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Reactive Power Management and Voltage Control

The Optimal Power Flow (OPF)

and its solution by the Interior Point Approach

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1 Model representation of linear network elements

1.1 Nport nodes

We assume that properties of the so-called Nport nodes is uniquely determined by its NPort equations.

The most general linear formulation which gives the algebraic dependencies among the NPort voltages U_{Nport_i} of the Nport nodes i and the NPort currents flowing into the NPort I_{Ntor_i} can be represented as follows:

$$M_{U_i} U_{Nport_i} + M_{I_i} I_{Nport_i} = b_i \tag{1}$$

M_{U_i} and M_{I_i} represent numerically given, complex matrices with dimension NxN, where N is the number of ports of the Nport nodes ii.

Vectors U_{Nport_i} und I_{Nport_i} have the dimension N and represent complex variables. The elements in U_{Ntor_i} and I_{Ntor_i} represent voltages and the currents (flowing into the NPort) at the N ports of the Nport node i. This is shown in Fig. 1.

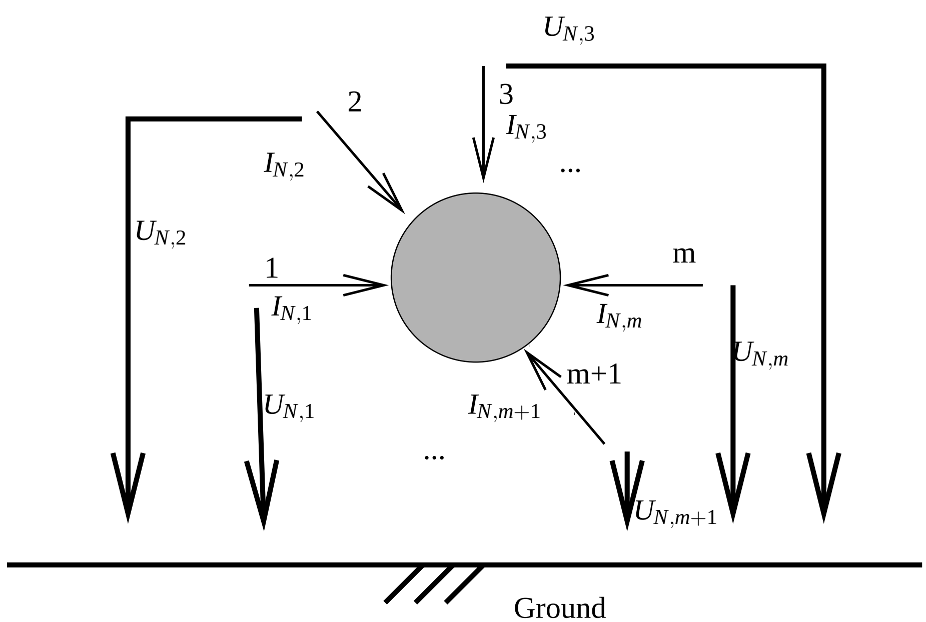


Figure 1: Nport voltages and Nport currents

Each Nport has its own constant and numerically given parameters which are part of the matrices M_{U_i} and M_{I_i} and the elements of the vector b_i.

1.2 Kirchhoff nodes

1.2.1 Sum of current = 0 and voltage potential identity

Currents always leave the Kirchhoff-nodes (this is only a convention) and enter the Nport node:

$$\sum_{i \in k} \underline{I}_{ki} = 0 \;\; (\text{k: all Kirchhoff-nodes}) \tag{2}$$

where the index i refers to all nodes k of connected NPort-node ports i.

In addition we can say that:

$$\underline{U}_k = \underline{U}_i \;\; \forall i \in k (\text{k: all Kirchhoff nodes}) \tag{3}$$

where the index i refers to all nodes k of connected NPort-node ports i.

1.2.2 Incidence matrix representation of Kirchhoff laws

(2) and (3) can be put in matrix form as follows:

$$\mathbf{C} \cdot U_{Kirchhoff} - U_{Nport} = 0 \tag{4}$$

where U_{Nport} is the vector of all NPort voltages. The vector $U_{Kirchhoff}$ represents all complex voltages of all Kirchhoff nodes.

The matrix C has the dimension *Number of Nport-node ports* $\cdot$ *Number of Kirchhoff nodes*.

