Introduction

Model: G Real plant G_p $\Rightarrow$ $G_p = G + E$

Nominal stability (NS): system is stable with no model uncertainty
Nominal performance (NP): system satisfies performance specifications with no model uncertainty
Robust stability (RS): system is stable for "all" perturbed plants
Robust performance (RP): system satisfies performance specifications for all perturbed plants

Transfer function: $G(s) = \dfrac{\beta_{n_z} s^{n_z} + \ldots + \beta_1 s + \beta_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}$ where $\begin{cases} n = \text{order of the system} \\ n_Z = \text{order of the numerator} \\ n - n_Z = \text{relative order (pole excess)} \end{cases}$

strictly proper: $G(s) \to 0 \quad (s \to \infty)$
semi-proper: $G(s) \to D \neq 0 \quad (s \to \infty)$
} proper A system G(s) is improper if $G(s) \to \infty \quad (s \to \infty)$

↑ D from state space realization

Scaling

Consider a SISO-System with $\hat{y} = \hat{G}\hat{u} + \hat{G}_d \hat{d}$, $\hat{e} = \hat{y} - \hat{r}$ with disturbance $\hat{d}$, reference $\hat{r}$, input $\hat{u}$

Chose $d = \hat{d}/\hat{d}_{max}$, $u = \hat{u}/\hat{u}_{max}$, $y = \hat{y}/\hat{e}_{max}$, $r = \hat{r}/\hat{e}_{max}$, $e = \hat{e}/\hat{e}_{max}$ such that all values $\in [0,1]$

max. expected max. allowed

For a MIMO-System you use <u>diagonal</u> matrices since each variable in vectors $\{\hat{d}, \hat{r}, \hat{u}, \hat{e}\}$ has its own max.:

$d = D_d^{-1}\hat{d}$, $u = D_u^{-1}\hat{u}$, $y = D_e^{-1}\hat{y}$, $e = D_e^{-1}\hat{e}$, $r = D_e^{-1}\hat{r}$

$\Rightarrow \quad D_e y = \hat{G} D_u u + \hat{G}_d D_d d$, $D_e e = D_e y - D_e r$ $\Rightarrow$ $G = D_e^{-1}\hat{G} D_u$, $G_d = D_e^{-1}\hat{G}_d D_d$ $\left(D_x := \hat{x}_{max}\right)$

Often $\tilde{r} = \hat{r}/\hat{r}_{max} = D_r^{-1}\hat{r}$ $\Rightarrow$ $r = R\tilde{r}$ $\left(R = D_e^{-1} D_r = \hat{r}_{max}/\hat{e}_{max}\right.$ largest expected change in reference (R≥1)$\left.\right)$

$e = y - r = G u + G_d d - R\tilde{r}$ where we can treat reference changes as disturbances with $G_d = -R$

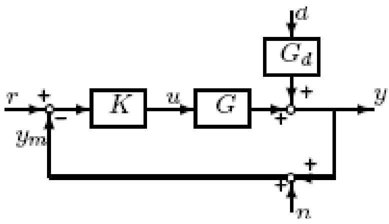
If an expected/allowed variation is not symmetric about 0 then the largest variation should be used for $\hat{d}_{max}$ and the smallest for $\hat{u}_{max}, \hat{e}_{max}$, e.g. $-5 \leq \{\hat{d}, \hat{u}\} \leq 10 \Rightarrow \hat{d}_{max} = 10$, $\hat{u}_{max} = 5$

Objective: $\boxed{\text{for } |d| \leq 1 \text{ and } |\tilde{r}| \leq 1 \text{ manipulate } u \text{ with } |u| \leq 1 \text{ such that } |e| = |y - r| \leq 1}$

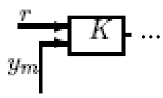
worst case: d=1, $\tilde{r}$=1

Notation

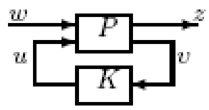
One degree-of-freedom control configuration:



Two degrees of freedom c.c.:



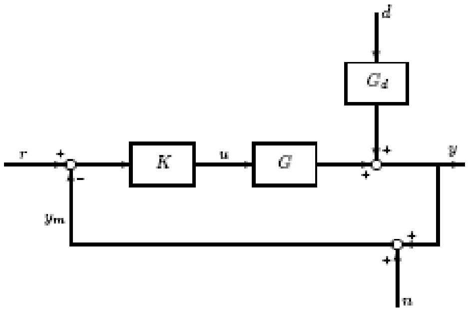
General control configuration



w includes disturbances/noise/...

Use Feedback because of unstable plants (see internal stability), model & signal uncertainty
Nominal performance/stability → without uncertainty Robust perform./stability → with uncertainty

Classical Feedback Control



Feedback Control

$$y = GK(r-y-n) \quad \Rightarrow \quad y = \underbrace{(I+GK)^{-1}GK}_{T} r + \underbrace{(I+GK)^{-1}}_{S} G_d d - Tn$$

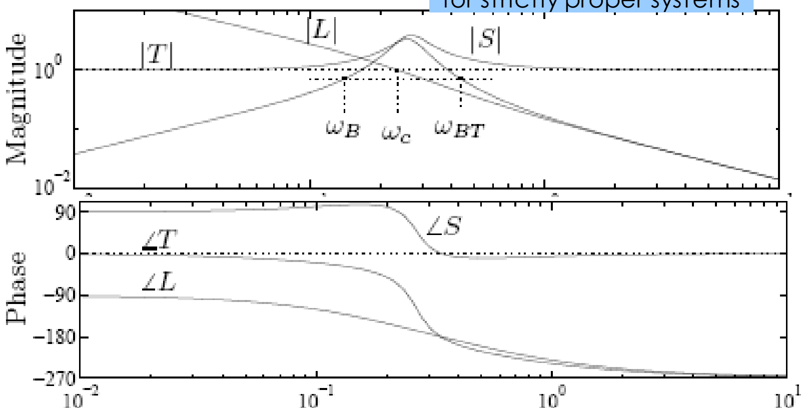
Where S is the sensitivity function and T the complementary sensitivity fct.

Control error: $e = y - r = -Sr + SG_d d - Tn$

Plant input: $u = KSr - KSG_d d - KSn$

Loop transfer function $L := GK \quad \Rightarrow \quad S = (I+L)^{-1} \;,\; T = (I+L)^{-1}L$ and $S+T=I$ ← this changes to S+HT=I when there is a matrix H in the feedback loop

S → 1 at high frequencies for strictly proper systems



crossover frequency $\omega_C \rightarrow |L(j\omega_C)| = 1$,

bandwidth $\omega_B \rightarrow |S(j\omega_B)| = 1/\sqrt{2}$ *for the 1st time from below*

(high BW = fast but sensitive to noise)

complementary bandwidth $\omega_{BT} \rightarrow |T(j\omega_{BT})| = 1/\sqrt{2}$ *for the 1st time f. above*

Plot for fix functions G(s), K(s) → see p.2-4

$\omega < \omega_B$	effective control
$\omega_B < \omega < \omega_{BT}$	control without performance improvement
$\omega_{BT} < \omega$	control has no effect

For PM<90° $\omega_B < \omega_c < \omega_{BT}$

Closed-loop Performance

$M_S = \max_\omega |S(j\omega)| = \|S\|_\infty \quad , \quad M_T = \max_\omega |T(j\omega)| = \|T\|_\infty$ Typically $M_S \le 2 \;,\; M_T < 1.25 \quad \Rightarrow \quad \begin{matrix} GM > 2 \\ PM > 30° \end{matrix}$

Poor performance/robustness if $M_{\{S,T\}} > 4$

$$GM = 1/|L(j\omega_{180})| \quad , \quad PM = \arg L(j\omega_c) + 180° \quad \Leftrightarrow \quad GM \ge M_S/(M_S - 1) \quad , \quad PM \ge 1/M_S$$

PM = possible lag before instability, max. possible time delay (Totzeit) $\theta_m = PM/\omega_c$

Controller Design

Main approaches:

1. Loop shaping (magnitude of $L(j\omega)$), shaping of closed-loop TF (S,T,K S)
2. Signal-based approach → Linear Quadratic Gaussian (LQG) control
3. Numerical optimization (computationally difficult)

Loop Shaping

$e = -Sr + SG_d d - Tn$ Should be small

Good disturbance rejection → large L
Good command following → large L
Nominal & Robust Stability → small L

Mitigation of measurement noise on outputs → L small
Small magnitude of input signals → K small and L small
$\Big\}$ trade-off

usually |L|>1 at low frequencies, |L|<1 at high frequencies (above crossover)

Fundamentals: We need a large $|L|$ in the bandwidth region (good feedback control), $|L|$ should fall sharply with frequency but for stability we need $|L(j\omega_{180})| < 1$ with a large ω_{180} (high bandwidth).
At low frequencies the shape of $|L|$ depends on the disturbances/references (at least one integrator for each integrator in $r(s)$) $\boxed{\Rightarrow \text{ typically slope of -1 (-20dB/dec) around } \omega_c \text{ and larger roll-off at higher freq.}}$

Inverse-based Controller

$L(s)=\frac{\omega_c}{s}$, $K(s)=\frac{\omega_c}{s}G^{-1}(s)$ $L(s)$ must contain all RHP-zeros of $G(s)$

\+ PM=90° , GM= ∞ , integrator in the loop (steady state error for reference step → 0)
\- disturbance rejection critically , RHP zeros cannot be inverted , only for plants with pole excess < 2

Ex: System where $|G_d(j\omega)|>1$ up to $\omega\approx 10$ ⇒ chose $\omega_c=10$ (feedback control needed up to $\omega=10$, as low as possible due to noise sensitivity)

Loop Shaping for Disturbance Rejection

$e=y=SG_d d$ to achieve $|e(\omega)|\le 1$ for $|d(\omega)|=1$ (worst-case disturbance) we require
$|SG_d(j\omega)|<1 \quad \forall\omega \quad \Leftrightarrow \quad |1+L|\ge|G_d| \;\forall\omega \quad \approx \quad |L|\ge|G_d| \;\forall\omega$

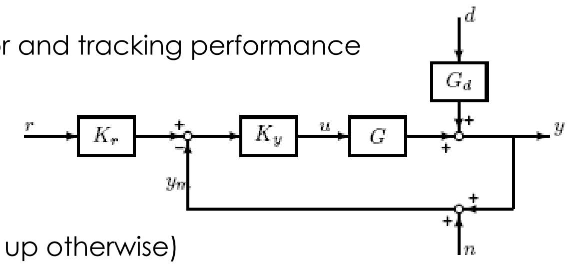
chose $|L_{min}|\approx|G_d| \quad \Rightarrow \quad |K_{min}|\approx|G^{-1}G_d| \quad \Rightarrow \quad |K|=\left|\frac{s+\omega_I}{s}\right||G^{-1}G_d|$ improves low-frequency performance by adding integrator or lead/lag → no steady state offset

Additional modifications: 1. slope of | L | = -1 for transient behaviour with acceptable gain and phase margins
2. increase loop gain at low frequencies to improve settling time and reduce offset → add integrator $(s+\omega_I)/s$ which is effective up to ω_I and should not add too much negative phase at ω_c
3. let L roll off faster at higher frequencies (beyond bandwidth) to reduce the use of manipulated inputs, make the controller realizable and reduce noise effects

Ex: $K=G^{-1}G_d=0.5(0.05s+1)^2 \overset{proper}{=} 0.5 \underset{\omega_I=0.2\omega_c}{\overset{2.}{\Rightarrow}} 0.5(s+2)/s \underset{Lead-lag}{\overset{3.}{\Rightarrow}} 0.5(s+2)/s\cdot(0.05s+1)/(0.005s+1)$
the $(0.05s+1)^2$ term only comes into effect at 1/0.05=20 which is beyond the desired ω_C=10

Two Degrees of Freedom Design

Use prefilter K_r (improves tracking) in order to meet both regulator and tracking performance



Design K_y , $L=GK_y$, $T=L(I+L)^{-1}$, $y=T_{ref}r$
↓ improv e perform. ↑ disturbance rejection ↑ desired TF

⇒ $K_r=T^{-1}T_{ref}$ practical choice $K_r(s)=\frac{\tau_{lead}s+1}{\tau_{lag}s+1}$
(slows response down if $\tau_{lead}<\tau_{lag}$, speeds it up otherwise)

Closed-loop shaping

$|L(j\omega)|\gg 1 \quad \Rightarrow \quad S\approx L^{-1}$, $T\approx 1$; $|L(j\omega)|\ll 1 \quad \Rightarrow \quad S\approx 1$, $T\approx L$ can't infer anything at ω_c

H_∞: the set of transfer functions with bounded ∞ -norm $\|f(s)\|_\infty=\max_\omega|f(j\omega)|=\lim_{p\to\infty}\left(\int_{-\infty}^{\infty}|f(j\omega)|^p d\omega\right)^{1/p}$
= set of stable and proper transfer functions

H_2: set of stable and strictly proper transfer functions: $\|f(s)\|_2=\sqrt{1/(2\pi)\int_{-\infty}^{\infty}|f(j\omega)|^2 d\omega}$

Weighted Sensitivity

Specifications: minimum bandwidth frequency ω_B^* , min. tracking error, shape of S, system type (max. steady-state tracking error A), max. peak magnitude of S: $\|S(j\omega)\|_\infty\le M$

upper bound: $|S(j\omega)|<1/|w_P(j\omega)| \;\forall\omega \quad \Leftrightarrow \quad |w_P S|<1 \;\forall\omega \quad \Leftrightarrow \quad \|w_P S\|_\infty<1$

Typical performance weight: $w_P(s)=\frac{s/M+\omega_B^*}{s+\omega_B^* A}$ steeper slope: $w_P(s)=\frac{(s/M^{1/2}+\omega_B^*)^2}{(s+\omega_B^* A^{1/2})^2}$

↑ 1/ | w_P | is equal to A ≤ 1 at low frequencies and M ≥ 1 at high frequencies
For this weight $L=\omega_B^*/s$ yields an S which exactly matches the bound

Performance Limitations in SISO Systems

Input-Output Controllability

How well can a plant be controlled? What control structure (which variables, etc.) should be used?

Keep the outputs within specified bounds displacements from their references in spite of unknown (bounded) variations like disturbances and plant changes/uncertainties using available inputs and measurements.

Assume scaling: $y(t) \in [r-1, r+1] \quad \forall \quad \{d(t) \in [-1,1] \ , \ r(t) \in [-R, R]\}$ using an input $u(t) \in [-1,1]$

$|e(\omega)| \leq 1 \quad \forall \quad \{|d(\omega)| \leq 1 \ , \ |r(\omega)| \leq R(\omega)\}$ using an input $|u(\omega)| \leq 1$

for simplicity we assume $R(\omega) = R \quad (\omega \leq \omega_r)$ and $R(\omega) = 0 \quad (\omega > \omega_r)$

because of $e = y - r = G u + G_d d - R \tilde{r}$ (see p.1) we can apply results for disturbances also to references by replacing G_d by $-R$

Perfect Control & Plant Inversion

$$y = G u + G_d d \ \underset{\text{control}}{\overset{\text{perfect}}{=}} \ r \quad \Rightarrow \quad \underbrace{u = G^{-1} r - G^{-1} G_d d}_{\text{perfect feedforward controller}} \ \overset{u = K(r-y)}{\Rightarrow} \ u = K S r - K S G_d d$$

because of $T = (1 + G K)^{-1} G K = G K (1 + G K)^{-1} = G K S$ we have $u = G^{-1} T r - G^{-1} T G_d d$

Where feedback is effective $(T \approx I)$ feedback input with feedback control is the same as perfect control input without feedback control.

Perfect control <u>cannot</u> be achieved if G contains RHP-zeros (G^{-1} unstable) or time delay (G^{-1} predictive), is uncertain or has more poles than zeros (G^{-1} unrealizable) or if $|G^{-1} G_d|$ or $|G^{-1} R|$ is large (>1).

