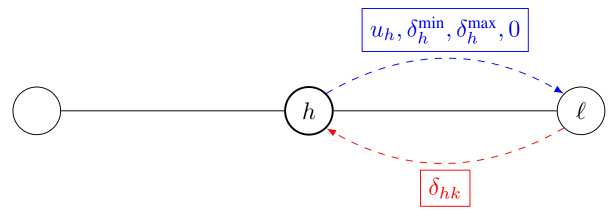
Send

i) Compensator h measures voltage u_h;
ii) compensator h computes $\delta_h^{\min}, \delta_h^{\max}$ as

$$\delta_h^{\min} = -q_h^{\max} - q_h, \qquad \delta_h^{\max} = q_h^{\max} - q_h;$$

iii) compensator h randomly chooses a neighbor ℓ in the communication graph $\mathcal{H}$ and sends the message $\boxed{u_h, \delta_h^{\min}, \delta_h^{\max}, |Z_{hl}^{\text{eff}}|}$ to it;
iv) compensator h waits for a response message from ℓ;
v) compensator h receives the message $\boxed{\delta_{hk}}$ from ℓ and actuates the system:

$$q_h^+ = q_h + \delta_{hk}.$$

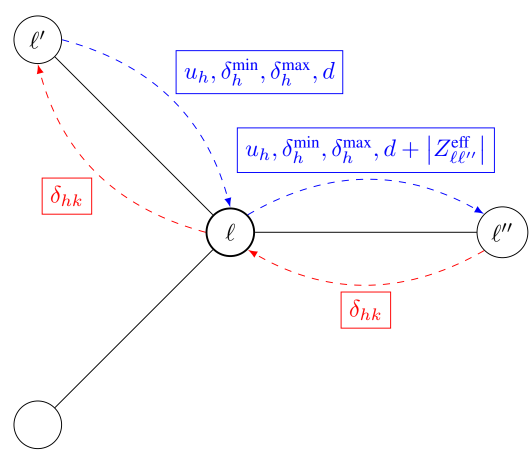


On the other hand, when a compensator receives a message $\boxed{u_h, \delta_h^{\min}, \delta_h^{\max}, d}$:

- if there are no other neighbors in the clustering graph $\mathcal{H}$, then **Respond** is executed;
- otherwise
  - with probability $\epsilon > 0$, **Forward** is executed;
  - with probability $1 - \epsilon$, **Respond** is executed.

Forward

i) Compensator ℓ has received the message $\boxed{u_h, \delta_h^{\min}, \delta_h^{\max}, d}$ from node ℓ';
ii) compensator ℓ randomly chooses a neighbor $\ell'' \ne \ell'$ in the communication graph $\mathcal{H}$ and forwards the message $\boxed{u_h, \delta_h^{\min}, \delta_h^{\max}, d + \left|Z_{\ell\ell''}^{\text{eff}}\right|}$ to it;
iii) compensator ℓ waits for a response message $\boxed{\delta_{hk}}$ from ℓ'';
iv) compensator ℓ receives the message $\boxed{\delta_{hk}}$ from ℓ'' and forwards it back to node ℓ'.



Respond

i) Compensator k has received the message $\boxed{u_h, \delta_h^{\min}, \delta_h^{\max}, d}$ from node ℓ;

ii) compensator k computes $\delta_k^{\min}, \delta_k^{\max}$ as

$$\delta_k^{\min} = -q_k^{\max} - q_k, \qquad \delta_k^{\max} = q_k^{\max} - q_k;$$

iii) compensator k measures voltage u_k;

iv) compensator k computes the optimal step δ_{hk} as

$$\delta_{hk} = \left[\frac{\nu_{hk}}{d}\right]_{\max\{\delta_h^{\min}, \delta_k^{\min}\}}^{\min\{\delta_h^{\max}, \delta_k^{\max}\}}, \tag{6}$$