The following is always valid with this matrix C:

$$\mathbf{C}^T \cdot I_{Nport} - I_{Kirchhoff} = 0 \tag{5}$$

However, since by definition we have

$$I_{Kirchhoff} = 0 \tag{6}$$

we have:

$$\mathbf{C}^T \cdot I_{Nport} = 0 \tag{7}$$

In the following, we assume the following given relationship for all Nport nodes which can be modelled by a linear relationship between Nport voltages and currents:

$$diag(M_{U_i}) U_{Nport} + diag(M_{I_i}) I_{Nport} = b \tag{8}$$

where $diag(M_{U_i})$ is a block-diagonal form of all NxN - matrices M_{U_i}, $diag(M_{I_i})$ is the block-diagonal form of all NxN - matrices M_{I_i} and the vector b represents the block vector form of all individual NPort vectors b_i .

2 Mathematical optimal power flow dispatch problem formulation

2.1 The power flow model as equality constraint set of the OPF

In this text it is assumed that details of the regular power flow problem and its solution are known.

The set of **regular power flow** equations is as follows:

	Equations	No. of equations
1,2,3	$\begin{bmatrix} \begin{bmatrix} diag(M_{I_i}^{Net}) & 0 \end{bmatrix} & \begin{bmatrix} diag(M_{U_i}^{Net}) & 0 \end{bmatrix} & 0 \\ C^T & 0 & 0 \\ 0 & -E & C \end{bmatrix} \cdot \begin{bmatrix} I_{Nport} \\ U_{Nport} \\ U_{Kirchhoff} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	NPNet(complex) + NP(complex) + NODES(complex)
4	$P_{Nport} + jQ_{Nport} - diag(U_{Nport_i}) I_{Nport}^* = 0$	NP(complex)
5	$P_{generator/load}^0 - P_{Nport}^{generator/load} = 0$	NPGE + NPLD - 1 (real)
6	$Q_{load}^0 - Q_{Nport}^{load} = 0$	NPLD (real)
7	$U_{generator}^0 - \left\| U_{Nport}^{generator} \right\| = 0$	NPGE(real)
8	$angle^0 - \angle(U_{Kirchhoff_n}) = 0$	1 (real)

(9)

Here the symbols have the following meaning:

Symbol	Explanation
(complex)	Complex-number equations; this indicates that the number of real-variable equations (transformed from complex-number equations) is 2*(complex).
(real)	real-number equations
NP	Total number of ports of all N-port elements (all inclusive)
NPNet	Number of ports of all N-ports related to network elements (i.e. excluding generators and load 1-ports)
NODES	Total number of Kirchhoff nodes
NPGE	Number of ports of all Generator N-port (N=1) elements
NPLD	Number of ports of all Load N-port (N=1) elements

Note that it is assumed that NPLD + NPGE + NPNet = NP.

The following table summarizes the unknown and known variables of these equations (1)

... (8) of (9):

Known variables	Unknown variables	Number
	I_{Nport}	$NP(complex)$
	U_{Nport}	$NP(complex)$
	P_{Nport}	$NP(real)$
	Q_{Nport}	$NP(real)$
	$U_{Kirchhoff}$	$NODES(complex)$
$P^0_{generator/load}$		$NPGE(real) + NPLD(real) - 1$
$U^0_{generator}$		$NPGE(real)$
Q^0_{load}		$NPLD(real)$
$angle^0$		1

These two tables describe a conventional power flow problem. In order to understand **the difference of these ordinary power flow equations to the equality set of the optimal power flow (OPF)** note the variables which have an upper index ^{0}: $P^0_{Generator_i}$ (Given active power values at generator nodes), $|V|^0_{Generator_i}$ (given voltage magnitude values at generator nodes), $P^0_{Load_i}$ and $Q^0_{Load_i}$ (given active and reactive power values at load nodes), $angle^0$ (given slack node complex voltage angle).

All these variables are assumed to be numerically known **in the ordinary power flow** model. With these given variables a set of non-linear equations results with identical number of unkown variables and equations. With this set of equations a Newton-Raphson solution process can be executed. The number of equations and unknowns is identical, i.e. $6 * NP + 2 * NODES$.

However, in the optimal power flow model some or even all of these variables with an upper index of ^{0} will not be given any more and will be declared as unknowns. It is up to the optimization algorithm to determine optimal values for these unknown variables.