Fundamental Limitations on Sensitivity

$S + T = 1 \quad \forall \omega$ ideally S small (small control error) and T small (no sensitivity to noise) → not possible

Bode Sensitivity Integral: (1st waterbed formula)

If the open-loop transfer function $L(s)$ is rational , has a relative degree of ≥ 2 and N_p RHP-poles p_i the closed-loop is stable if $\int_0^\infty \ln|S(j\omega)| d\omega = \pi \cdot \sum_{i=1}^{N_p} \Re(p_i)$

Idea: for some ω $\quad |L(j\omega) - (-1)| < 1 \quad \Leftrightarrow \quad |S| > 1$ → nichtminimalphasig

Stable plant $\Rightarrow \int_0^\infty \ln|S(j\omega)| d\omega = 0 \quad \Leftrightarrow$ 1 

A=B

Unstable plant $\Rightarrow$ $A < B$ (pay a price for stabilizing the system)

Weighted sensitivity integral: (2nd waterbed formula)

If $L(s)$ has a single real RHP-zero z and N_p RHP-poles p_i. The closed-loop is stable if $\int_0^\infty \ln|S(j\omega)| \cdot w(z,\omega) d\omega = \pi \cdot \prod_{i=1}^{N_p} \left| \frac{p_i + z}{p_i - z} \right|$ where the weight function

$w(z,\omega) = \frac{2z}{z^2 + \omega^2} = \frac{2}{z} \frac{1}{1 + (\omega/z)^2}$ "cuts off" contributions of $\ln|S|$ at $\omega > z$

if there are two complex conjugated zeros $(z = x \pm j y)$ use

$$w(z, \omega) = \frac{x}{x^2 + (y - \omega)^2} + \frac{x}{x^2 + (y + \omega)^2}$$

For a <u>stable</u> plant $\int_0^z \ln|S(j\omega)| d\omega \approx 0 \quad (|S| \approx 1 @ \omega > z)$

"when you reduce $|S|$ at low frequencies a large peak at high frequencies is unavoidable"

Interpolation Constraints

If p/z is a RHP-pole/zero of $L(s)$, then $T(p) = 1 \ , \ S(p) = 0 \ , \ T(z) = 0 \ , \ S(z) = 1$ peaks/valleys in 3D-plot

Fundamental Limitations: Bounds on Peaks

Maximum modulus principle: if $f(s)$ is stable (e.g. analytic in the complex RHP), then the maximum value of $|f(s)|$ for s in the RHP is attained on the region's boundary (i.e. somewhere along the $j\omega$-axis)

$\Rightarrow \quad f(s)$ stable $\quad \Leftrightarrow \quad \|f(j\omega)\|_\infty = \max_\omega |f(j\omega)| \geq |f(s_0)| \quad \forall s_0 \in$ RHP

$\Rightarrow \quad f(s) = w_P(s)S(s) = w_T(s)T(s)$ with weighted (compl.) sensitivity

Weighted sensitivity peak: If $G(s)$ has a RHP-zero z and $w_P(s)$ is any stable weight function, then for closed-loop stability we need $\|w_P S\|_\infty \geq |w_P(z)S(z)| = |w_P(z)|$

Weighted compl. sens. peak: If $G(s)$ has a RHP-pole p and $w_T(s)$ is any stable weight function, then for closed-loop stability we need $\|w_T T\|_\infty \geq |w_T(p)T(p)| = |w_T(p)|$

Combined RHP-poles & zeros: G(s) has N_z RHP-zeros z_j and N_p RHP-poles p_i. For closed-loop stability we

need $\|w_P S\|_\infty \geq c_{1j}|w_P(z_j)| \quad , \quad c_{1j} = \prod_{i=1}^{N_p} \frac{|z_j + \overline{p_i}|}{|z_j - p_i|} \geq 1$ for each zero z_j (1)

and $\|w_T T\|_\infty \geq c_{2i}|w_T(p_i)| \quad , \quad c_{2i} = \prod_{j=1}^{N_z} \frac{|\overline{z_j} + p_i|}{|z_j - p_i|} \geq 1$ for each pole p_i (2)

For $w_P = w_T = 1$ we get $\|S\|_\infty \geq \max_j c_{1j} \quad , \quad \|T\|_\infty \geq \max_i c_{2i}$

$\Rightarrow$ large peaks for S and T are unavoidable if RHP poles and zeros are close

Proof of (1): If there are RHP-zeros create an all-pass $S_a(s) = \prod_i \frac{s - p_i}{s + \overline{p_i}} \quad \Rightarrow \quad S = S_a S_m$

(2) similar, see p.183

where S_m is the "minimum-phase version" of S and $|S_a(j\omega)| = 1 \quad \forall \omega$

If there is a RHP-zero at z, we get from the maximum modulus principle $\|w_P S\|_\infty = \max_\omega |w_P S(j\omega)| = \max_\omega |w_P S_m(j\omega)| \geq |w_P(z) S_m(z)|$

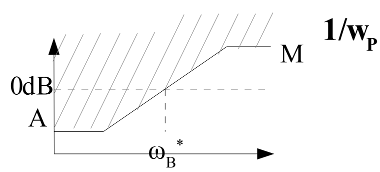
where $S_m(z) = S(z)S_a(z)^{-1} = 1/S_a(z) = c_1$

Bandwidth Limitation II

Performance requirement: $|S(j\omega)| < 1/|w_P(j\omega)| \quad \forall \omega \quad \Leftrightarrow \quad \|w_P S\|_\infty < 1$

from the weighted sensitivity peak theorem we have that $\|w_P S\|_\infty \geq |w_P(z)| \quad \Rightarrow \quad |w_P(z)| < 1$

for $w_P(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A}$ and a real zero at z we get $\omega_B^*(1 - A) < z\left(1 - \frac{1}{M}\right)$



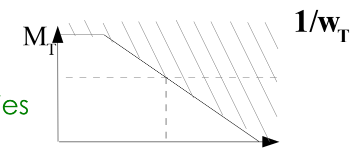
Bsp: $\begin{array}{l} A = 0 \\ M = 2 \end{array} \quad \Rightarrow \quad \omega_B^* < \frac{z}{2}$ $\qquad M = M_S = \|S\|_\infty$

Limitations Imposed by RHP-poles

Specification: $|T(j\omega)| < 1/|w_T(j\omega)| \quad \forall \omega \quad \Leftrightarrow \quad \|w_T T\|_\infty < 1$

from the weighted complementary sensitivity peak theorem we have $\|w_T T\|_\infty \geq |w_T(p)| \quad \Rightarrow \quad |w_T(p)| < 1$

for $w_T(s) = \frac{s}{\omega_{BT}^*} + \frac{1}{M_T}$ we get $\omega_{BT}^* > p\frac{M_T}{M_T - 1}$



↑ roll-off ≥ 1 at high frequencies (any real system) and |T| < M_T at low frequencies

Bsp: $M_T = 2 \quad \Rightarrow \quad \omega_{BT}^* > 2p$ which is approximately achieved by $\omega_c > 2p$

Combined RHP-poles and RHP-zeros

RHP-zero: $\omega_c < z/2$ RHP-pole: $\omega_c > 2p$

RHP-pole & RHP-zero: $z > 4\mathrm{p}$ for acceptable performance and robustness

Sensitivity peaks: with RHP-pole p and RHP-zero z we always have: $\|S\|_\infty \geq c$, $\|T\|_\infty \geq c$, $c = \dfrac{|z+p|}{|z-p|}$

<u>Ex</u>: Balancing a rod System with poles $\{0, 0, \sqrt{(M+m)\,g/(M\,l)}\}$ and zero $\sqrt{g/l}$
→ for m<<M pole~zero, this system is impossible to stabilize
→ for m>>M the system is difficult to stabilize because p>z (e.g. $m/M = 0.1 \Rightarrow \|S\|_\infty \geq 42$, $\|T\|_\infty \geq 42$
→ only small frequencies controllable, for large frequencies disturbances are amplified by ~40)

Limitations Imposed by Input Constraints

Perfect control (e=0) for input $u = G^{-1} r - G^{-1} G_d d$

Disturbance Rejection ($r=0, |d(\omega)|=1, |u(\omega)|<1$) implies $|G^{-1}(j\omega) G_d(j\omega)| < 1 \;\; \forall \omega \;\; \Leftrightarrow \;\; |G_d| < |G|$

Command tracking ($d=0, |r(\omega)|=R \;\; \forall \omega < \omega_r, |u(\omega)|<1$) implies $|G^{-1}(j\omega)\, R| < 1 \;\; \forall \omega \leq \omega_r \;\; \Leftrightarrow \;\; |G| > R$

For <u>acceptable control</u> ($|e(j\omega)|<1$) these requirements change to
$$\begin{aligned} &|G| > |G_d| - 1 && \forall \omega, |G_d(\omega)| > 1 \\ &|G| > |R| - 1 < 1 && \forall \omega \leq \omega_r \end{aligned} \qquad (*)$$

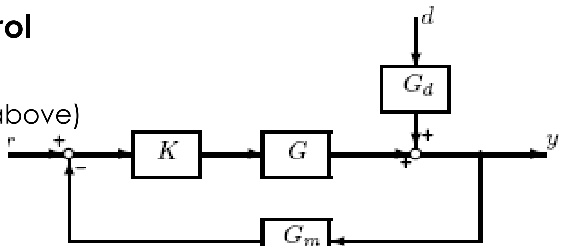
Summary: Controllability Analysis with Feedback Control

$y = G u + G_d d$, $y_m = G_m y$

ω_c = gain crossover frequency (where $|L(j\omega)|$ crosses 1 from above)

ω_d = frequency where $|G_d(j\omega_d)|$ crosses 1 from above

$G_m(0) = 1$ (perfect steady-state measurement)



(1) Speed of response to reject disturbances: We require $\omega_c > \omega_d \;\; \Rightarrow \;\; |S(j\omega)| \leq |1/G_d(j\omega)| \;\; \forall \omega$

(2) Speed of response to track reference changes: We require $|S(j\omega)| \leq 1/R$ up to ω_r (where tracking is required)

(3,4) Input constraints arising from dist. & setpoints: see $(*)$

(5) Time delay θ in $G(s) G_m(s)$: We approximately require $\omega_c < 1/\theta$

(6) Tight control at low frequencies with RHP-zero: We require $\omega_c < z/2$ for a real RHP-zero in $G(s) G_m(s)$

(7) Phase lag constraint (most practical cases, e.g. PID): We require usually $\omega_c < \omega_u$ ($arg[G G_m(j\omega_u)] = -180°$)

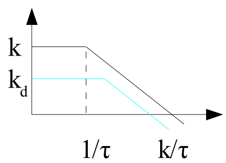
(8) Real open-loop unstable pole in G(s) at s=p: We need hight feedback gains and $\omega_c > 2\mathrm{p}$
(for unstable plants we need $|G| > |G_d|$ up to frequency p)

N=-1 ⇒ -90° phase = 90° PM at ω_c
at $\omega = 1/\theta$ and $\omega = z/2$ we have -55° phase ⇒ 35° PM

Applications of Controllability Analysis: First-Order Delay Process

Problem statement: $G(s) = k \dfrac{e^{-\theta s}}{1 + \tau s}$, $G_d(s) = k_d \dfrac{e^{-\theta_d s}}{1 + \tau_d s}$, $|k_d| > 1$ + measurement delays θ_m, θ_{md}

Specification: $|e| < 1$ for $|u| < 1, |d| < 1$:
$$\begin{aligned} |u| < 1 &\overset{(3)}{\Rightarrow} k > k_d \;\; , \;\; k/\tau > k_d/\tau_d \\ |e| < 1 &\overset{(1)}{\Rightarrow} \omega_d \approx k_d/\tau_d < \omega_c \\ &\overset{(5)}{\Rightarrow} \omega_c < 1/\theta_{tot} \quad \underbrace{(\theta_{tot} = \theta + \theta_m)}_{\text{total loop delay}} \end{aligned}$$



⇒ Feedback: $\theta + \theta_m < \tau_d / k_d$ Feedforward: $\theta + \theta_{md} - \theta_d < \tau_d / k_d$

Uncertainty and Robustness for SISO Systems

Introduction to Robustness

A control system is robust if it is insensitive to differences between the actual system and the model

Approach:
1. Determine uncertainty set (mathematical representation)
2. Check robust stability (RS) → does the system remain stable for all plants in the uncert. set?
3. Check robust performance (RP) → if RS is satisfied, check whether perform. specs are met

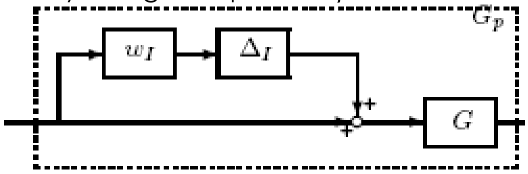
Notation: Π = uncertainty set $\quad G(s)\in\Pi$ = nominal plant model
$\{G_p(s), G'(s)\}\in\Pi$ = particular perturbed plant models

Classes of Uncertainty

Parametric uncertainty: Structure of the model & order known, some parameters unknown

Neglected/unmodeled dynamics uncertainty: Missing dynamics (usually at high frequencies)

Lumped uncertainty: Sources of parametric and/or unmodeled dynamics uncertainty combined into a single lumped structure

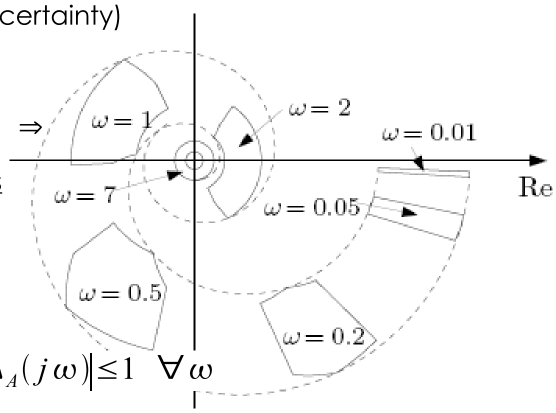


Multiplicative uncertainty Π_I: $G_p(s)=G(s)(1+w_I(s)\Delta_I(s))$
where $|\Delta_I(j\omega)|\le 1 \quad \forall\omega \quad \Leftrightarrow \quad \|\Delta_I\|_\infty \le 1$ ($\Delta_I(s)$ is <u>any</u> stable transfer function $|\Delta_I(j\omega)|\le 1 \quad \forall\omega$)

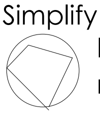
Inverse multiplicative uncertainty Π_{iI}: $G_p(s)=G(s)(1+w_{iI}(s)\Delta_{iI}(s))^{-1}$, $|\Delta_{iI}(j\omega)|\le 1 \quad \forall\omega$
(better suited for pole uncertainty)

Representing Uncertainty in the Frequency Domain

<u>Ex</u>: $G_p(s)=k/(\tau s+1)e^{-\theta s}$ $\qquad 2\le\{k,\theta,\tau\}\le 3 \quad \Rightarrow$



Simplify graph by replacing unshaped <u>uncertainty regions</u> by disc approximations where uncerties have a maximum length (radius) $|w_A(j\omega)|$ but arbitrary phases. → conservative approach since we now also consider perturbations that will never occurr.



Additive Uncertainty: Π_A: $\quad G_p(s)=G(s)+w_A(s)\Delta_A(s) \qquad |\Delta_A(j\omega)|\le 1 \quad \forall\omega$

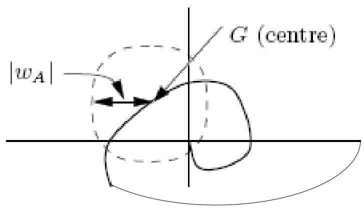
where Δ_A is <u>any</u> stable transfer function which at each frequency is no larger than one in magnitude

Alternative: multiplicative uncertainty Π_I: $\quad G_p(s)=G(s)(1+w_I(s)\Delta_I(s)) \qquad |\Delta_I(j\omega)|\le 1 \quad \forall\omega$

Π_A and Π_I are equivalent if $|w_I(j\omega)|=|w_A(j\omega)|/|G(j\omega)| \quad \Rightarrow \quad |G w_I|=|w_A|$

the dashed circle is not allowed to enclose (0,0) because in that case <u>any</u> phase would be possible for $G(s)=0$ which would be too uncertain

$|w_A|>G \quad \Rightarrow \quad |w_I|>1$



Obtaining the Weight for Complex Uncertainty

Select a nominal model $G(s)$

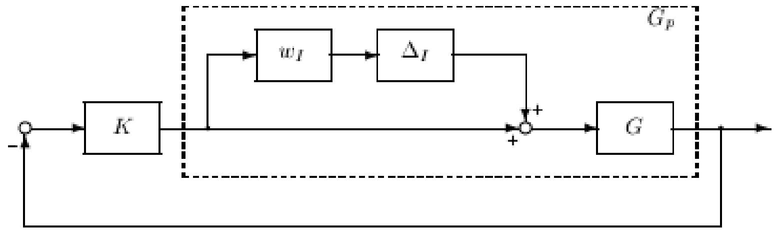
Multiplicative (relative) uncertainty: $|w_I(j\omega)|\ge l_I(\omega)=\max\limits_{G_p\in\Pi}\left|\dfrac{G_p(j\omega)-G(j\omega)}{G(j\omega)}\right|$

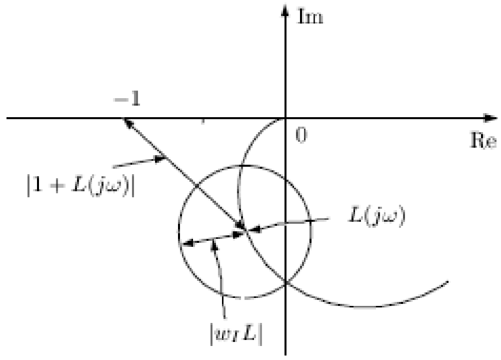
Additive uncertainty: $|w_A(j\omega)|\ge l_A(\omega)=\max\limits_{G_p\in\Pi}|G_p(j\omega)-G(j\omega)|$