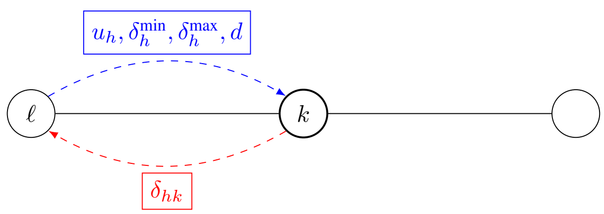
where $[\cdot]_a^b = \min\{\max\{\cdot, a\}, b\}$ and where

$$\begin{aligned}\nu_{hk} = &\frac{|u_k|^2 - |u_h|^2}{2}\sin\theta \\ &+ |u_h||u_k|\sin(\angle u_h - \angle u_k)\cos\theta;\end{aligned} \tag{7}$$

v) compensator k responds to node ℓ with the message $\boxed{\delta_{hk}}$;

vi) compensator k actuates the system:

$$q_k^+ = q_k - \delta_{hk}.$$



C. Convergence

To show the effectiveness of this algorithm in solving the optimization problem (5), we need the following lemmas.

Lemma 1: Let $P_{hk}(t)$ be the probability that node h and k execute the procedures **Send** and **Respond**, respectively, at iteration t. Then

$$P_{hk}(t) = P_{hk} > 0.$$

Proof: As remarked before, the probability of h executing the procedure **Send** at time t is independent from the specific time instant. Therefore $P_{hk}(t) = P_{hk}$.

The statement that $P_{hk} > 0$ for all h, k descends from the fact that the communication graph $\mathcal{H}$ is connected and that $\epsilon > 0$. Therefore, in the language of Markov chains, each compensator is *accessible* from any other compensator via a proper sequence of *transitions* (i.e., **Forward** steps), and thus the probability of such sequence of transitions is strictly greater than zero. ∎

We introduce the maps $T_{hk},\ h, k \in \mathcal{C}$, which corresponds to the update of the system state q when node h and node k execute the **Send** and **Respond** procedures, respectively:

$$T_{hk}(q) = q + \delta_{hk}\left(\mathbf{1}_h - \mathbf{1}_k\right).$$

The following lemmas hold.

Lemma 2: For all q and for any $h, k \in \mathcal{C}$,

$$J(T_{hk}(q)) \le J(q).$$

Moreover, $J(T_{hk}(q)) = J(q)$ if and only if q is already the solution of the optimization subproblem

$$\begin{aligned} \min_{q_h, q_k} \quad & J(q) \\ \text{subject to} \quad & \sum_{v \in \mathcal{V}} q_v = 0 \\ & q \in \mathcal{B}, \end{aligned} \tag{8}$$

and in this case $T_{hk}(q) = q$.

Proof: Consider the generic update

$$q + \delta\left(\mathbf{1}_h - \mathbf{1}_k\right)$$

which involves only node h and k. We have

$$\begin{aligned} &J(q + \delta(\mathbf{1}_h - \mathbf{1}_k)) \\ &\quad= \frac{\cos\theta}{2}(q + \delta(\mathbf{1}_h - \mathbf{1}_k)^T X(q + \delta(\mathbf{1}_h - \mathbf{1}_k)) \\ &\qquad= \delta^2\frac{\cos\theta}{2}(\mathbf{1}_h - \mathbf{1}_k))^T X(\mathbf{1}_h - \mathbf{1}_k) \\ &\qquad\quad+ \delta\cos\theta(\mathbf{1}_h - \mathbf{1}_k)^T Xq + \frac{\cos\theta}{2}q^T Xq. \end{aligned} \tag{9}$$

According to the model presented in [16] and recalled in Section II, we have that the coefficient of the δ^2-term in (9) can be expressed by using (3) as

$$\frac{\cos\theta}{2}(\mathbf{1}_h - \mathbf{1}_k)^T X(\mathbf{1}_h - \mathbf{1}_k) = \frac{\cos\theta}{2}\left|Z_{hk}^{\text{eff}}\right|. \tag{10}$$

Moreover, one can verify that ν_{hk} as defined in (7) corresponds to

$$\nu_{hk} = \operatorname{Im}\left[e^{-j\theta}\frac{1}{2}(\bar{u}_h + \bar{u}_k)(u_h - u_k)\right].$$