In order to have slightly easier formulas, it will be assumed in this paper that **all** power flow variables related to **generators** which have an index ^{0} in the power flow equations above will be unknown variables with the exception of the slack node angle $angle^0$ which cannot be assumed to be free (phasor theory asks for at least one exactly defined voltage angle). This means that $P^0_{Load_i}$ and $Q^0_{Load_i}$ (given active and reactive power values at load nodes) are numerically given at all load nodes and are not unknown variables.

Using the terms of optimization the set of equations which remain valid is called the **OPF equality constraint set**. This set of equations is summarized as follows:

	Equations	No. of equations
$1/2/3_{OPT}$	$\begin{bmatrix} \begin{bmatrix} diag(M^{Net}_{I_i}) & 0 \end{bmatrix} & \begin{bmatrix} diag(M^{Net}_{U_i}) & 0 \end{bmatrix} & 0 \\ C^T & 0 & 0 \\ 0 & -E & C \end{bmatrix} \cdot \begin{bmatrix} I_{Nport} \\ U_{Nport} \\ U_{Kirchhoff} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$	
		NPNet(complex)
		+ NP(complex)
		+ NODES(complex)
4_{OPT}	$P_{Nport} + jQ_{Nport} - diag(U_{Nport_i})I^*_{Nport} = 0$	NP(complex)
5_{OPT}	$P_{generator} - P^{generator}_{Nport} = 0$	NPGE (real)
6_{OPT}	$P^0_{load} - P^{load}_{Nport} = 0$	NPLD (real)
7_{OPT}	$Q^0_{load} - Q^{load}_{Nport} = 0$	NPLD (real)
8_{OPT}	$U_{generator} - \left\lvert U^{generator}_{Nport} \right\rvert = 0$	NPGE(real)
9_{OPT}	$angle^0 - \angle(U_{Kirchhoff_n}) = 0$	1 (real)

(10)

The following table summarizes the unknown and known variables of these equations (1_{OPT}) ... (9_{OPT}) of (10):

Known variables	Unknown variables	Number
	I_{Nport}	$NP(complex)$
	U_{Nport}	$NP(complex)$
	P_{Nport}	$NP(real)$
	Q_{Nport}	$NP(real)$
	$U_{Kirchhoff}$	$NODES(complex)$
	$P_{generator}$	$NPGE(real)$
	$U_{generator}$	$NPGE(real)$
P^0_{load}		$NPLD(real)$
Q^0_{load}		$NPLD(real)$
$angle^0$		1

In this equation set the number of equations is $6*NP+2*NODES$ and the number of unknown variables is $6*NP+2*NODES+2*NPGE$.

Thus a high degree of freedom exists: There are theoretically an infinite number of states which satisfy these equality constraints.

Thus whenever one talks about the power flow as equality constraint set of the optimal power flow one has to be aware that many generator related variables which are assumed to be known in the regular power flow, are unknown variables in the optimal power flow.

In a typical optimization problem, this degree of freedom will be reduced by the choice of the **inequality constraint set and an objective function** the contours of which should point to the solution with an extreme objective function value.

2.2 Mathematical formulation of operational constraints

2.2.1 Introduction

In the preceding section a power system model and needed equality constraints have been formulated. Satisfying these equality constraints with any numerical set of variables means that the physical characteristics of the power system under certain conditions are satisfied. The problem is that many of these physically possible states do not make operational sense or are not operationally possible. Thus in order to model the power system behavior more realistically additional constraints have to be formulated. Different types of operational constraints can be formulated:

- Physical damage to network equipment must be prevented since power system equipment is often very expensive and hard to repair.
- Laws dictate mandatory standards to be satisfied by all utilities, e.g. the voltage amplitude at a node must be within certain upper and lower limits, or: a power end user must have power available at almost 100% during the year, or: if any network element is unvoluntarily outaged the power system must be brought back to an acceptable network state within a given time period.
- Physically given limits for power system sources (any generator has its upper power limit).
- Power consumption at certain discrete time steps force the power to flow from the generators via the transmission system to predefined geographical places, i.e. the power consumed by loads at certain nodes at certain times is given and must be considered.
- Contracts among utilities determine precisely at what times how much power must be imported or exported from one utility to another. This imposes limitations on the power system operation and also on its model.
- Operational limits being computed by other power system problem analysis areas like network stability determine e.g. the maximum allowable complex voltage angle shift from node i to another node j.
- Human operators cannot implement more than say 10 % of all possible controls manually within a give short time period.