Unmodelled dynamics: $w_I(s)=\dfrac{\tau s+r_0}{(\tau/r_\infty)s+1}$ where r_0 = magnitude at low frequencies
r_∞ = magnitude at high frequencies

and $1/\tau$ is the frequency where the relative uncertainty reaches 100%

SISO Robust Stability (RS) with Multiplicative Uncertainty





$L_p = G_p K = GK(1+w_I \Delta_I) = L + w_I L \Delta_I \quad |\Delta_I(j\omega)| \le 1 \quad \forall\omega$

Encirclements of -1 are avoided if <u>none</u> of the discs cover -1

$RS \quad \Leftrightarrow \quad |w_I L| < |1+L| \quad \forall\omega \quad \Leftrightarrow \quad \left|\frac{w_I L}{1+L}\right| < 1 \quad \forall\omega \quad \Leftrightarrow \quad |w_I T| < 1 \quad \forall\omega \quad \overset{def}{\Leftrightarrow} \quad \|w_I T\|_\infty < 1$

$\Leftrightarrow \quad |T| < 1/|w_I| \quad \forall\omega$ limits bandwidth

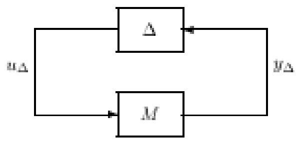
same setup as above but direction of loop changed

Inverse mult. uncertainty: $RS \quad \Leftrightarrow \quad |1+L_p| > 0 \quad \forall L_p, \omega \quad \Leftrightarrow \quad |1+L(1+w_{iI}\Delta_{iI})^{-1}| > 0 \quad \forall\omega, |\Delta_{iI}| \le 1$

$\Leftrightarrow \quad |1+w_{iI}\Delta_{iI}+L| > 0 \, \forall\omega, |\Delta_{iI}| \le 1 \quad \overset{worst\_case}{\Leftrightarrow} \quad |1+L| - |w_{iI}| > 0 \quad \forall\omega \quad \Leftrightarrow \quad |S| < 1/|w_{iI}|$

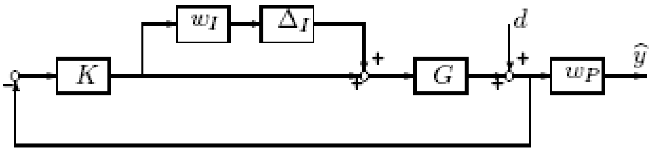
MΔ-Structure: $M = w_I K(1+GK)^{-1} G = w_I T \quad \Rightarrow \quad RS \quad \Leftrightarrow \quad |1+M\Delta| > 0 \quad \forall\omega, |\Delta| \le 1$

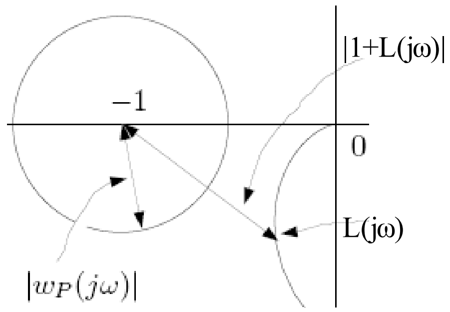
$\overset{worst\_case}{\Rightarrow} \quad RS \quad \Leftrightarrow \quad 1-|M| > 0 \quad \forall\omega \quad \Leftrightarrow \quad |M(j\omega)| < 1 \quad \forall\omega$



SISO Robust Performance (RP) with Multiplicative Uncertainty

Nominal performance: $NP \quad \Leftrightarrow \quad |w_P S| < 1 \quad \forall\omega \quad \Leftrightarrow \quad |w_P| < |1+L| \quad \forall\omega$





the NP-condition must be satisfied for <u>all</u> possible plants:

$NP \quad \overset{def}{\Leftrightarrow} \quad |w_P S| < 1 \quad \forall\omega \quad \Leftrightarrow \quad |w_P| < |1+L| \quad \forall\omega$ " $L(j\omega)$ must be at least $|w_P|$ from -1 "

$RP \quad \overset{def}{\Leftrightarrow} \quad |w_P S_p| < 1 \quad \forall\{S_p, \omega\} \quad \Leftrightarrow \quad |w_P| < |1+L_p| \quad \forall\{L_p, \omega\} \qquad (\Leftrightarrow \quad |\hat{y}/d| < 1 \quad \forall\Delta_I)$

Set of possible loop transfer functions: $L_p = G_p K = L(1+w_I\Delta_I) = L + w_I L \Delta_I$

$RP \quad \overset{graphical}{\underset{derivation}{\Leftrightarrow}} \quad |w_P| + |w_I L| < |1+L| \quad \forall\omega \quad \Leftrightarrow \quad |w_P(1+L)^{-1}| + |w_I L(1+L)^{-1}| < 1 \quad \forall\omega \quad \Leftrightarrow \quad \max_\omega (|w_P S| + |w_I T|) < 1$

The Relationship Between NP, RS and RP

$NP \quad \Leftrightarrow \quad |w_P S| < 1 \quad \forall\omega \quad \Leftrightarrow \quad |w_P| < |1+L| \quad \forall\omega$

$RS \quad \Leftrightarrow \quad |w_I T| < 1 \quad \forall\omega$

$RP \quad \Leftrightarrow \quad |w_P S| + |w_I T| < 1 \quad \forall\omega$

- A prerequisite for RP is that we satisfy NP and RS (applies for SISO and MIMO and for any uncertainty)
- For SISO systems, if we satisfy RS and NP we have at each frequency $|w_P S| + |w_I T| \le 2\max\{|w_P S|, |w_I T|\} < 2$
  $\Rightarrow$ within a factor of at most 2 we will automatically get RP when NP and RS are satisfied
- $|w_P S| + |w_I T| \ge \min\{|w_P|, |w_I|\} \quad \Rightarrow$ we cannot have both $|w_P| > 1$ (good performance) and $|w_I| > 1$ (more than 100% uncertainty) at the same frequency because where $|w_I| > 1$ the uncertainty allows RHP-zeros (from Δ which show up in G_p) which results in bad perf.

Elements of Linear System Theory

System Descriptions

State Space Representation

$$\dot{x}(t)=A\,x(t)+B\,u(t)+B_d\,d(t) \qquad \overset{\tilde{x}=S\,x}{\Rightarrow} \qquad \tilde{A}=S\,A\,S^{-1} \quad \tilde{B}=S\,B$$
$$y(t)=C\,x(t)+D\,u(t)+D_d\,d(t)+n(t) \qquad \tilde{C}=C\,S^{-1} \quad \tilde{D}=D$$

with disturbances d and measurement noise n

Dynamical system response: $x(t)=e^{A(t-t_0)}x(t_0)+\int_{t_0}^{t}e^{A(t-\tau)}B\,u(\tau)\,d\tau$ where $e^{At}=I+\sum_{k=1}^{\infty}(At)^k/k!$

$\tilde{A}=S\,A\,S^{-1}=\Lambda=diag\{\lambda_i\} \Rightarrow e^{\Lambda t}=S^{-1}diag\{e^{\lambda_i(A)t}\}S$ where $e^{\lambda_i t}$ is the mode associated w/ the EV $\lambda_i(A)$

Impulse response: $g(t)=\underset{\text{output} \uparrow\uparrow \text{input}}{\begin{bmatrix}\ddots & \ddots\\ \ddots & g_{ij}(t)\end{bmatrix}}=C\,e^{At}B+D\,\delta(t) \quad t\geq 0 \quad \Rightarrow \quad y(t)=g*u=\int_0^t g(t-\tau)u(\tau)\,d\tau$

Transfer Function Representation

$y(s)=G(s)u(s) \qquad G(s)=C(sI-A)^{-1}B+D \;=\; (C\,adj(sI-A)B+D\,det(sI-A))/det(sI-A)$

where $det(sI-A)=\prod_{i=1}^{n}\lambda_i(sI-A)=\prod_{i=1}^{n}(s-\lambda_i(A))$

PID-Controller

ideal controller $K(s)=K_c(1+(\tau_I s)^{-1}+\tau_D s)$ not realizable because of differentiator → improper

proper PID controller: $K(s)=K_c\big(1+1/(\tau_I s)+\tau_D s/(1+\epsilon\,\tau_D s)\big) \quad \epsilon\leq 0.1$

with common realizations

1. Jordan canonical form
2. Observability canonical form
3. Controllability canonical form
4. Observer canonical form

1. $\{A,B,C,D\}=\left\{\begin{bmatrix}0 & 0\\ 0 & -1/(\epsilon\tau_D)\end{bmatrix},\begin{bmatrix}K_c/\tau_I\\ K_c/(\epsilon^2\tau_D)\end{bmatrix},[1\quad -1],K_c\frac{1+\epsilon}{\epsilon}\right\}$

2. $\{A,B,C,D\}=\left\{\begin{bmatrix}0 & 0\\ 0 & -1/(\epsilon\tau_D)\end{bmatrix},\begin{bmatrix}K_c(\tau_I^{-1}-1/(\epsilon^2\tau_D))\\ K_c/(\epsilon^3\tau_D^2)\end{bmatrix},[1\quad 0],K_c\frac{1+\epsilon}{\epsilon}\right\}$

3. $\{A,B,C,D\}=\left\{\begin{bmatrix}0 & 0\\ 1 & -1/(\epsilon\tau_D)\end{bmatrix},\begin{bmatrix}1\\ 0\end{bmatrix},\left[K_c(\tau_I^{-1}-\frac{1}{\epsilon^2\tau_D})\quad \frac{K_c}{\epsilon^3\tau_D^2}\right],K_c\frac{1+\epsilon}{\epsilon}\right\}$

4. $\{A,B,C,D\}=\left\{\begin{bmatrix}-1/(\epsilon\tau_D) & 1\\ 0 & 0\end{bmatrix},\begin{bmatrix}K_c/(\epsilon\tau_I\tau_D)\\ K_c(\epsilon^2\tau_D-\tau_I)/(\epsilon^2\tau_I\tau_D)\end{bmatrix},[1\quad 0],K_c\frac{1+\epsilon}{\epsilon}\right\}$

State Controllability and State Observability

If $C=[B\quad AB\quad A^2B\quad \dots\quad A^{n-1}B]$ has rank n (n=#states) the system is controllable $\qquad$ W_C needs to be invertible

$u(t)=-B^T e^{A^T(t_1-t)}W_C^{-1}(t_1)(e^{At_1}x_0-x_1) \qquad W_C(t)=\int_0^t \underbrace{e^{A\tau}B}_{\Omega}\underbrace{B^Te^{A^T\tau}}_{\Omega^T}d\tau$ brings x from x_0 to $x_1=x(t)$

$P=W_c(\infty) \quad \Leftrightarrow \quad AP+PA^T=-BB^T \quad \Rightarrow$ controllable iff $P>0$

A has eigenvalues p_i and left eigenvectors q_i $(q_i^H A=p_i q_i^H) \Rightarrow$ controllable iff $q_i^H B\neq 0 \quad \forall i$

→ controllability says nothing about if a final state is stable or if the necessary inputs are achievable

If $O=[C\quad CA\quad \dots\quad CA^{n-1}]^T$ has rank n the system is observable

stable system → $Q=\int_0^{\infty}e^{A^T\tau}C^TCe^{A\tau}d\tau$ must have full rank $\qquad A^TQ+QA=-C^TC$

A has eigenvalues p_i and eigenvectors t_i $(At_i=p_it_i) \Rightarrow$ observable iff $Ct_i\neq 0 \quad \forall i$

→ system might not be observable if obtaining x(0) requires high-order derivatives (sensitive to noise)

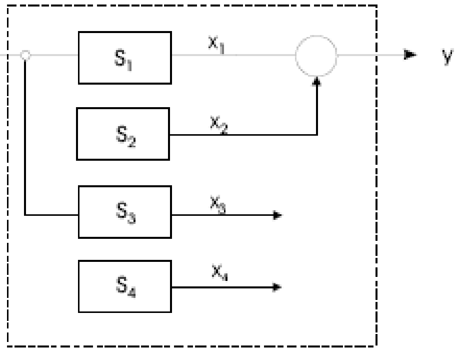
Minimal realization = state-space realization is state controllable and state observable

Kalman's Decomposition

coordinate transformation into
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\\ \dot{x}_4\end{bmatrix}=\begin{bmatrix}A_{11} & A_{12} & 0 & 0\\ 0 & A_{22} & 0 & 0\\ A_{31} & A_{32} & A_{33} & A_{34}\\ 0 & A_{42} & 0 & A_{44}\end{bmatrix}\cdot\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix}+\begin{bmatrix}B_1\\ 0\\ B_3\\ 0\end{bmatrix}u$$

S_1=minimal realization (controll- & observable)

$$y=\begin{bmatrix}C_1 & C_2 & 0 & 0\end{bmatrix}\cdot\begin{bmatrix}x_1 & x_2 & x_3 & x_4\end{bmatrix}^T$$



Stability

any unstable linear system can be stabilized by feedback control if it contains no hidden unstable modes.

A system is (internally) stable if none of its components contains hidden unstable modes and if it is BIBO-stable
A system is state stabilizable if all unstable modes are state controllable.
A system is state detectable if all unstable modes are state observable.
A system with unstabilizable/undetectable modes is said to contain hidden unstable modes.

Poles

The poles p_i of a system are the eigenvalues $\lambda_i(A)$

The pole or characteristic polynomial is defined as $\phi(s)=det(sI-A)=\prod_{i=1}^{n}(s-p_i)$

A linear dynamic system is stable iff all poles are in the open LHP $\Leftrightarrow$ $\Re\{\lambda_i(A)\}<0\ \ \forall i$ $\Leftrightarrow$ A is Hurwitz

$\phi(s)$ corresponding to a minimal realization of a system $G(s)$ is the least common denominator of all non-identically-zero minors of all orders of $G(s)$ (MacFarlane and Karcanias Theorem)

Minor of order $n=det(G_{n\times n}(s))$ ...of order $m=det(G^i_{m\times m})$ where $G^i=G$ with columns/rows removed $(m<n)$

Ex: $G(s)=\frac{1}{1.25\underbrace{(s+1)(s+2)}_{\phi(s)}}\begin{bmatrix}s-1 & s\\ -6 & s-2\end{bmatrix}$ $\Rightarrow$ minors of order 1 = $\{s-1, s, -6, s-2\}/(1.25\phi(s))$; minor of order 2 = det G(s) = $1/(1.25^2\phi(s))$

Zeros

Zeros are values of s at which $G(s)$ loses rank ($G(z_i)=0$ only valid for SISO systems)

from SS realizations: $u=u_z e^{zt}$, $x=x_z e^{zt}$, $y=0$ $\Rightarrow$ $\begin{bmatrix}zI-A & -B\\ C & D\end{bmatrix}\cdot\begin{bmatrix}x_z\\ u_z\end{bmatrix}=0$ zeros=solutions of det[..]=0
MATLAB: tzero(A,B,C,D)

from TF: $rank(G(z_i))<rank(G(s))$ $\Rightarrow$ $z(s)=\prod_{i=1}^{n_z}(s-z_i)$ multivariable zeros have no relationship with the zeros of the transfer function elements

z(s) corresponding to a minimal realization of the system is the greatest common divisor of all the numerators of all order-r minors of G(s) where r = rank (G(s)), provided that these minors have been adjusted in such a way as to have the pole polynomial Φ(s) as their denominators

$$det(G)=\frac{\alpha(z)}{\phi(z)}\overset{!}{=}0$$

Zero Directions / Pole Directions

Locations of zeros/poles depend on I/O-scaling, directions not

$\exists z,\ rank(G(z))<rank(G(s))$ $\Rightarrow$ $\exists\{u_z, y_z\}\neq\vec{0},$ $\begin{cases}G(z)u_z=0 & \text{with input zero direction } u_z\\ y_z^H G(z)=0 & \text{with output zero direction } y_z\end{cases}$

$G(z)=U\Sigma V^H=[u_1\ \ \dots\ \ u_k]\cdot diag(\sigma_i)\cdot[v_1\ \ \dots\ \ v_n]^T$

u_z=last column of V	y_z=last column of U
u_p=first column of V	y_p=first column of U

If G(s) has a pole at s=p, then $G(p)u_p=\infty$, $y_p^H G(p)=\infty$ where $\{u_p, y_p\}$ are input/output pole direct.