Therefore, by using (2), we have that

$$\begin{aligned} \nu_{hk} = \operatorname{Im}&\left[e^{-j\theta}\left(U_N + \frac{1}{U_N}e^{-j\theta}\frac{(\mathbf{1}_h + \mathbf{1}_k)^T}{2}\bar{X}s \right.\right.\\ &\left.+ \frac{1}{U_N}\frac{\bar{\lambda}_h(U_N) + \bar{\lambda}_k(U_N)}{2}\right)\left(\frac{1}{U_N}e^{j\theta}(\mathbf{1}_h - \mathbf{1}_k)^T X\bar{s}\right.\\ &\left.\left.+ \frac{1}{U_N}\left(\lambda_h(U_N) - \lambda_k(U_N)\right)\right)\right] \\ = &\operatorname{Im}\left[(\mathbf{1}_h - \mathbf{1}_k)^T X\bar{s}\right] + \lambda_{hk}(U_N) \\ = &-(\mathbf{1}_h - \mathbf{1}_k)^T Xq + \lambda_{hk}(U_N), \end{aligned}$$

where $\lambda_{hk}(U_N)$ is a scalar quantity which is infinitesimal for large U_N.

The first order term in the δ-polynomial in (9) can therefore be approximated by $-\cos\theta\,\nu_{hk}$, yielding, together with (10),

$$J(q+\delta(\mathbf{1}_h-\mathbf{1}_k)) = \frac{\cos\theta}{2}\left|Z_{hk}^{\text{eff}}\right|\delta^2 - \cos\theta\,\nu_{hk}\delta + J(q). \tag{11}$$

The expression (11) for power losses has a minimum in

$$\delta^* = \frac{\nu_{hk}}{\left|Z_{hk}^{\text{eff}}\right|}.$$

From the triangular inequality on graph electric distances, it follows that in the algorithm execution d is always larger or equal than $\left|Z_{hk}^{\text{eff}}\right|$. Therefore, unless $\delta^* = 0$ (i.e. q is the solution of (8) and thus $T_{hk}(q) = q$), δ_{hk} as computed in (6) always satisfies

$$0 < \delta_{hk} \leq \delta^*$$

and thus, by standard convexity arguments,

$$J(T_{hk}(q)) = J(q + \delta_{hk}(\mathbf{1}_h - \mathbf{1}_k)) < J(q).$$

■

Lemma 3: Let $q \neq q^{\text{opt}}$, where q^{opt} is the solution of the optimization problem (5). Then there exists a pair of compensators $h, k \in \mathcal{C}$ such that

$$T_{hk}(q) \neq q.$$

Proof: If q is not the constrained minimum of the optimization problem (5), then there exists a feasible update Δq such that

$$\begin{cases} \mathbf{1}^T \Delta q = 0 & \text{(12a)} \\ q + \Delta q \in \mathcal{B} & \text{(12b)} \\ \nabla J(q)^T \Delta q < 0 & \text{(12c)} \end{cases}$$

where (12a)-(12b) characterize feasibility of the update Δq, while (12c) characterizes suboptimality of the state q with respect to optimization problem (5) (see for example [24, Chapter 4.2.3]). We then build a finite family of vectors $\{\Delta q^{(i)}\}$ with the following iterative procedure. Let $i = 1$. While $\Delta q \neq 0$:

1) let $h(i), k(i) \in \mathcal{C}$ be two indices such that $\Delta q_{h(i)} > 0$ and $\Delta q_{k(i)} < 0$;
2) let $\eta_i = \min\{|\Delta q_{h(i)}|, |\Delta q_{k(i)}|\}$;
3) let $\Delta q^{(i)} = \eta_i(\mathbf{1}_{h(i)} - \mathbf{1}_{k(i)})$;
4) let Δq be updated as $\Delta q \leftarrow \Delta q - \Delta q^{(i)}$;
5) let $i \leftarrow i + 1$.

