Operatoren in Koordinatensystemen

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1 Kartesische Koordinaten

$$\nabla\phi(x,y,z) = \begin{bmatrix} \partial_x\phi \\ \partial_y\phi \\ \partial_z\phi \end{bmatrix}$$

$$\text{div}\vec{V}(x,y,z) = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z}$$

$$\text{rot}\vec{V}(x,y,z) = \begin{bmatrix} \partial_y V_z - \partial_z V_y \\ \partial_z V_x - \partial_x V_z \\ \partial_x V_y - \partial_y V_x \end{bmatrix}$$

$$\Delta\phi(x,y,z) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

2 Polar- und Zylinderkoordinaten

$$\nabla\phi(r,\varphi,z) = \frac{\partial\phi}{\partial r}\cdot\vec{e}_r + \frac{1}{r}\cdot\frac{\partial\phi}{\partial\varphi}\cdot\vec{e}_\varphi + \frac{\partial\phi}{\partial z}\cdot\vec{e}_z$$

$$\text{div}\vec{V}(r,\varphi,z) = \frac{1}{r}\cdot\frac{\partial(r\cdot V_r)}{\partial r} + \frac{1}{r}\cdot\frac{\partial V_\varphi}{\partial\varphi} + \frac{\partial V_z}{\partial z}$$

$$\begin{aligned}\text{rot}\vec{V}(r,\varphi,z) =& \left[\frac{1}{r}\cdot\frac{\partial V_z}{\partial\varphi} - \frac{\partial V_\varphi}{\partial z}\right]\cdot\vec{e}_r + \left[\frac{\partial V_r}{\partial z} - \frac{\partial V_z}{\partial r}\right]\cdot\vec{e}_\varphi \\ &+ \frac{1}{r}\cdot\left[\frac{\partial(r\cdot V_\varphi)}{\partial r} - \frac{\partial V_r}{\partial\varphi}\right]\cdot\vec{e}_z\end{aligned}$$

$$\Delta\phi(r,\varphi,z) = \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\varphi^2} + \frac{\partial^2\phi}{\partial z^2}$$

3 Kugelkoordinaten

$$\nabla\phi(r,\theta,\varphi) = \frac{1}{r}\cdot\frac{\partial\phi}{\partial\theta}\cdot\vec{e}_\theta + \frac{1}{r\sin\theta}\cdot\frac{\partial\phi}{\partial\varphi}\cdot\vec{e}_\varphi + \frac{\partial\phi}{\partial r}\cdot\vec{e}_r$$

$$\text{div}\vec{V}(r,\theta,\varphi) = \frac{1}{r\sin\theta}\frac{\partial(V_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial V_\varphi}{\partial\varphi} + \frac{1}{r^2}\frac{\partial(r^2\cdot V_r)}{\partial r}$$

$$\begin{aligned}\text{rot}\vec{V}(r,\theta,\varphi) =& \left[\frac{1}{r\sin\theta}\cdot\frac{\partial V_r}{\partial\varphi} - \frac{1}{r}\cdot\frac{\partial(r\cdot V_\varphi)}{\partial r}\right]\cdot\vec{e}_\theta \\ &+ \frac{1}{r}\cdot\left[\frac{\partial(r\cdot V_\theta)}{\partial r} - \frac{\partial V_r}{\partial\theta}\right]\cdot\vec{e}_\varphi \\ &+ \frac{1}{r\sin\theta}\cdot\left[\frac{\partial(V_\varphi\sin\theta)}{\partial\theta} - \frac{\partial V_\theta}{\partial\varphi}\right]\cdot\vec{e}_r\end{aligned}$$

$$\Delta\phi(r,\theta,\varphi) = \frac{2}{r}\cdot\frac{\partial\phi}{\partial r} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r^2\sin\theta}\cdot\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial\phi}{\partial\theta}\right] + \frac{1}{r^2\sin^2\theta}\cdot\frac{\partial^2\phi}{\partial\varphi^2}$$

4 Rechenregeln für Operatoren

- $\text{div}(\text{rot}\vec{V}) = 0$
- $\text{div}(\phi\cdot\vec{V}) = \langle\nabla\phi,\vec{K}\rangle + \phi\cdot\text{div}\vec{K}$
- $\text{div}(\vec{V}_1\times\vec{V}_2) = \langle\vec{V}_2,\text{rot}\vec{V}_1\rangle - \langle\vec{V}_1,\text{rot}\vec{V}_2\rangle$
- $\text{rot}(\phi\cdot\vec{V}) = \nabla\phi\times\vec{V} + \phi\cdot\text{rot}\vec{V}$
- $\text{rot}(\nabla\phi) = 0$
- $\Delta\phi = \nabla^2\phi = \text{div}(\nabla\phi)$
- $\text{rot}(\text{rot}\vec{V}) = \text{div}(\text{div}\vec{V}) - \Delta\vec{V}$