Ex: G has zero $z=4$ and pole $p=-2$

$$G(z)=G(4)=\frac{1}{6}\begin{bmatrix}3 & 4\\ 4.5 & 6\end{bmatrix}=\frac{1}{6}\begin{bmatrix}0.55 & \underbrace{\begin{matrix}-0.83\\ 0.55\end{matrix}}_{y_z}\\ 0.83 & \end{bmatrix}\cdot\begin{bmatrix}9.01 & 0\\ 0 & 0\end{bmatrix}\cdot\begin{bmatrix}0.6 & \underbrace{\begin{matrix}-0.8\\ 0.6\end{matrix}}_{u_z}\\ 0.8 & \end{bmatrix}^H \quad \rightarrow 0$$

$$G(p+\epsilon)=G(-2+\epsilon)=\frac{1}{\epsilon^2}\begin{bmatrix}-3+\epsilon & 4\\ 4.5 & 2(-3+\epsilon)\end{bmatrix}=\frac{1}{\epsilon^2}\begin{bmatrix}\underbrace{\begin{matrix}-0.55\\ 0.83\end{matrix}}_{y_p} & \begin{matrix}-0.83\\ -0.55\end{matrix}\end{bmatrix}\cdot\begin{bmatrix}9.01 & 0\\ 0 & 0\end{bmatrix}\cdot\begin{bmatrix}\underbrace{\begin{matrix}0.6\\ -0.8\end{matrix}}_{u_p} & \begin{matrix}-0.8\\ -0.6\end{matrix}\end{bmatrix}^H \quad \rightarrow\infty$$

Pole Directions from State-Space Realizations

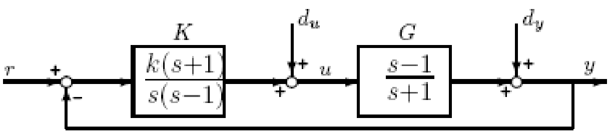
right/left eigenvectors: $At_p = pt_p \;,\; q_p^H A = pq_p^H \;\Rightarrow$ pole directions $y_p = Ct_p \;,\; u_p = B^H q_p$

Remarks on Poles and Zeros

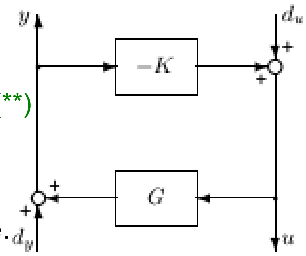
1. For square systems the poles/zeros of G(s) are essentially the p/z of det G(s) (only without cancellations!)
2. Poles/zeros at the same point but with other directions do not cancel or otherwise interact
3. There are no zeros if the outputs contain direct information about all states (from y we can obtain x) zeros usually appear when there are fewer inputs or outputs than states
4. Feedback control (G/(I+KG) moves poles, feedforward control (GK) can cancel poles, parallel compensation (G+K) cannot affect the poles in G
5. With feedback, the zeros of G/(I+KG) are the zeros of G plus the poles of K, feedforward can cancel zeros, but no RHP-zeros due to internal stability. Parallel compensation (G+K) can move zeros using extra input

Ex: $G(s) = \dfrac{z(s)}{\phi(s)} \;\Rightarrow\; T(s) = \dfrac{k\,z(s)}{\phi(s) + k\,z(s)} = k\dfrac{z_{cl}(s)}{\phi_{cl}(s)}$ zero locations are unchanged, pole locations are changed by feedback

Internal Stability of Feedback Systems



$$u = (I + KG)^{-1} d_u - K(I + GK)^{-1} d_y$$
$$y = G(I + KG)^{-1} d_u + (I + GK)^{-1} d_y$$
(*) (**)



A feedback system (**) is internally stable iff all four closed-loop transfer matrices are stable.

If there are no RHP pole-zero cancellations between $G(s)$ and $K(s)$ (that is all RHP-poles in G and K are contained in the minimal realizations of GK and KG) the feedback system is internally stable iff one of the four closed-loop transfer function matrices (*) is stable

Implications: 1. If $G(s)$ has a RHP-zero at z, then $L = GK \;,\; T = GK(I + GK)^{-1} \;,\; SG = (I + GK)^{-1} G \;,$ $L_I = KG \;,\; T_I = KG(I + KG)^{-1}$ will each have a RHP-zero at z.

2. If $G(s)$ has a RHP-pole at p, then L and L_I also have a RHP-pole at p, while $S = (I + GK)^{-1} \;,\; KS = K(I + GK)^{-1}$ and $S_I = (I + KG)^{-1}$ have a RHP-zero at p.

3. Perfect control implies $S \approx 0 \;,\; T \approx 1$ which is impossible for RHP-zeros and possible for RHP-poles (they cause problems when tight (high gain) control is not possible)

RHP pole-zero cancellations between two transfer functions (e.g. L and S=1/(I+L)) does not necessarily imply internal instability. RHP p/z-cancellations are only not allowed between separate physical components.

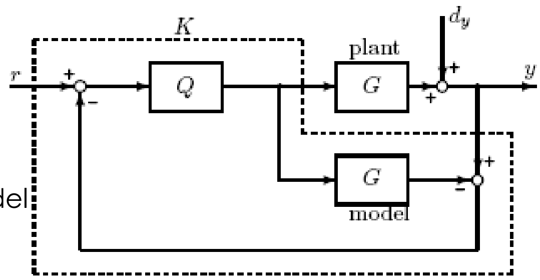
Stabilizing Controllers

For a stable plant $G(s)$ the negative feedback system (**) is internally stable iff $Q = K(I + GK)^{-1}$ is stable.
$\Rightarrow \;(I + GK)^{-1} = I - GQ \;,\; (I + KG)^{-1} = I - QG \;,\; G(I + KG)^{-1} = G(I - QG)$ are also stable if G is stable.

All stabilizing negative feedback controllers for the stable plant $G(s)$ are given by
$K = (I - QG)^{-1} Q = Q(I - GQ)^{-1}$ where Q is any stable transfer function matrix.

Although Q may generate unstable controllers K there is no danger of getting a RHP-pole in K that cancels a RHP-zero in G

Internal Model Control (IMC) parametrization: → →
the controller Q can be designed in an open-loop fashion since the feedback signal only contains information about the difference between the actual output and the output predicted from the model



Stability Analysis in the Frequency Domain

$L(s)=C_{ol}(sI-A_{ol})^{-1}B_{ol}+D_{ol}$ where the poles of $L(s)$ are the roots of $\phi_{ol}(s)=det(sI-A_{ol})$

$SS_{ol} \Rightarrow \phi_{cl}=det(sI-A_{cl})=det(sI-A_{ol}+B_{ol}(I+D_{ol})^{-1}C_{ol})$

$det(I+L(s))=det(I+C_{ol}(sI-A_{ol})^{-1}B_{ol}+D_{ol}) \overset{Schur's}{\underset{formula}{\Rightarrow}} det(I+L(s))=\frac{\phi_{cl}(s)}{\phi_{ol}(s)}\cdot\underbrace{det(I+D_{ol})}_{const.}$ calculations see 5.37-38

If there are no cancellations between ϕ_{ol} and ϕ_{cl} the closed-loop poles are solutions of $det(I+L(s))=0$.

MIMO Nyquist Stability Criteria

proof on p. 161/2

P_{ol} = #unstable poles in $L(s)$. The closed-loop system with loop transfer function $L(s)$ and negative feedback is stable iff the Nyquist plot of $det(I+L(s))$ (*the image of det(I+L(s)) as s goes clockwise around the Nyquist D-contour*) makes P_{ol} anti-clockwise encirclements of the origin and does not pass through (0/0).

If this is not satisfied there are $P_{cl}=N+P_{ol}$ unstable closed-loop poles where N =#clockwise encirclements.

Small Gain Theorem

small gain $\rightarrow$ stable BUT small gain not given $\not\rightarrow$ unstable

Consider a system with a stable loop transfer function $L(s)$. Then the closed-loop system is stable if the spectral radius $\rho(L(j\omega))=\max_i|\lambda_i(L(j\omega))|<1 \quad \forall\omega$ (system gain <1 in all directions) proof 5.40

For SISO-Systems $\rho(L(j\omega))=|L(j\omega)|$

The closed-loop system with stable loop transfer function $L(s)$ is stable if $\|L(j\omega)\|<1 \quad \forall\omega$ small gain
where $\|L\|$ denotes any matrix norm satisfying $\|AB\|\le\|A\|\cdot\|B\|$, for example $\bar{\sigma}(L)$ $(|\lambda|\le\|L\|)$

This theorem is usually conservative because phase information is not considered.

System Norms

the H_2-norm is only finite for **strictly**-proper systems

Given information about the allowed input signals $w(t)$, how large can the outputs $z(t)$ become?

2-norm: $\|z(t)\|_2=\sqrt{\sum_i\int_{-\infty}^{\infty}|z_i(\tau)|^2 d\tau}$ *not an induced norm that satisfies the multiplicative property*

H_2 norm: $\|G(s)\|_2=\sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}tr(G(j\omega)^H G(j\omega))d\omega}=\|g\|_2=\sqrt{\int_0^{\infty}tr(g^T(\tau)g(\tau))d\tau}=\sqrt{\sum_{ij}\int_0^{\infty}|g_{ij}(\tau)|^2 d\tau}$

$G(s)=C(sI-A)^{-1}B \Rightarrow \|G(s)\|_2=\sqrt{tr(B^T Q B)} = \sqrt{tr(C P C^T)}$

where Q is the observability Gramian and P is the controllability Gramian as calculated on p.9

Other notations/interpretations:

$\|G(s)\|_2=\sqrt{\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_i\sigma_i^2(G(j\omega))d\omega}$, $\|G(s)\|_2=\max_{w(t)\ unit\ impulses}\|z(t)\|_2$, $\|G(s)\|_2^2=\sqrt{\sum_{i=1}^{m}\|z_i(t)\|_2^2}$

H_∞ norm: $\|G(s)\|_\infty=\max_\omega\bar{\sigma}(G(j\omega))$

Performance interpretations: 1. The peak of the transfer function “magnitude”
2. Worst-case steady-state gain for sinusoidal inputs at any freq.
3. Induced (worst-case) 2-norm in the time domain

$$\|G(s)\|_\infty=\max_{w(t)\neq 0}(\|z(t)\|_2/\|w(t)\|_2)=\max_{\|w(t)\|_2=1}\|z(t)\|_2$$

the H_∞ norm is the smallest value of γ such that the Hamiltonian matrix H has no eigenvalues on the imaginary axis, where $H=\begin{bmatrix} A+BR^{-1}D^TC & BR^{-1}B^T \\ -C^T(I+DR^{-1}D^T)C & -(A+BR^{-1}D^TC)^T \end{bmatrix}$, $R=\gamma^2 I-D^T D$

Optimizing performance: minimize H_∞ : push down peak of largest singular value (“worst direction, worst frequency”)
minimize H_2 : push down whole thing (“average direction, average frequency”)

Introduction to Multivariable Control

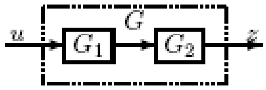
Transfer Functions for MIMO Systems

MIMO rule: Start from output and write down the blocks as you meet them when moving backwards (against the signal flow) taking the most direct path towards the input. If you exit from a feedback loop include a term $(I-L)^{-1}$ (or $(I+L)^{-1}$ for negative feedback) where L is the transfer function around that loop (evaluated against signal flow starting at the point of exit from the loop).

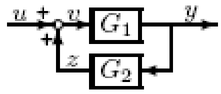
Cascade rule (a): $G=G_2 G_1$

Feedback rule (b): $v=(I-L)^{-1}u$, $L=G_2 G_1$

Push-through rule: $G_1(I-G_2 G_1)^{-1}=(I-G_1 G_2)^{-1} G_1$



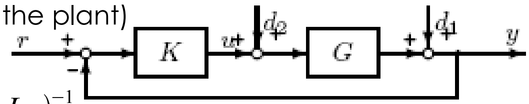
(a) Cascade system



(b) Positive feedback system

Negative Feedback Control Systems

Output transfer function: $L_O=G K$ (breaking the loop at the output of the plant)



Output sensitivity: $S_O=(I+L_O)^{-1}$

Output complementary sensitivity: $T_O=I-S_O=(I+L_O)^{-1} L_O=L_O(I+L_O)^{-1}$

Input: $L_I=K G$ (r → u)

Input sensitivity: $S_I=(I+L_I)^{-1}$

Input complementary sensitivity: $T_I=I-S_I=L_I(I+L_I)^{-1}$

Relationships: $T=L(I+L)^{-1}=(I+L^{-1})^{-1}=(I+L)^{-1}L$ $\quad (I+L)^{-1}+(I+L)^{-1}L=S+T=I$

$G K(I+G K)^{-1}=G(I+K G)^{-1}K=(I+G K)^{-1}G K$ $\quad G(I+K G)^{-1}=(I+G K)^{-1}G$

Multivariable Frequency Response Analysis

Sinusoidal input to channel j: $d_j(t)=d_{j0}\sin(\omega t+\alpha_j)$ starting at $t=-\infty$ → output $y_i(t)=y_{i0}\sin(\omega t+\beta_i)$

Phasor notation: $d_j(\omega)=d_{j0}e^{j\alpha_j}$, $y_i(\omega)=y_{i0}e^{j\beta_i}$ $\Rightarrow$ $y(\omega)=G(j\omega)d(\omega)$

Amplification (gain): $y_{i0}/d_{j0}=|g_{ij}(j\omega)|$ Phase shift: $\beta_i-\alpha_j=\arg g_{ij}(j\omega)$

Directions in Multivariable Systems

The gain of $G(s)$ is $\dfrac{\|y(\omega)\|_2}{\|d(\omega)\|_2}=\dfrac{\|G(j\omega)d(\omega)\|_2}{\|d(\omega)\|_2}=\dfrac{\sqrt{\sum_i |y_i(\omega)|^2}}{\sqrt{\sum_j |d_j(\omega)|^2}}=\dfrac{\sqrt{y_{10}^2+y_{20}^2+\ldots}}{\sqrt{d_{10}^2+d_{20}^2+\ldots}}$

The gain depends on ω and is independent of $\|d(\omega)\|_2$. It depends on the direction of the input d.

2-norm measures length of vectors

Eigenvalues measure the gain for the special case when the inputs and outputs are in the same direction.

$\Rightarrow$ $\rho(G)=|\lambda_{max}(G)|$ does not satisfy the properties $\|G_1+G_2\|\le\|G_1\|+\|G_2\|$ and $\|G_1 G_2\|\le\|G_1\|\cdot\|G_2\|$

Singular Value Decomposition

$G=U\Sigma V^H$ where Σ contains non-negative singular values σ_i arranged in descending order along its main diagonal (other elements = 0), U and V are unitary $(A^H=A^{-1})$ matrices. $\|Ux\|_2=\|x\|_2 \quad \forall x$

column vectors of U = output directions: $\|u_i\|_2=\sqrt{|u_{i1}|^2+\ldots+|u_{il}|^2}=1$, $u_i^H u_i=1$, $u_i^H u_j=0 \quad \forall i\ne j$

column vectors of V = input directions: (same properties as u_i)

$G v_i=\sigma_i u_i$ where σ_i is the gain of the matrix G in a direction: $\sigma_i(G)=\|G v_i\|_2=\|G v_i\|_2/\|v_i\|_2$

max Gain: $\bar{\sigma}(G)=\max_{d\ne 0}(\|G d\|_2/\|d\|_2)=\max_{\|d\|_2=1}\|G d\|_2$ min Gain: $\underline{\sigma}(G)=\min_{d\ne 0}(\|G d\|_2/\|d\|_2)=\min_{\|d\|_2=1}\|G d\|_2$

"strongest"/"weakest" directions: $\bar{u}:=u_1$, $\bar{v}:=v_1$, $\underline{u}:=u_k$, $\underline{v}:=v_k$ $\Rightarrow$ $D\bar{v}=\bar{\sigma}\bar{u}$, $G\underline{v}=\underline{\sigma}\,\underline{u}$

interactive system: all inputs affect all outputs

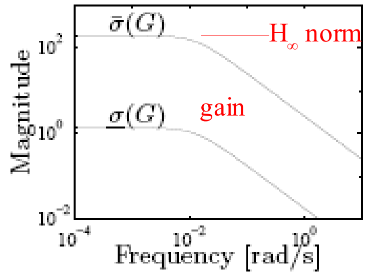
ill-conditioned system: some combinations of inputs have a strong effect on the outputs, others only weak → quantified by condition number $\bar{\sigma}/\underline{\sigma}$ (small $\Rightarrow$ uncertainty is not a problem)

we want $\underline{\sigma}$ >1 at all frequencies where control is required
more outputs than inputs → 3rd + 4th + ... vector = uncontrollable output directions
more inputs than outputs → 3rd + 4th + ... vector = input directions without effect

Singular Values for Performance

1-degree-of-freedom: e= –Sr

$\underline{\sigma}(S(j\omega)) \leq \frac{\|e(\omega)\|_2}{\|r(\omega)\|_2} \leq \underbrace{\overline{\sigma}(S(j\omega))}_{\text{worst case}}$ for performance $\frac{\|e(\omega)\|_2}{\|r(\omega)\|_2}$ should be small for any direction of $r(\omega)$