Notice that at any iteration i, $\mathbf{1}^T \Delta q = 0$. Therefore the existence of the two indices $h(i), k(i)$ is guaranteed, as long as $\Delta q \neq 0$. The procedure ends in no more than $|\mathcal{C}|$ steps, because at each iteration at least one of the components of Δq is set to zero. It is also easy to see that

$$\Delta q = \sum_i \Delta q^{(i)}.$$

We now show that, for any i, $\Delta q^{(i)}$ is a feasible update step for the pair $h(i), k(i)$. Indeed, let $\Delta q^{(i)} = \eta_i(\mathbf{1}_{h(i)} - \mathbf{1}_{k(i)})$. Notice that $\mathbf{1}^T \Delta q^{(i)} = 0$. According to the procedure, $0 < \Delta q_{h(i)}^{(i)} \leq \Delta q_{h(i)}$ and $\Delta q_{k(i)} \leq \Delta q_{k(i)}^{(i)} < 0$. Therefore, via the convexity of the box constraint set $\mathcal{B}$, $\Delta q^{(i)}$ is a feasible update step for the pair $h(i), k(i)$.

As, according to (12c),

$$\nabla J(q)^T \Delta q = \nabla J(q)^T \sum_i \Delta q^{(i)} < 0,$$

there must exist an element $\Delta q^{(i)}$ such that $\nabla J(q)^T \Delta q^{(i)} < 0$. Therefore, q is not the solution of the optimization subproblem (8) corresponding to nodes $h(i)$ and $k(i)$, and thus, by Lemma 2, $T_{hk}(q) \neq q$. ■

Finally, to prove the convergence of the proposed algorithm, we need to introduce the concept of *set-valued* maps. A set-valued map $T : X \rightrightarrows X$ associates to an element of X a subset of X. An evolution of the dynamical system determined by a set-valued map T is a sequence $\{x_t\}_{t \in \mathbb{Z}_{\geq 0}}$ with the property that $x_{t+1} \in T(x_t)$ for all $t \in \mathbb{Z}_{\geq 0}$. A set W is strongly positively invariant for T if $T(w) \subset W$ for all $w \in W$. The following theorem holds.

Lemma 4 (Theorem 4.5 in [25]): Let (X, d) be a metric space. Given a collection of maps $T_1, \ldots, T_\ell$, define the set-valued map $T : X \rightrightarrows X$ by $T(x) = \{T_1(x), \ldots, T_\ell(x)\}$. Given a stochastic process $\sigma : \mathbb{Z}_{\geq 0} \to \{1, \ldots, \ell\}$, consider an evolution $\{x_n\}_{n \in \mathbb{Z}_{\geq 0}}$ of T satisfying

$$x_{n+1} = T_{\sigma(n)}(x_n).$$

Assume that

i) there exists a compact set $W \subseteq X$ that is strongly positively invariant for T;
ii) there exists a function $U : W \to \mathbb{R}$ such that $U(w') < U(w)$, for all $w \in W$ and $w' \in T(w) \backslash \{w\}$;
iii) the maps T_i, for $i \in \{1, \ldots, \ell\}$, and U are continuous on W; and
iv) there exists $p \in]0, 1[$ and $h \in \mathbb{N}$ such that, for all $i \in \{1, \ldots, \ell\}$ and $n \in \mathbb{Z}_{\geq 0}$

$$\mathbb{P}[\sigma(n+h) = i | \sigma(n), \ldots, \sigma(1)] \geq p.$$

If $x_0 \in W$, then there exists $c \in \mathbb{R}$ such that almost surely the evolution $\{x_n\}_{n \in \mathbb{Z}_{\geq 0}}$ approaches the set

$$(N_1 \cap \cdots \cap N_\ell) \cap U^{-1}(c),$$

where $N_i = \{w \in W | T_i(w) = w\}$ is the set of fixed points of T_i in W, $i \in \{1, \ldots, \ell\}$.

With this tool, we can finally state the following convergence result.

Theorem 5: Consider the algorithm described in Section IV. If the probability ϵ is strictly greater than zero, then the system is driven to the solution of the constrained optimization problem (5).