$\Rightarrow \quad \overline{\sigma}(S(j\omega)) < 1/|w_P(j\omega)| \ \ \forall\omega \quad \Leftrightarrow \quad \overline{\sigma}(w_P S) < 1 \ \ \forall\omega \quad \Leftrightarrow \quad \|w_P S\|_\infty < 1$

where the H_∞ norm is defined as the peak of the maximum singular value of the frequency response $\|M(s)\|_\infty = \max_\omega \overline{\sigma}(M(j\omega))$

typically singular values are small at low frequencies (effective feedback) and approach 1 at high frequencies (any real system is proper)

Bandwidth ω_B : frequency where $\overline{\sigma}(S)$ crosses $1/\sqrt{2} = 0.7$ from below

Since $S = (I+L)^{-1}$, the singular values inequality $\underline{\sigma}(A) - 1 \leq 1/\left(\overline{\sigma}(I+A)^{-1}\right) \leq \underline{\sigma}(A) + 1$ yields

$\underline{\sigma}(L) - 1 \leq \frac{1}{\overline{\sigma}(S)} \leq \underline{\sigma}(L) + 1 \quad \Rightarrow \quad \begin{matrix} low\ \omega: & \underline{\sigma}(L) \gg 1 & \Rightarrow & \overline{\sigma}(S) \approx 1/\underline{\sigma}(L) \ , \ \underline{\sigma}(L(j\omega_B)) \in [0.41, 2.41] \\ high\ \omega: & \overline{\sigma}(L) \ll 1 & \Rightarrow & \overline{\sigma}(S) \approx 1 \end{matrix}$ ↑ control is effective

Introduction to MIMO Robustness

<u>Example</u>: $G(s) = \frac{1}{s^2+a^2}\begin{bmatrix} s-a^2 & a(s+1) \\ -a(s+1) & s-a^2 \end{bmatrix}$, $a = 10$ with minimal realization $\left[\begin{array}{cc|cc} 0 & a & 1 & 0 \\ -a & 0 & 0 & 1 \\ \hline 1 & a & 0 & 0 \\ -a & 1 & 0 & 0 \end{array}\right]$

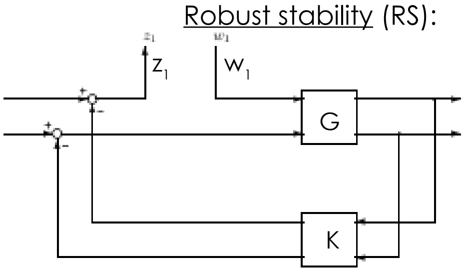
Poles at $s = \pm ja$ for stabilization: $K = I$ $T(s) = GK(I+GK)^{-1} = \frac{1}{s+1}\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}$

<u>Nominal stability</u> (NS): Two closed loop poles at $s = -1$ and $A_{cl} = A - BKC = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

<u>Nominal performance</u> (NP): $\underline{\sigma}(L) \leq 1 \ \ \forall\omega$ poor performance in low gain direction

g_{12}, g_{21} large $\Rightarrow$ strong interaction

<u>Robust stability</u> (RS): check one loop at a time $\frac{z_1}{w_1} = L_1(s) = \frac{1}{s} \quad \Rightarrow \quad \begin{matrix} GM = \infty \\ PM = 90° \end{matrix}$ seems good



no good robustness because perturbations $u'_i = (1+\epsilon_i)u_i$ result in

$det(sI - A'_{cl}) = s^2 + \underbrace{(2+\epsilon_1+\epsilon_2)}_{a_1}s + \underbrace{1+\epsilon_1+\epsilon_2+(a^2+1)\epsilon_1\epsilon_2}_{a_0}$ is stable for

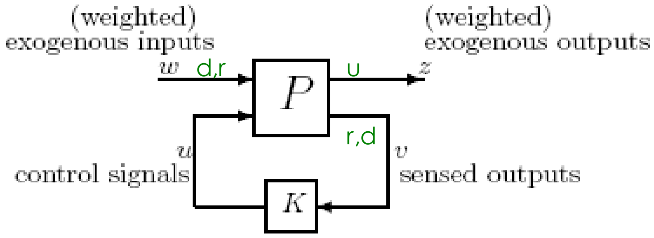
$-1 < \epsilon_1 < \infty \ , \ \epsilon_2 = 0$ and $-1 < \epsilon_2 < \infty \ , \ \epsilon_1 = 0$ but only stable for small simultaneous changes ($\{a_0, a_1\} \overset{!}{>} 0 \ \Leftrightarrow$ stability) is not good

$\|S\|_\infty, \|T\|_\infty$ large $\Rightarrow$ indicates robustness problems

$SISO: \ NP + RS \Rightarrow RP$, $MIMO: \ NP + RS \nRightarrow RP$ (RP can't be achieved by decoupling controller)

Multivariable plants can display a sensitivity to uncertainty which is fundamentally different from SISO systems.

General Control Problem Formulation

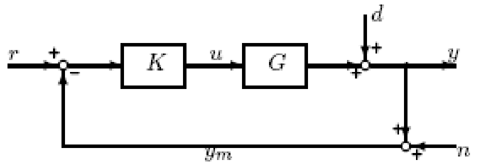


scaled: $|w|=1$ positive feedback used

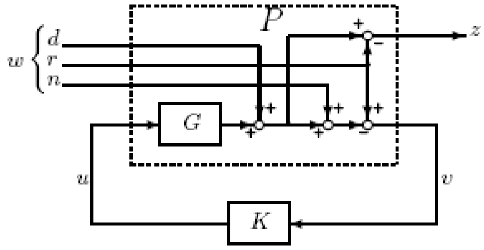
certain values might be in z and v, etc.

Generalized plant P contains G,G_d+structure

Try to minimize some norm of the transfer function from w to z → find a controller K which based on the information in v, generates a control signal u which counteracts the influence of w on z, thereby minimizing the closed-loop norm from w to z.



⇒



one degree-of-freedom control configuration

↓ performance is specified in terms of the actual output, not the measured output y_m

$$w=\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}=\begin{bmatrix} d \\ r \\ n \end{bmatrix} \qquad \begin{aligned} z &= e = y-r = Gu+d-r = Iw_1-Iw_2+Gu \\ v &= r-y_m = r-y-n = r-Gu-d-n = -Iw_1+Iw_2-Iw_3-Gu \end{aligned}$$

$$\Rightarrow P=\begin{array}{c} \left[\begin{array}{ccc|c} I & -I & 0 & G \\ \hline -I & I & -I & -G \end{array}\right] \begin{array}{c} z \\ v \end{array} \\ \begin{array}{cc} \quad w & \qquad\qquad u \end{array} \end{array}$$

is the transfer function from $[w \quad u]^T$ to $[z \quad v]^T$

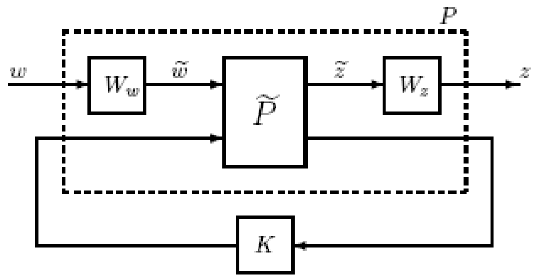
MATLAB: *Robust Control toolbox*

```
systemnames='G';
inputvar='[d(1);r(1);n(1);u(1)]';
input_to_G='[u]';
outputvar='[G+d-r; r-G-d-n]';
sysoutname='P';
sysic;
```

Including Weights in P

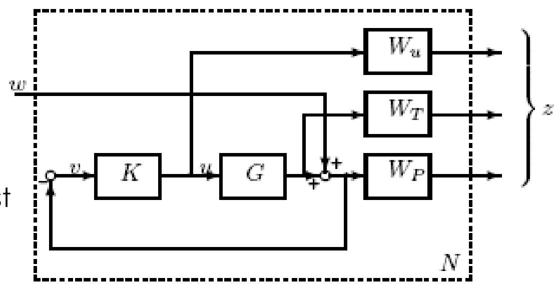
→ gives a meaningful controller synthesis problem.

The weighting matrices are usually frequency dependent and selected such that weighted signals w and z are of magnitude 1, that is, the norm from w to z should be less than 1.



Ex: stacked problem: bound $\bar{\sigma}(S)$ for performance, $\bar{\sigma}(T)$ for robustness and to avoid sensitivity to noise and $\bar{\sigma}(KS)$ to penalize large inputs ($u=KSw$) → solve $\min_K \|N(K)\|_\infty$

where $N=\begin{bmatrix} W_u KS \\ W_T T \\ W_P S \end{bmatrix} \Rightarrow z=Nw$ whose H_∞ norm must be minimized ($w=-r$ or $w=d_y$)



$$\Rightarrow \begin{aligned} z_1 &= W_u u \\ z_2 &= W_T G u \\ z_3 &= W_P w + W_P G u \\ v &= -w - Gu \end{aligned} \quad \Rightarrow \quad P=\left[\begin{array}{c|c} 0 & W_u I \\ 0 & W_T G \\ W_P I & W_P G \\ \hline -I & -G \end{array}\right]$$

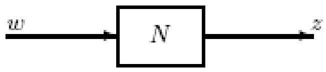
Partitioning the Generalized Plant P

$$P=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \Leftrightarrow \begin{matrix} z=P_{11}w+P_{12}u \\ v=P_{21}w+P_{22}u \end{matrix}$$

Ex: stacked problem → $P_{11}=\begin{bmatrix} 0 \\ 0 \\ W_P I \end{bmatrix}$, $P_{12}=\begin{bmatrix} W_u I \\ W_T G \\ W_P G \end{bmatrix}$, $\begin{matrix} P_{21}=-I \\ P_{22}=-G \end{matrix}$

if K is a nxm-matrix, P_{22} is a mxn matrix

Closing the Loop to get N



For analysis obtain K into the interconnection structure
→ combine P with $u=Kv$ and eliminate $\{u,v\}$ from the equations
$\Rightarrow \quad N=P_{11}+P_{12}K(I-P_{22}K)^{-1}P_{21} \quad =: \quad F_l(P,K)$

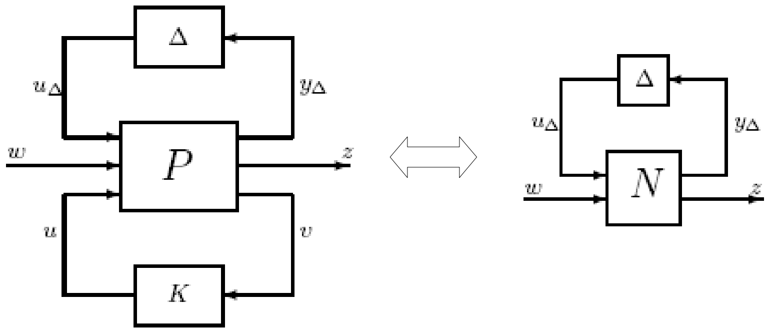
linear fractional transformation (LFT) of P

MATLAB: `N=lft(P,K)`

Deriving P from N → 6.4.6 (p. 6.40 – 6.41)

A General Control Configuration Including Model Uncertainty

Δ is a block-diagonal matrix that includes all possible perturbations (representing uncertainty) to the system. It is normalized such that $\|\Delta\|_\infty \leq 1$



for controller synthesis for robust performance analysis

upper LFT: $z=F_u(N,\Delta)w \;=\; [N_{22}+N_{21}\Delta(I-N_{11}\Delta)^{-1}N_{12}]w$

Almost any control problem with uncertainty can be represented like this. First represent each source of uncertainty by a perturbation block Δ_i which is normalized such that $\|\Delta_i\| \leq 1$. Then "pull out" each of these blocks from the system so that an input and an output can be associated with each Δ_i . Finally, collect these perturbation blocks into a large block-diagonal matrix having perturbation inputs and outputs.

Relative Gain Array (RGA)

$RGA=G\times(G^{-1})^T$ where $\times$ means element-by-element multiplication

- For decentralized control we prefer pairings for which the RGA-number at crossover frequencies is close to 1
- For stable plants avoid pairing on negative steady-state ($\omega=0$) RGA-elements (sub-controllers might cause instability)
  → for stability prefer pairings corresponding to an RGA-number close to 0 at crossover frequencies
- Plants with large RGA-elements around the crossover frequency are fundamentally difficult to control because of sensitivity to uncertainty in the input channels → decouplers should not be used
- large RGA-elements imply sensitivity to element-by-element uncertainty
- If the sum of the elements in a column of RGA are small (<<1) the corresponding input might be deleted
- If all elements in a row of RGA are small (<<1) then the corresponding output cannot be controlled

If results are unclear: iterate until the matrix only contains 1 and 0: RGA=RGA(RGA)

RGA_{ij} = Output i paired with input j

Limitations on Performance in MIMO Systems

Fundamental Limits on Sensitivity

$S+T=I \quad \Rightarrow \quad |1-\bar{\sigma}(S)| \leq \bar{\sigma}(T) \leq 1+\bar{\sigma}(S) \quad , \quad |1-\bar{\sigma}(T)| \leq \bar{\sigma}(S) \leq 1+\bar{\sigma}(T)$

$\Rightarrow \quad \bar{\sigma}(S)$ is small iff $\bar{\sigma}(T)$ is large and vice versa

If $G(s)$ has a RHP-zero at z with output direction y_z, then for internal stability of the feedback system T must have a RHP-zero in the same direction and $S(z)$ an eigenvalue of 1 corresponding to the left EV y_z.
If $G(s)$ has a RHP-pole at p … y_p, … T must have an eigenvalue of 1 w/ right EV y_p and $S(p)y_p=0$.

A peak on $\bar{\sigma}(S)$ $(\bar{\sigma}(T))$ larger than 1 is unavoidable if the plant has a RHP-zero (RHP-pole)

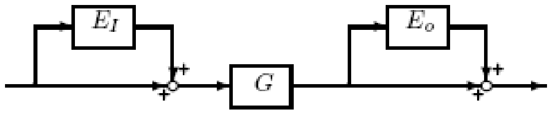
Weighted sensitivity: RHP-zero at $s=z \quad \Rightarrow \quad \|w_P(s)S(s)\|_\infty = max_\omega \bar{\sigma}(w_P S(j\omega)) \geq |w_P(z)|$

Weighted complementary sensitivity: RHP-pole at $s=p \quad \Rightarrow \quad \|w_T(s)T(s)\|_\infty = max_\omega \bar{\sigma}(w_T T(j\omega)) \geq |w_T(p)|$

In MIMO systems we can move effect of RHP zeros to different outputs by appropriate control (selecting an appropriate T(s) → when shifting a zero to one output only usually coupling effects increase *→ p.237*

Limitations Imposed by Uncertainty

G' = G + uncertainty



Additional problem: Plant directional uncertainty

Output uncertainty: $G'=(I+E_O)G \quad \Leftrightarrow \quad E_O=(G'-G)G^{-1}$

Input uncertainty: $G'=G(I+E_I) \quad \Leftrightarrow \quad E_I=G^{-1}(G'-G)$

The diagonal elements of E_I, E_O are <u>always</u> present in real systems

see A.143

Feedforward control: $y=T_r r \quad , \quad T_r=I \quad , \quad T'_r=G'K_r \quad \Rightarrow \quad y'-y=(T'_r-T_r)r=(G'-G)G^{-1}T_r r=E_O T_r r$

Feedback control: With one degree-of-freedom feedback control the nominal transfer function is $y=Tr$ where $T=L(I+L)^{-1}$ (ideally $T=I$). The change in response with model error is $y'-y=(T'-T)r=S'E_O T r$. Thus, $y'-y=S'E_O T r=S'E_O y$

→ with feedback control the effect of the uncertainty is reduced by a factor S' relative to that w/ feedforw.
→ feedback effective in crossover region (||S|| and ||T||≈ 1). At high freq. L is usually small → T small → T-T' small

Upper Bound on σ̄(S')

Output uncertainty: $S'=S(I+E_O T)^{-1}$

Input uncertainty: $S'=S(I+GE_I G^{-1}T)^{-1}=SG(I+E_I T_I)^{-1}G^{-1}=(I+TK^{-1}E_I K)^{-1}S=K^{-1}(I+T_I E_I)^{-1}KS$

Assume $\{G, G'\}$ are stable $\Rightarrow$ $\{S, S'\}$ are stable $\Rightarrow$ $\{(I+E_O T)^{-1}, (I+E_I T_I)^{-1}\}$ are stable

$\bar{\sigma}(E_I) \leq |w_I| \quad , \quad \bar{\sigma}(E_O) \leq |w_O|$ (typically $|w_I|$ or $|w_O|$ =0.2 at low frequencies and >1 at higher frequencies)

Upper bound for output uncertainty: $\bar{\sigma}(S') \;\leq\; \bar{\sigma}(S)\bar{\sigma}((I+E_O T)^{-1}) \;\leq\; \dfrac{\bar{\sigma}(S)}{1-|w_O|\bar{\sigma}(T)}$

Upper bounds for input uncertainty:

$$\bar{\sigma}(S') \;\leq\; \gamma(G)\bar{\sigma}(S)\bar{\sigma}((I+E_I T_I)^{-1}) \;\leq\; \gamma(G)\frac{\bar{\sigma}(S)}{1-|w_I|\bar{\sigma}(T_I)}$$

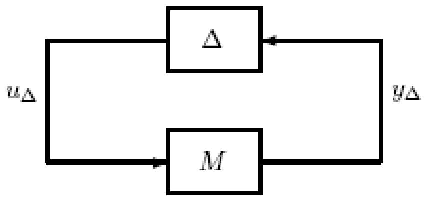
$$\bar{\sigma}(S') \;\leq\; \gamma(K)\bar{\sigma}(S)\bar{\sigma}((I+T_I E_I)^{-1}) \;\leq\; \gamma(K)\frac{\bar{\sigma}(S)}{1-|w_I|\bar{\sigma}(T_I)}$$

where $\gamma(G)=\bar{\sigma}(G)/\underline{\sigma}(G) \quad , \quad \gamma(K)=\bar{\sigma}(K)/\underline{\sigma}(K)$

If $\gamma(G)\approx 1$ the system is insensitive to input uncertainties, irrespective of the controller.
If we use a "round" controller $\gamma(K)\approx 1$ the sensitivity function is <u>not</u> sensitive to input uncertainty.

$$\bar{\sigma}((I+ET)^{-1})=\frac{1}{\underline{\sigma}(I+ET)} \leq \frac{1}{1-\bar{\sigma}(ET)}$$

Robust Stability and Performance Analysis



General Control Configuration with Uncertainty

$\Delta = diag\{\Delta_i\}$ For analysis we use the $N\Delta$ -structure (see p.16).
For robust stability analysis we use the $M\Delta$ -structure.

$$N \;=\; F_l(P,K) \;=\; P_{11}+P_{12}K(I-P_{22}K)^{-1}P_{21}$$
$$F \;=\; F_u(N,\Delta) \;=\; N_{22}+N_{21}\Delta(I-N_{11}\Delta)^{-1}N_{12} \quad\Rightarrow\quad z=F\,w$$

To analyze robust stability of F we can rearrange the system into the $M\Delta$ -structure where $M=N_{11}$ is the transfer function from the output to the input of the perturbations.

Representing Uncertainty

Each individual perturbation is assumed to be stable and is normalized: $\bar{\sigma}(\Delta_i(j\omega))\le 1 \quad \forall\omega$
For a scalar perturbation we have $|\delta_i(j\omega)|\le 1 \quad \forall\omega$.

Since the maximum singular value of a block diagonal matrix is equal to the largest of the maximum singular values of the individual blocks, it then follows for $\Delta=diag\{\Delta_i\}$ that $\bar{\sigma}(\Delta_i(j\omega))\le 1 \quad \forall\{\omega,i\} \quad\Leftrightarrow\quad \|\Delta\|_\infty\le 1$

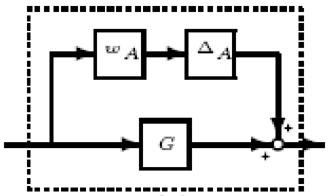
Δ has <u>structure</u>, and therefore in the robustness analysis we do <u>not</u> want to allow all Δ that is satisfied but only the subset where Δ=diag{Δ$_i$}.

Unstructured Uncertainty

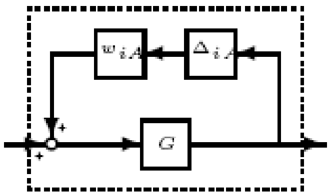
unstructured uncertainty = the use of a "full" complex perturbation matrix Δ where any Δ(jω) with $\bar{\sigma}(\Delta(j\omega))\le 1$ is allowed

Feedforward forms of unstructured uncertainty:

$a)$	Π_A:	$G_p=G+E_A$	$E_a=w_A\Delta_A$
$b)$	Π_I:	$G_p=G(I+E_I)$	$E_I=w_I\Delta_I$
$c)$	Π_O:	$G_p=(I+E_O)G$	$E_O=w_O\Delta_O$



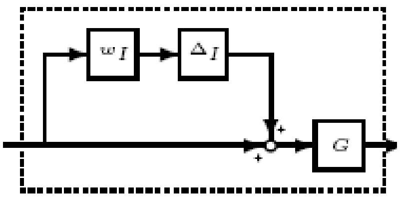
additive (a)



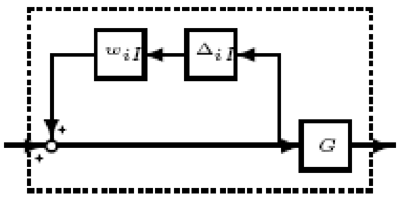
(d)

Feedback (inverse) forms of unstructured uncertainty:

$d)$	Π_{iA}:	$G_p=G(I-E_{iA}G)^{-1}$	$E_{ia}=w_{iA}\Delta_{iA}$
$e)$	Π_{iI}:	$G_p=G(I-E_{iI})^{-1}$	$E_{iI}=w_{iI}\Delta_{iI}$
$f)$	Π_{iO}:	$G_p=(I-E_{iO})^{-1}G$	$E_{iO}=w_{iO}\Delta_{iO}$

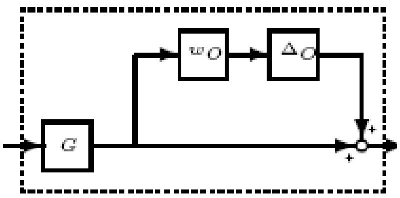


input (b) multiplicative

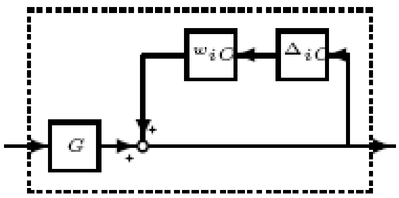


(e)

Δ denotes the normalized perturbation and E the "actual" perturbation. Sometimes instead of scalar weights $E=w\Delta$ we can also use matrix weights $E=W_2\Delta W_1$.



output (c) multiplicative



(f)

Lumping Uncertainty into a Single Perturbation

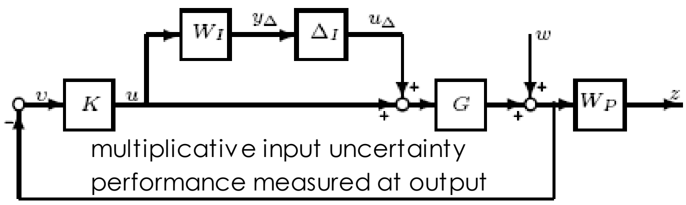
Output uncertainty is usually less restrictive than input uncertainty in terms of control performance
→ try $G_p=(I+w_O\Delta_O)G \quad\Rightarrow\quad l_O(\omega)=\max_{G_p\in\Pi}\bar{\sigma}((G_p-G)G^{-1})\le|w_O(j\omega)|$ if $|w_O|\ge l_O$ is reasonable (<1 where we want control)

Shifting uncertainty: assume input uncer.: $G_p=G(I+E_I) \quad\Rightarrow\quad l_I(\omega)=\max_{E_I}\bar{\sigma}(G^{-1}(G_p-G))=\max_{E_I}\bar{\sigma}(E_I)$

shift to output: $l_O(\omega)=\max_{E_I}\bar{\sigma}((G_p-G)G^{-1})=\max_{E_I}\bar{\sigma}(GE_IG^{-1})=|w_I|\gamma(G(j\omega))$

which is much larger than when using input uncertainty if the condition number γ is large

Obtaining P,N and M



$$P=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}=\left[\begin{array}{cc|c} 0 & 0 & W_I \\ W_P G & W_P & W_P G \\ \hline -G & -I & -G \end{array}\right]$$

$$\Rightarrow \quad N=\begin{bmatrix} -W_I K G(I+KG)^{-1} & -W_I K(I+GK)^{-1} \\ W_P G(I+KG)^{-1} & W_P(I+GK)^{-1} \end{bmatrix}$$

$$\Rightarrow \quad M=N_{11}=-W_I K G(I+KG)^{-1}$$

Alternatively: M is the transfer function from u_Δ to y_Δ, N is the transfer function from $[u_\Delta \quad w]^T$ to $[y_\Delta \quad z]^T$

if K is a nxm-matrix, P_{22} is a mxn matrix

Robust Stability and Robust Performance

Robust stability analysis: With a given controller K we determine whether the system remains stable for all plants in the uncertainty set.

Robust performance analysis: If RS is satisfied, we determine how "large" the transfer function from exogenous inputs w (normalized disturbances and references) to outputs z (normalized errors) may be for all plants in the uncertainty set.

$$F:=F_u(N,\Delta) \quad (z=F\,w)$$

using the NΔ-Structure we require:

$NS \Leftrightarrow N$ is internally stable

$NP \Leftrightarrow \|N_{22}\|_\infty<1$ and NS w → z without Δ

$RS \Leftrightarrow F$ is stable $\forall \Delta$, $\|\Delta\|_\infty \le 1$ and NS

$RP \Leftrightarrow \|F\|_\infty<1 \quad \forall \Delta$, $\|\Delta\|_\infty \le 1$ and NS

← sometimes $F=w_P S_p$ with performance weight w_P and a set of perturbed sensitivity functions S_p

allowed perturbations: $\|\Delta\|_\infty \le 1$

Robust Stability of the MΔ-structure

Suppose that Δ is stable, the system with $z=F_u(N,\Delta)w$ is nominally stable (with Δ=0) → N is stable

⇒ the only source of instability is $(1-N_{11}\Delta)^{-1}|_{N_{11}=M}$ ⇒ stability of N Δ is equivalent to stability of M Δ

Determinant stability condition: If {M, Δ} are stable. Consider the convex set of perturbations Δ, such that if Δ' is an allowed perturbation, then so is cΔ' where $c \in \mathbb{R}$, $|c|\le 1$.

Δ=0 : det(I-MΔ)=1
if there is a circle around 0 there must also exist one through 0

Then the MΔ-system is stable for all allowed perturbations (we have RS) iff the Nyquist plot of $det(I-M(s)\Delta(s))$ does not encircle the origin $\forall \Delta$

$$\Leftrightarrow \quad det(I-M\Delta)\neq 0 \quad \forall\{\omega,\Delta\} \quad \Leftrightarrow \quad \lambda_i(M\Delta)\neq 1 \quad \forall\{i,\omega,\Delta\}$$

RS for Complex Unstructured Uncertainty

Assume that the nominal system M(s) is stable (NS) and that the perturbations Δ(s) are stable. Then the MΔ-system is stable for all perturbations Δ satisfying $\|\Delta\|_\infty \le 1$ (i.e. we have RS) iff $\bar{\sigma}(M(j\omega))<1 \quad \forall \omega \quad \Leftrightarrow \quad \|M\|_\infty<1$

RS with Structured Uncertainty: Motivation

$D=diag\{d_i I_i\}$ where I_i is an identity matrix of the same dimension as the i'th perturbation block Δ_i , d_i is a scalar.

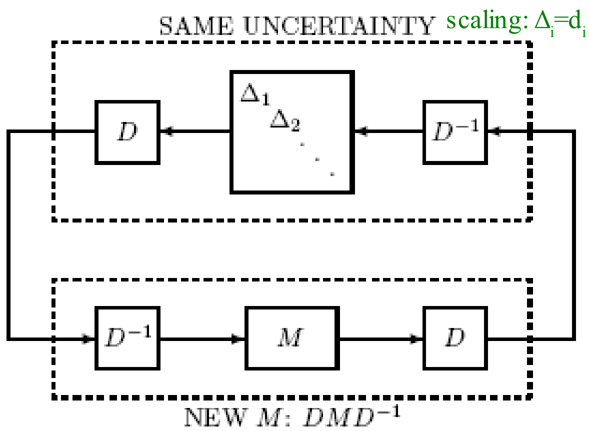
not iff as above (unstructured)

RS if $\bar{\sigma}(M(j\omega))<1 \quad \forall \omega \quad \Rightarrow \quad \bar{\sigma}(D M D^{-1})<1 \quad \forall \omega$

is obtained by minimizing at each frequency the scaled singular value ⇒ RS if $\min_{D(\omega)\in D} \bar{\sigma}(D(\omega)M(j\omega)D(\omega)^{-1})<1 \quad \forall \omega$

may be significantly smaller than $\bar{\sigma}$

where D is the set of block-diagonal matrices whose structure is compatible to that of Δ, i.e. $\Delta D=D\Delta$



The Structured Singular Value

test RS/RP without having to search through the infinite set of allowed perturbations Δ

The structured singular value (μ / SSV) is a function which provides a generalization of the singular value $\bar{\sigma}$ and the spectral radius ρ . We will use the SSV to get necessary and sufficient conditions for RS and RP.

Definition 1: Find the smallest structured Δ (measured in terms of $\bar{\sigma}(\Delta)$) which makes $det(I-M\Delta)=0$

$$\Rightarrow \quad \mu(M)=\frac{1}{\bar{\sigma}(\Delta)} \quad \Leftrightarrow \quad \mu(M)=\frac{1}{\min_{\Delta}\{\bar{\sigma}(\Delta) | det(I-M\Delta)=0 \quad for\ structured\ \Delta\}}$$

$\mu(M)$ depends also on the allowed structure for Δ $\Rightarrow$ sometimes notation $\mu_{\Delta}(M)$ is used

If Δ is unstructured (= a full matrix) the smallest Δ which yields singularity has $\bar{\sigma}(\Delta)=1/\bar{\sigma}(M)$
$\Rightarrow \quad \mu(M)=\bar{\sigma}(M)$

Definition 2: If M is a given complex matrix and $\Delta=diag\{\Delta_i\}$, denote a set of complex matrices with $\bar{\sigma}(\Delta)\le 1$ and with a given block-diagonal structure (some blocks may be repeated or restricted to be real) → the structured singular value (real, nonnegative function) is defined by

$$\mu(M)=\frac{1}{\min\{k_m | det(I-k_m M\Delta)=0 \quad for\ structured\ \Delta,\ \ \bar{\sigma}(\Delta)\le 1\}}$$

μ>1: <u>all</u> uncertainty blocks must be decreased in magnitude by a factor μ to guarantee stability

If no such structured Δ exists then $\mu(M)=0$

A value of $\mu=1$ means that there exists a perturbation with $\bar{\sigma}(\Delta)=1$ which is just large enough to make $I-M\Delta$ singular. A larger value of μ is "bad" as it means that a smaller perturbation makes $I-M\Delta$ singular, whereas a smaller value of μ is "good".

the Δ corresponding to k_m will always have $\sigma(\Delta)=1$ (Proof: p.335)

Robust Stability with Structured Uncertainty

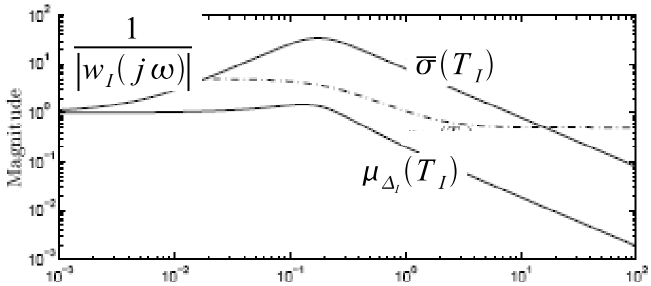
Δ is a set of norm-bounded block-diagonal perturbations, $\{M,\Delta\}$ are stable
$\Rightarrow$ the $M\Delta$ -system is stable for all allowed perturbations with $\bar{\sigma}(\Delta)\le 1 \ \ \forall\omega$ iff $\boxed{\mu(M(j\omega))<1 \ \ \forall\omega}$

$\Rightarrow \quad RS \ \Leftrightarrow \ \mu(M(j\omega))\bar{\sigma}(\Delta(j\omega))<1 \ \ \forall\omega$
"generalized small gain theorem" (with structure of Δ)

$$G(s)=\frac{1}{\tau s+1}\begin{bmatrix}-87.8 & 1.4\\ -108.2 & -1.4\end{bmatrix}, \quad K(s)=\frac{1+\tau s}{s}\begin{bmatrix}-0.0015 & 0\\ 0 & -0.075\end{bmatrix}$$

$$w_I(s)=\frac{s+0.2}{0.5s+1}$$

<u>Ex</u>:



mult. uncertainty in <u>each</u> manipulated input

$\Rightarrow$ since $\bar{\sigma}(T_I)>1/|w_I(j\omega)|$ the system would not be stable for full-block input uncertainty. In case of structured Δ we can see that RS is satisfied and the use of the singular value would be conservative.

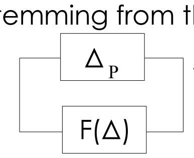
Robust Performance

With an H_∞ performance objective, the RP-condition is identical to a RS-condition with an additional (fictitious) perturbation block Δ_P (which is always a full matrix).