Proof: Consider the set-valued map $T = \{T_{hk},\ h, k \in \mathcal{C}\}$. Let W be the compact set $J^{-1}(q(0))$. J is strongly positively invariant for T by Lemma 2, therefore condition i) of Lemma 4 is satisfied. By the same Lemma 2, condition ii) is also satisfied, while condition iii) can be easily verified by inspection.

Moreover, because of Lemma 1, condition iv) is also satisfied, and therefore Lemma 4 applies. Thus, the evolution of the state q almost surely approaches, as t goes to infinity, the set

$$N = \{q \,|\, T_{hk}(q) = q \ \forall h, k \in \mathcal{C}\},$$

which, according to Lemma 3, corresponds to the singleton $\{q^{\text{opt}}\}$, where q^{opt} is the solution of the optimization problem (5). ■

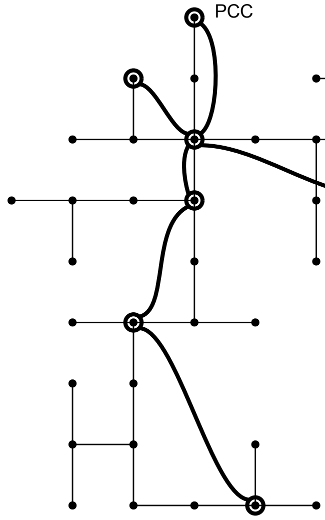


Fig. 1. The thin lines represent the graph $\mathcal{G}$, which describes the electrical topology of the micro-grid (in this case, an adaptation of the IEEE37 testbed). Circled nodes are compensators (belonging to $\mathcal{C}$, while the other nodes represent loads. The thick lines represent the edges of the communication graph $\mathcal{H}$, describing which compensators can communicate via PLC.

V. Simulations

In this section we present numerical simulations of the proposed randomized algorithm. To do so, we considered a 4.8 kV testbed inspired by the standard testbed IEEE 37 [26]. We however assumed that load are balanced, and therefore all currents and voltages can be described in a single-phase phasorial notation.

As shown in Figure 1, some of the nodes (including the PCC node 0) belong to the set of compensators, i.e. they are capable of injecting a commanded amount of reactive power. The amount of total reactive power injected by each compensator is limited to a portion of the total rating of the same node in the IEEE37 testbed. The nodes which are not compensators, are a blend of constant-power, constant-current, and constant-impedance loads, with a total power demand of about 2 MW of active power and 1 MVAR of reactive power (see [26] for the testbed data).

The length of the power lines range from a minimum of 25 meters to a maximum of almost 600 meters. The impedance of the power lines differs from edge to edge (for example, resistance ranges from 0.182 Ω/km to 1.305 Ω/km). However, the inductance/resistance ratio exhibits a smaller variation, ranging from $\angle z(e) = 0.47$ to $\angle z(e) = 0.59$. This justifies our assumption that $\angle z(e)$ can be considered constant across the network.

Power distribution losses in this grid are reported in the following table, where J^{opt} has been obtained via numerical optimization performed on the exact micro-grid model, and represent the minimum losses that can be achieved by properly choosing the amount of reactive power injected by each compensator (and retrieved from the PCC).

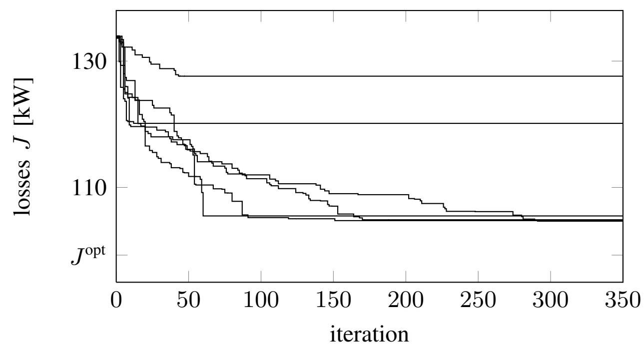


Fig. 2. Multiple instances of the algorithm when $\epsilon = 0$, i.e. there is no multi-hop strategy. Different randomly generated initial conditions correspond to different steady state configurations of the compensators, none of which results to be optimal.