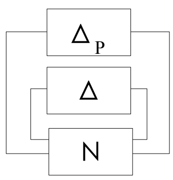
"Pull out" uncertain perturbations and rearrange the uncertain system into the $N\Delta$ -structure, assume that N is (internally) stable, then $\quad RP \ \Leftrightarrow \ \|F\|_\infty=\|F_u(N,\Delta)\|_\infty<1 \quad \forall\|\Delta\|_\infty\le 1 \ \Leftrightarrow \ \mu_{\hat{\Delta}}(N(j\omega))<1 \ \ \forall\omega$

where μ is computed with respect to the structure $\hat{\Delta}=diag\{\Delta,\Delta_P\}$, Δ is the true uncertainty and Δ_P is a full complex perturbation stemming from the H_∞ norm performance specification (w/ dimensions of F^T)

$RP \ \Leftrightarrow \ \begin{matrix}\|F(\Delta)\|_\infty<1\\ \forall\|\Delta\|_\infty\le 1\end{matrix} \ \overset{*}{\Leftrightarrow}$

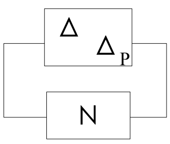


is RS $\forall\begin{cases}\|\Delta_P\|_\infty\le 1\\ \|\Delta\|_\infty\le 1\end{cases} \ \Leftrightarrow$



is RS $\forall\begin{cases}\|\Delta_P\|_\infty\le 1\\ \|\Delta\|_\infty\le 1\end{cases} \ \Leftrightarrow$

$\Leftrightarrow$



is RS $\forall\|\hat{\Delta}\|_\infty\le 1 \ \Leftrightarrow \ \mu_{\hat{\Delta}}(N)<1 \ \ \forall w$

* see "RS for Complex Unstructured Uncertainty" on p.19

Summary of µ-conditions for NP, RS and RP

Rearrange the uncertain system into the $N\Delta$ -structure, where the block-diag. perturbations satisfy $\|\Delta\|_\infty \le 1$
Introduce $F = F_u(N,\Delta) = N_{22} + N_{21}\Delta(I - N_{11}\Delta)^{-1}N_{12}$ and let RP be $\|F\|_\infty \le 1 \;\; \forall$ allowable perturbations

$$
\begin{aligned}
\Rightarrow \quad & NS && \Leftrightarrow && N \text{ (internally) stable} \\
& NP && \Leftrightarrow && \bar{\sigma}(N_{22}) = \mu_{\Delta_P} < 1 \;\; \forall \omega \;\wedge\; NS \\
& RS && \Leftrightarrow && \mu_\Delta(N_{11}) < 1 \;\; \forall \omega \;\wedge\; NS \\
& RP && \Leftrightarrow && \mu_{\hat{\Delta}}(N) < 1 \;\; \forall \omega \;\wedge\; NS
\end{aligned}
$$

Δ is a block-diagonal matrix, Δ_P is always a full complex matrix. NS must be tested separately in all cases.

Ex: $\mu = 1.1 \;\; \Rightarrow$ our RP-requirement would be satisfied exactly if we reduced both the performance requirement and the uncertainty by a factor of 1.1

To find the worst-case weighted performance for a given uncertainty, one needs to keep the magnitude of the perturbations fixed ($\bar{\sigma}(\Delta) \le 1$) and compute $\mu^s(N(j\omega)) = \max_{\bar{\sigma}(\Delta)\le 1} \bar{\sigma}(F_l(N,\Delta)(j\omega))$ skewed-µ

To find μ^S numerically, we scale the performance part of N by a factor $k_m = 1/\mu^S$ and iterate on k_m until $\mu = 1 \;\Rightarrow\; \mu^S(N)$ solves $\mu(K_m N) = 1$, $K_m = \begin{bmatrix} I & 0 \\ 0 & 1/\mu^S \end{bmatrix}$ at each frequency

Note that μ underestimates how bad/good the actual worst-case performance is. This follows because $\mu^S(N)$ is always further from 1 than $\mu(N)$.

µ-Synthesis and DK-Iteration

The structured singular value µ is a very powerful tool for the analysis of RS with a given controller. However, one may also seek µ-synthesis to find the controller that minimizes a certain µ-condition.

$$\mu(N) \le \min_D \bar{\sigma}(DND^{-1}) \quad \Rightarrow \quad \min_K \left(\min_D \|DN(K)D^{-1}\|_\infty \right)$$

define upper bound for µ then find controller that minimizes peak value over frequency

K-Step: Synthesize an H_∞ controller for $\min_K \|DN(K)D^{-1}\|_\infty$ with fixed D (begin with $D = I$)

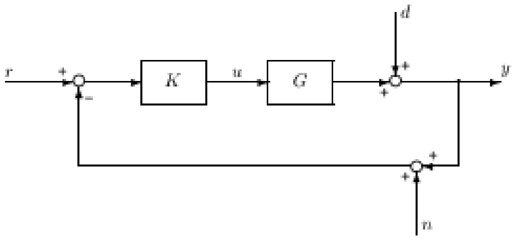
D-Step: Find $D(j\omega)$ to minimize at each frequency $\bar{\sigma}(DND^{-1}(j\omega))$ with fixed N

Step 3: Fit the magnitude of each element of $D(j\omega)$ to a stable and minimum phase TF $D(s)$

→ iterate until satisfactory performance is achieved, $\|DND^{-1}\|_\infty < 1$ or until the H_∞ norm stops decreasing

Controller Design

Trade-offs in MIMO Feedback Design



$y = Tr + Sd - Tn \qquad u = KS[r - n - d]$

Closed-loop objectives:
1. disturbance rejection $\overline{\sigma}(S)$ small — often needed at low freq.
2. noise attenuation $\overline{\sigma}(S)$ small — often needed at high freq.
3. reference tracking $\overline{\sigma}(T) \approx \underline{\sigma}(T) \approx 1$
4. input usage (control energy) reduction $\overline{\sigma}(KS)$ small
5. robust stability (w/ additive perturbation) $\overline{\sigma}(KS)$ small
6. robust stability (w/ multiplicative output perturb.) $\overline{\sigma}(T)$ small

→ feedback design is a trade-off over requirements 1 to 6

Over specified frequency ranges, we can approximate the closed-loop requirements by open-loop objectives:

1. disturbance rejection $\underline{\sigma}(GK)$ large valid for frequencies at which $\underline{\sigma}(GK) \gg 1$
2. noise attenuation $\overline{\sigma}(GK)$ small valid for frequencies at which $\overline{\sigma}(GK) \ll 1$
3. reference tracking $\underline{\sigma}(GK)$ large valid for frequencies at which $\underline{\sigma}(GK) \gg 1$
4. control energy reduction $\overline{\sigma}(K)$ small valid for frequencies at which $\overline{\sigma}(GK) \ll 1$
5. robust stability (additive pert.) $\overline{\sigma}(K)$ small valid for frequencies at which $\overline{\sigma}(GK) \ll 1$
6. robust stability (mult. outp. pert.) $\overline{\sigma}(GK)$ small valid for frequencies at which $\overline{\sigma}(GK) \ll 1$

To shape singular values of GK by selecting K is a relatively easy task, but to do this in a way which also guarantees closed-loop stability is in general difficult.

General Control Problem Formulation

$$\begin{bmatrix} z \\ v \end{bmatrix} = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \cdot \begin{bmatrix} w \\ u \end{bmatrix}$$

state-space realization:
$$P \stackrel{S}{=} \begin{array}{c c} & \begin{array}{ccc} x & w & u \end{array} \\ \begin{array}{c} x \\ z \\ v \end{array} & \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & D_{21} & D_{22} \end{array}\right] \end{array}$$

Lower functional transformation: $F_l(P, K) = P_{11} + P_{12} K (I - P_{22} K)^{-1} P_{21}$ $\qquad z = F_l(P, K) w$

H_2 and H_∞ control involve the minimization of the H_2 and H_∞ norms of $F_l(P, K)$.

H₂ Optimal Control

The standard H_2 optimal control problem is to find a stabilizing controller K which minimizes

$$\|F(s)\|_2 = \sqrt{\frac{1}{2\pi} \int_{-\infty}^{\infty} tr[F(j\omega) F(j\omega)^H] d\omega} \qquad F = F_l(P, K)$$

For a particular problem the generalized plant P will include the plant model, the interconnection structure, and the designer specified weighting functions. (see "LQG: A Special H₂ Optimal Controller" on p.23)

Stochastic interpretation: suppose in the general control configuration that the exogeneous input w is white noise of unit intensity $\Rightarrow \quad E\{w(t) w(\tau)^T\} = I \delta(t - \tau)$

expected power in the error signal $z \quad \Rightarrow \quad E\left\{ \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} z(t)^T z(t) dt \right\}$

$$= tr\, E\{z(t) z(t)^T\} \stackrel{\substack{Parseval's \\ Therorem}}{=} \|F\|_2^2 = \|F_l(P, K)\|_2^2$$

⇒ by minimizing the H_2 norm the output (or error) power of the generalized system is minimized

LQG: A Special H₂ Optimal Controller

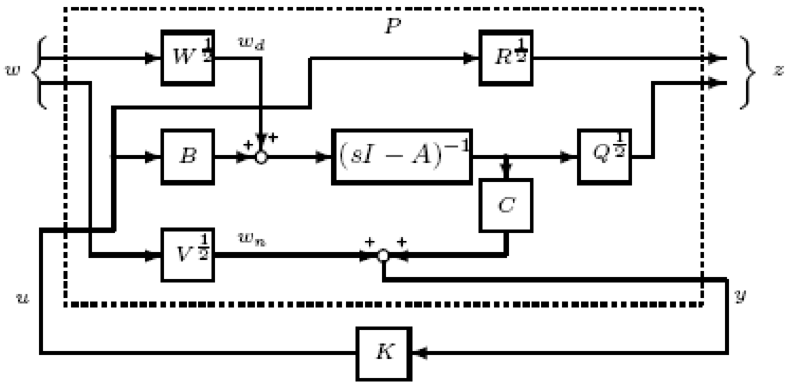
$\dot{x}=Ax+Bu+w_d$, $y=Cx+w_n$ $\qquad E\left\{\begin{bmatrix} w_d(t) \\ w_n(t) \end{bmatrix}\begin{bmatrix} w_d(\tau)^T & w_n(\tau)^T \end{bmatrix}\right\}=\begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix}\delta(t-\tau)$

Find $u=Ky$ such that $J=E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T [x^T Q x+u^T R u]dt\right\}$ is minimized with $Q=Q^T\geq 0$, $R=R^T\geq 0$

⇒ define $z=\begin{bmatrix} Q^{1/2} & 0 \\ 0 & R^{1/2} \end{bmatrix}\cdot\begin{bmatrix} x \\ u \end{bmatrix}$, $\begin{bmatrix} w_d \\ w_n \end{bmatrix}=\begin{bmatrix} W^{1/2} & 0 \\ 0 & V^{1/2} \end{bmatrix}w$ where w is white noise

⇒ the LQG cost function is $J=E\left\{\lim_{T\to\infty}\frac{1}{T}\int_0^T z(t)^T z(t)\,dt\right\}=\|F_l(P,K)\|_2^2$ where $z(s)=F_l(P,K)w(s)$

and the generalized plant is given by $P=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \stackrel{s}{=} \left[\begin{array}{c|cc|c} A & W^{1/2} & 0 & B \\ \hline Q^{1/2} & 0 & 0 & 0 \\ 0 & 0 & 0 & R^{1/2} \\ \hline C & 0 & V^{1/2} & 0 \end{array}\right]$



H∞ Optimal Control

The standard H_∞ optimal control problem is to find all stabilizing controllers K which minimize

$$\|F_l(P,K)\|_\infty=\max_\omega \overline{\sigma}(F_l(P,K)(j\omega)) = \max_{w(t)\neq 0}\frac{\|z(t)\|_2}{\|w(t)\|_2} \qquad \text{where } \|z(t)\|_2=\sqrt{\int_0^\infty \sum_i |z_i(t)|^2 dt}$$

It is often computationally simpler to design a sub-optimal controller. Let γ_{min} be the minimum value of $\|F_l(P,K)\|_\infty$ over all stabilizing controllers K. Then the H_∞ sub-optimal control problem is:

given a $\gamma>\gamma_{min}$, find all stabilizing controllers such that $\|F_l(P,K)\|_\infty<\gamma$

By reducing γ iteratively an optimal solution is approached. general algorithm: see p.391

Mixed-Sensitivity H∞ Control

Transfer function shaping problems in which the sensitivity function $S=(I+GK)^{-1}$ is shaped along with one or more other closed-loop transfer function such as KS or the complementary sensitivity function $T=I-S$.

To optimize performance, minimize $\|w_1 S\|_\infty$ (S: d → z)
to minimize control inputs minimize $\|w_2 K S\|_\infty$ (KS: d → u)

$w_1(s)$: scalar low-pass (disturbance is low-frequency signal)
$w_2(s)$: scalar high-pass with $\omega_c \approx$ desired closed-loop bandwidth

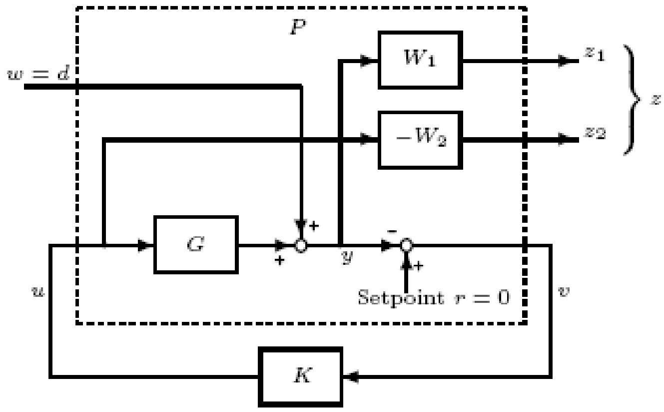
Compromise: minimize $\left\| \begin{bmatrix} w_1 S \\ w_2 K S \end{bmatrix} \right\|_\infty$

imagine the disturb. d as a single exogenous input and define an error signal $z=\begin{bmatrix} W_1 y \\ -W_2 u \end{bmatrix}=\begin{bmatrix} W_1 S \\ W_2 K S \end{bmatrix} \cdot w$

$$\Rightarrow \quad P_{11}=\begin{bmatrix} W_1 \\ 0 \end{bmatrix}, \quad P_{12}=\begin{bmatrix} W_1 G \\ -W_2 \end{bmatrix}, \quad \begin{matrix} P_{21}=-I \\ P_{22}=-G \end{matrix}$$

where the partitioning is such that $\begin{bmatrix} z_1 \\ z_2 \\ v \end{bmatrix}=\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} w \\ u \end{bmatrix}$

$$\Rightarrow \quad F_l(P,K)=\begin{bmatrix} W_1 S \\ W_2 K S \end{bmatrix}$$

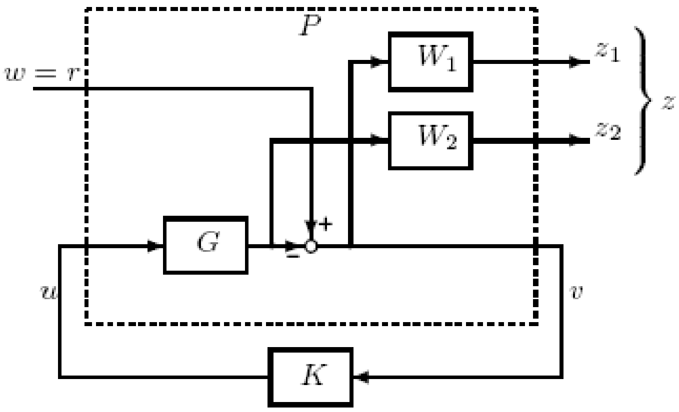


S/T-Mixed Sensitivity Minimization Problem

Another useful mixed sensitivity optimization problem is to find a stabilizing controller which minimizes

$$\left\| \begin{bmatrix} W_1 S \\ W_2 T \end{bmatrix} \right\|_\infty \quad \Rightarrow \quad P_{11}=\begin{bmatrix} W_1 \\ 0 \end{bmatrix} \quad P_{12}=\begin{bmatrix} -W_1 G \\ W_2 G \end{bmatrix} \quad \begin{matrix} P_{21}=I \\ P_{22}=-G \end{matrix}$$

The ability to be able to shape T is desirable for tracking problems and noise attenuation. It is also important for robust stability with respect to multiplicative perturbations at the plant output.



Appendix A: Matrix Theory and Norms

Properties: $A^H = \bar{A}^T$ $(AB)^T = B^T A^T$ $(AB)^H = B^H A^H$ $(AB)^{-1} = B^{-1} A^{-1}$ $A^{-1} = \frac{adj\, A}{det\, A}$

$c_{ij} = [adj\, A]_{ij} = (-1)^{i+j} det\, A^{ij}$ where A^{ij} is the submatrix formed by del. row i and column j

$det\, A = \sum_{i=1}^{n} a_{ij} c_{ij} = \sum_{j=1}^{n} a_{ij} c_{ij}$

A is symmetric if $A^T = A$ and Hermitian if $A^H = A$ if A is non-singular $det(A^{-1}) = 1/(det\, A)$

$\{A_1, A_2\} \in \mathbb{C}^{n \times n}$, $c \in \mathbb{R}$ $\Rightarrow$ $det(A_1 A_2) = det(A_2 A_1) = det\, A_1 \cdot det\, A_2$ $det(cA) = c^n det(A)$

if $A_1 A_2$ and $A_2 A_1$ are square $det(I + A_1 A_2) = det(I + A_2 A_1)$

Schur's formula: $det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = det(A_{11}) \cdot det(A_{22} - A_{21} A_{11}^{-1} A_{12}) = det(A_{22}) \cdot det(A_{11} - A_{12} A_{22}^{-1} A_{21})$

Eigenvalues: the solutions λ of the equation $det(A - \lambda I) = 0$

Eigenvectors: solutions of $(A - \lambda_i I) t_i = 0$ $\Leftrightarrow$ $A t_i = \lambda_i t_i$ (right) or $q_i^H (A - \lambda_i I) = 0$ $\Leftrightarrow$ $q_i^H A = \lambda_i q_i^H$ (left) A6-A8

Singular Value Decomposition

Unitary matrix: $U^H = U^{-1}$ $\|\lambda(U)\| = 1 \quad \forall i$

Any complex matrix $A_{l \times m}$ may be factorized into $A = U \Sigma V^H$ where $\{U_{l \times l}, V_{m \times m}\}$ are unitary matrices ($B^H = B^{-1}$) and $\Sigma_{l \times m}$ is of the form $\Sigma = \begin{bmatrix} \Sigma_1 \\ 0 \end{bmatrix}$ $(l \geq m)$ or $\Sigma = \begin{bmatrix} \Sigma_1 & 0 \end{bmatrix}$ $(l \leq m)$

$\Sigma_1 = diag\{\sigma_1, \sigma_2, \ldots, \sigma_k\}$, $k = min(l, m)$, $\bar{\sigma} = \sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_k = \underline{\sigma}$

The unitary matrices form orthonormal bases for the column (output) space and the row (input) space.
We define $\bar{u} = u_1$, $\underline{u} = u_k$ (column vector) and $\bar{v} = v_1$, $\underline{v} = v_k$ (row vector)

The singular values $\sigma_i(A) = \sqrt{\lambda_i A^H A} = \sqrt{\lambda_i A A^H}$ are unique, however the SVD is not, since all $U' = U S$, $V' = V S$, $S = diag\{e^{j\theta_i}\}$ are also solutions.

The rank of a matrix is the number of non-zero singular values of the matrix.

SVD of a Matrix Inverse

$A^{-1} = V \Sigma^{-1} U^H \overset{j=m-i+1}{\Rightarrow} \sigma_i(A^{-1}) = 1/\sigma_j(A)$, $u_i(A^{-1}) = v_j(A)$, $v_i(A^{-1}) = u_j(A)$, $\bar{\sigma}(A^{-1}) = 1/\underline{\sigma}(A)$

Singular Value Inequalities

$\underline{\sigma}(A) \leq |\lambda_i(A)| \leq \bar{\sigma}(A)$ $\bar{\sigma}(A^H) = \bar{\sigma}(A)$ $\bar{\sigma}(A^T) = \bar{\sigma}(A)$

$\bar{\sigma}(AB) \leq \bar{\sigma}(A) \cdot \bar{\sigma}(B)$ $\underline{\sigma}(A) \bar{\sigma}(B) \leq \bar{\sigma}(AB)$ $\bar{\sigma}(A) \underline{\sigma}(B) \leq \bar{\sigma}(AB)$ $\underline{\sigma}(A) \underline{\sigma}(B) \leq \underline{\sigma}(AB)$

$max\{\bar{\sigma}(A), \bar{\sigma}(B)\} \leq \bar{\sigma}\begin{bmatrix} A \\ B \end{bmatrix} \leq \sqrt{2}\, max\{\bar{\sigma}(A), \bar{\sigma}(B)\}$ $\bar{\sigma}\begin{bmatrix} A \\ B \end{bmatrix} \leq \bar{\sigma}(A) + \bar{\sigma}(B)$

$\bar{\sigma}\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = max\{\bar{\sigma}(A), \bar{\sigma}(B)\}$ $\sigma_i(A) - \bar{\sigma}(B) \leq \sigma_i(A+B) \leq \sigma_i(A) + \bar{\sigma}(B)$

$|\bar{\sigma}(A) - \bar{\sigma}(B)| \leq \bar{\sigma}(A+B) \leq \bar{\sigma}(A) + \bar{\sigma}(B)$ $\underline{\sigma}(A) - \bar{\sigma}(B) \leq \underline{\sigma}(A+B) \leq \underline{\sigma}(A) + \bar{\sigma}(B)$

$\underline{\sigma}(A) - 1 \leq \underline{\sigma}(I+A) \leq \underline{\sigma}(A) + 1$ $\underline{\sigma}(A) - 1 \leq [\bar{\sigma}(I+A)^{-1}]^{-1} \leq \underline{\sigma}(A) + 1$

Condition Number

$\gamma(A) = \sigma_1(A)/\sigma_k(A) = \bar{\sigma}(A)/\underline{\sigma}(A)$ where $k = min(l, m)$

Norms

Properties: $\|e\|\geq 0$, $\underbrace{\|e\|=0 \Leftrightarrow e=0}_{semi-norm\ if\ this\ is\ not\ given}$, $\|\alpha\cdot e\|=|\alpha|\cdot\|e\| \ \forall\ \alpha\in\mathbb{C}$, $\|e_1+e_2\|\leq\|e_1\|+\|e_2\|$

Vector norms: $\|a\|_p=\left(\sum_i |a_i|^p\right)^{1/p}$, $\|a\|_1=\sum_i |a_i|$, $\|a\|_2=\sqrt{\sum_i |a_i|^2}$ $\left(a^H a=\|a\|_2^2\right)$

$\|a\|_\infty=\|a\|_{max}=max_i |a_i|$, $\|a\|_{max}\leq\|a\|_2\leq\sqrt{m}\|a\|_{max}$, $\|a\|_2\leq\|a\|_1\leq\sqrt{m}\|a\|_2$

Matrix norms: has the above mentioned properties plus $\|A\,B\|\leq\|A\|\cdot\|B\|$

$\|A\|_{sum}=\sum_{i,j}|a_{ij}|$, $\|A\|_F=\sqrt{\sum_{i,j}\|a_{ij}\|^2}=\sqrt{tr(A^H A)}$, $\|A\|_{max}=max_{i,j}|a_{ij}|$ ← not a mat. norm

Induced norm: $z=A\,w \Rightarrow \|A\|_{ip}=\max_{w\neq 0}\frac{\|A\,w\|_p}{\|w\|_p} = \max_{\|w\|_p\leq 1}\|A\,w\|_p=\max_{\|w\|_p=1}\|A\,w\|_p$

$\|A\|_{i1}=max_j\left(\sum_i |a_{ij}|\right)$, $\|A\|_{i\infty}=max_i\left(\sum_j |a_{ij}|\right)$, $\|A\|_{i2}=\bar{\sigma}(A)=\sqrt{\rho(A^H A)}$

Implications:
1. $B:=w$ $\Rightarrow \|A\,w\|\leq\|A\|\cdot\|w\|$
2. $\|A\|\geq\max_{w\neq 0}\frac{\|A\,w\|}{\|w\|}$
3. $A:=z^H$, $B=w$ $\Rightarrow |z^H w|\leq\|z\|_2\cdot\|w\|_2$

Spectral Radius

$\rho(A)=\max_i|\lambda_i(A)|$ is not a norm! For any norm: $\rho(A)\leq\|A\|$

Some Matrix Norm Relationships

$\bar{\sigma}(A)\leq\|A\|_F\leq\sqrt{min(l,m)}\,\bar{\sigma}(A)$ $\|A\|_{max}\leq\bar{\sigma}(A)\leq\sqrt{l\,m}\,\|A\|_{max}$ $\bar{\sigma}(A)\leq\sqrt{\|A\|_{i1}\|A\|_{i\infty}}$

$m^{-1/2}\|A\|_{i\infty}\leq\bar{\sigma}(A)\leq\sqrt{l}\,\|A\|_{i\infty}$ $l^{-1/2}\|A\|_{i1}\leq\bar{\sigma}(A)\leq\sqrt{m}\,\|A\|_{i1}$ $max\{\bar{\sigma}(A),\|A\|_F,\|A\|_{i1},\|A\|_{i\infty}\}\leq\|A\|_{sum}$

$\|U_1 A\,U_2\|_F=\|A\|_F$ $\bar{\sigma}(U_1 A\,U_2)=\bar{\sigma}(A)$ $\|A\|_F=\sqrt{\sum_i \sigma_i^2(A)}$

$\min_D\|D\,A\,D^{-1}\|_{i1}=\min_D\|D\,A\,D^{-1}\|_{i\infty}=\rho(|A|)=\max_i|\lambda_i(|A|)|$ where D is a diagonal matrix $\left(\rho(A)\leq\rho(|A|)\right)$

Signal Norms

l_p norm: $\|z(t)\|_p=\left(\int_{-\infty}^{\infty}\sum_i |e_i|(\tau)^p\,d\tau\right)^{1/p}$ ∞ -norm (peak value): $\|z(t)\|_\infty=\max_\tau\left(\max_i|e_i(\tau)|\right)$

Finite energy signal: $E_z=\|z\|_2^2=\int_{-\infty}^{\infty}|z(t)|^2 dt <\infty$ $E_z<\infty \Rightarrow P_z<\infty$

Finite power signal: $P_z=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|z(t)|^2 dt <\infty$ Root-mean square value (RMS) = $\sqrt{P_z}$

Power-norm: $\|z(t)\|_{RMS}=\lim_{T\to\infty}\sqrt{\frac{1}{2T}\int_{-T}^{T}\sum_i|e_i(\tau)|^2 d\tau}$ (semi-norm)

energy signal: finite energy, zero avg. power ($0\leq E<\infty$, $P=0$)
power signal: infinite energy, finite avg. power ($0<P<\infty$, $E=\infty$)
energy = $\int|f|^2$, power = avg. energy

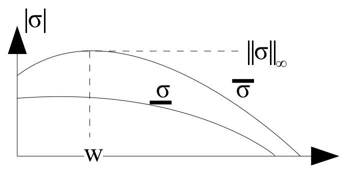
System Norms

H_∞: $G(s)$ is stable & proper, the matrix functions are analytic and bounded in the open right-half plane

$$\|G(s)\|_{H_\infty} = \|G(s)\|_\infty = \sup_{\Re(s)>0}\bar{\sigma}\{G(s)\} = \max_\omega\bar{\sigma}(G(j\omega)) = \max_\omega\max_{w(\sigma)}\frac{\|z\|_2}{\|w\|_2} = \max_{w(t)\neq 0}\frac{\|z(t)\|_2}{\|w(t)\|_2}\ \begin{matrix}\leftarrow \text{output}\\ \leftarrow \text{input}\end{matrix}$$

$$\|G(s)\|_{H_\infty} = \sup_{w\neq 0}\frac{E_z}{E_w} = \sup_{w\neq 0}\frac{P_z}{P_w} = \text{max energy/power gain}$$

$$\begin{bmatrix}z_1\\z_2\end{bmatrix}=G(s)\begin{bmatrix}w_1\\w_2\end{bmatrix} \quad , \quad G(s)=U\begin{bmatrix}\bar{\sigma} & 0\\ 0 & \underline{\sigma}\end{bmatrix}$$



H_2: $G(s)$ is strictly proper, $deg(N(s))<deg(D(s))$

H_2-space = all causal, finite energy signals.

H_2 is a closed subspace of $L_2(jR)$ with metric fct. $G(s)$ analytic in $\Re(s)>0$ → stable freq. sign.

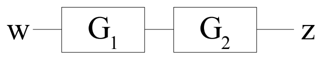
Perseval: $\|G(j\omega)\|_{H_2}^2=\|g(t)\|_{L_2}$ $\Rightarrow$ $E_f=\|f\|_{H_2}^2=\int_{-\infty}^{\infty}|f(t)|^2 dt=\frac{1}{2\pi}\int_{-\infty}^{\infty}|F(j\omega)|^2 d\omega=\|F(j\omega)\|_{H_2}^2$

Stability/Controllability Gramian

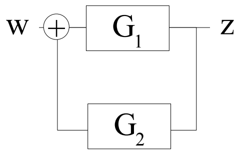
$L_O=\int_0^T e^{A(T-\tau)}BB^* e^{A(T-\tau)}d\tau$ $\quad A^*L_O+L_OA-C^*C=0$ $\quad \|G(s)\|_{H_2}=tr(B^*L_OB)=tr(CL_CC^*)$

$L_C=\int_0^T e^{A(T-\tau)}C^*C e^{A(T-\tau)}d\tau$ $\quad AL_C+L_CA^*+BB^*=0$

Transfer-Functions in MIMO-Systems

1. 

$z=G_2G_1w$

2. 

$z=G_1(I-G_2G_1)^{-1}w$

3. Push-through rule: $G_1(I-G_2G_1)^{-1}=(I-G_1G_2)^{-1}G_1$

Appendix B: Optimal Control

designs K to a given G

Cost function: $J(x)=\int_0^{\infty} x^TQx+u^TRu \; dt$ with tuning parameters {Q,R}

Optimal feedback: $u^*=-K^*x$ $\Rightarrow$ $\dot{x}=(A-BK)x=\bar{A}x$

$\Rightarrow J(x)=\int_0^{\infty}x^T(Q+K^TRK)x\,dt=\int_0^{\infty}x^T\bar{Q}x\,dt=\int_0^{\infty}x_0^Te^{\bar{A}^Tt}\bar{Q}e^{\bar{A}t}x_0\,dt=:x_0^TPx_0$

$P=\int_0^{\infty}e^{\bar{A}^Tt}\bar{Q}e^{\bar{A}t}dt \overset{partial}{\underset{integration}{=}} \ldots=-\bar{Q}\bar{A}^{-1}-\bar{A}P\bar{A}^{-1}$ $\Rightarrow$ $P\bar{A}+\bar{A}^TP=-\bar{Q}$ Lyapunov Eq.

optimality $\Rightarrow$ $\partial_K J(x)\overset{!}{=}0$ $\Rightarrow$ ... $\Rightarrow$ $\partial_K K^T(RK-B^TP)+(RK-B^TP)^T\partial_K K=0$

$\Rightarrow$ $K^*=R^{-1}B^TP$

Algebraic Ricotti Equation (ARE): $A^TP+PA-PBR^{-1}BP+Q=0$ $\left(\Leftarrow K^*\rightarrow P\bar{A}+\bar{A}P=-\bar{Q}\right)$

Properties of LQR:
- general assumption that system is <u>stabilizable</u> and <u>detectable</u>
- $\{R,Q\}>0$ $\Rightarrow$ closed-loop stable
- $PM>60°$, $GM\rightarrow\infty$
- state vector x <u>has to be known entirely</u> → observer design $\dot{\hat{x}}=A\hat{x}+Bu+(y-\hat{y})K_f$

$u=-K_r\hat{x}$

$\dot{x}=Ax-BK_r\hat{x}=(A-BK_r)x+BK_re$ $\quad (e=x-\hat{x})$

$\dot{e}=(Ax-BK_r\hat{x})-(A\hat{x}-BK_r\hat{x}+K_fCx-K_fC\hat{x})=(A-K_fC)e$

$$\Rightarrow \begin{bmatrix}\dot{x}\\ \dot{e}\end{bmatrix}=\begin{bmatrix}A-BK_r & BK_r\\ 0 & A-K_fC\end{bmatrix}\cdot\begin{bmatrix}x\\ e\end{bmatrix}$$

Separation Theorem: Poles of the closed-loop system are the poles of $A-BK_r$ + Poles of $A-K_fC$

$\Rightarrow$ controller and observer can be designed separately

Kalman: Combine LQR with Kalman filter → LQG (dual to LQR design)

$Q\rightarrow W$ $\quad R\rightarrow V$ $\quad B\rightarrow C^T$ $\quad x_0\rightarrow q$ $\quad K_r\rightarrow K_f^T$

$(E[ww^T])$ $\quad (E[vv^T])$ $\quad A\rightarrow A^T$

MATLAB:
[A,B,C,D]=ssdata(G);
[Ak,Bk,Ck,D]=lqg(A,B,C,D,QR,WV)
where QR="diag(Q,R)" and
WV="diag(W,V)"

LQG: + MIMO , "optimal" control , computation simple , {Q,R,W,V} more intuitive than {K_P I_I, I_D}

– indirect tuning , plant has to be known , robustness margins may disappear

– no integral action , no regulating , no reference tracking $\Rightarrow$ expansions available

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