Losses without compensation	133.7 kW
Fraction of delivered power	5.2 %
Minimum losses J^{opt}	99.2 kW
Fraction of delivered power	3.9 %
Losses reduction	25.8 %

We now simulate the behavior of the proposed algorithm when $\epsilon = 0$, i.e. there is no multi-hop strategy. Notice that the algorithm in this form corresponds to a possible implementation of the algorithm proposed in [27] for the unconstrained case. It is possible to see in Figure 2 that the algorithm do not converge to the optimum, due to the presence of saturation limits.

We now consider the case in which $\epsilon > 0$, i.e. with the proposed multi-hop strategy. It is possible to see from Figure 3 that the algorithm converges to the optimum for any choice of ϵ greater that zero. The speed of convergence of the algorithm, however, depends on the choice of ϵ: larger ϵ's seems to correspond to faster performance, at the price of an increased communication overhead and longer iteration times. A fair comparison of the rate of convergence for different probabilities ϵ requires the knowledge of the delays caused by each step of the multihop communication. The optimal trade-off between these aspects depends on the specific communication and control architecture deployed in the micro-grid.

VI. Conclusions

We proposed an implementation of the distributed approach proposed in [5] for the optimal reactive power compensation in smart micro-grids. The algorithm proposed in this paper deals effectively with the tight saturation limits that characterize the power inverters dispersed across the micro-grid. In order to guarantee convergence of the algorithm to the optimal state (minimal power distribution losses), a multi-hop coordination protocol has been introduced. The resulting algorithm is randomized, asynchronous, and robust to insertion or removal of nodes, with minimal reconfiguration.

The convergence results is also of interest *per se*: the same results apply to generic convex quadratic problems subject to one linear constraint and with a separable feasible set. Many

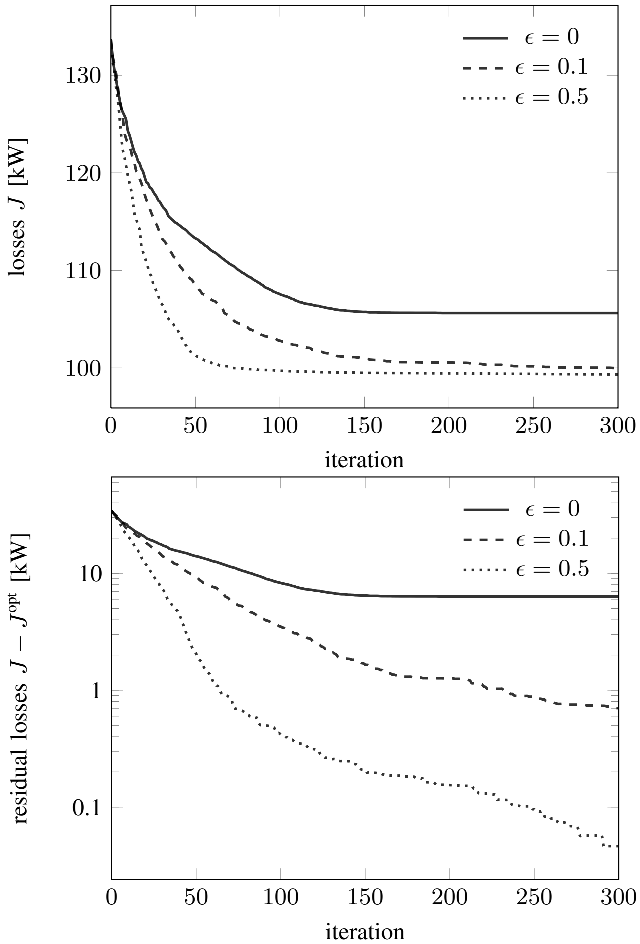


Fig. 3. Montecarlo analysis of the behavior of the algorithm for different values of ϵ. As soon as ϵ is greater than zero, the algorithm converges to the optimum. Also the rate of convergence depends on the choice of ϵ. In this comparison we assume that one iteration corresponds to a complete iteration of the algorithm, regardless of how many hops are required for its completion.

different applications can be casted into this framework, including learning in support vector machines, portfolio selection problems, load balancing in grid of processors, and others.

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