These examples show that a huge number of operational constraints exist which must be translated into mathematical constraint types.

These mathematical constraints are derived in the following subsections. Three main constraint groups are identified: The **transmission constraints**, representing all operational limits of the actual network. The **contingency constraints** are related to all operational aspects if any network element is outaged as compared to the actual network and its associated network state.

2.2.2 Transmission constraints

Transmission constraints are always related to the actual network, i.e. to a network with topology. In this network every N-port element has its own physical limits related to currents, voltages and powers.

N-port current magnitude - maximum limit I^{max}_{Nport} of port current

For example, the maximum current magnitude values for transmission lines and transformers are given due to limitation of the branch material. Excessive currents would damage the transmission elements.

Because N-port currents I_{Nport} are available as explicit unknowns in the OPF formulation part related to equality constraints (see preceding subsection), they can easily be formulated as follows:

$$|I_{Nport}| \leq I^{max}_{Nport} \tag{11}$$

These N-port related inequality constraints can be written using real variables. The complex variables are transformed into the cartesian coordinate system as follows:

$$I_{Nport_i} \quad = Ie_{Nport_i} + jIf_{Nport_i}$$

(11) can be rewritten with real variables as follows:

$$Ie^2_{Nport} + If^2_{Nport} \leq \left(I^{max}_{Nport}\right)^2 \tag{12}$$

Note, that this is a non-linear inequality constraint using real variables.

N-port MVA-power - maximum limit S^{max}_{Nport} at each port

The same reason as the one for branch maximum current limit, discussed before, is valid.

Because N-port active P_{Nport} and reactive powers Q_{Nport} are available as explicit unknowns in the OPF formulation part related to equality constraints (see preceding subsection), they can easily be formulated as follows:

$$P^2_{Nport} + Q^2_{Nport} \leq \left(S^{max}_{Nport}\right)^2 \tag{13}$$

Lower V^{min}_{Nport} and upper V^{max}_{Nport} voltage magnitude limits at ports of all N-ports

Very strict standards often dictate these limits. Too high or too low voltages (magnitudes) could cause problems with respect to end user power apparatus damage or instability in the power system. This could lead to unwanted and economically expensive partial unavailability of power for end users.

$$V^{min}_{Nport} \leq |U_{Nport}| \leq V^{max}_{Nport} \tag{14}$$

(14) is valid for every port of all N-ports of the network. These complex N-port related inequality constraints can be written using real va riables. The complex variables are transformed into the cartesian coordinate system as follows:

$$U_{Nport_i} \quad = e_{Nport_i} + jf_{Nport_i}$$

(14) can be rewritten with real variables as follows:

$$\left(V_{Nport}^{min}\right)^2 \leq e_{Nport}^2 + f_{Nport}^2 \leq \left(V_{Nport}^{max}\right)^2 \tag{15}$$

Note, that this is a non-linear inequality constraint using real variables.

There are ports (often a subset of the generator node ports) where the upper and lower voltage limits are identical, i.e. the voltage magnitude of these node is numerically given.

Lower $P_{Generator}^{min}$ and upper $P_{Generator}^{max}$ generator active power limits

The active power of a generator i is defined to be the real part of the complex variable $\underline{S}_{Generator_i}$. This quantity is physically limited in each generator.

$$P_{Generator}^{min} \leq P_{Generator} \leq P_{Generator}^{max} \tag{16}$$

(16) must be formulated for every generator 1-port element. Often the lower limit is zero.

Lower $Q_{Generator}^{min}$ and upper $Q_{Generator}^{max}$ generator reactive power limits

The reactive power of a generator i is an important measure of voltage magnitude quality, e.g. a low voltage indicates a local shortage of reactive power. Reactive power of generator 1-port elements is usually limited:

$$Q_{Generator}^{min} \leq Q_{Generator} \leq Q_{Generator}^{max} \tag{17}$$

(17), where $Q_{Generator}$ represents the unknown reactive power at every generator 1-port, must be formulated for every generator. Due to the fact, that the unknown $Q_{Generator}$ is not an explicit unknown variable in the equality constraint set of the preceding section, the following equality constraint set has to be (conceptually) added whenever the inequality constraints (17) must be valid in the OPF optimum:

$$Q_{generator} - Q_{Nport}^{generator} = 0 \tag{18}$$

The upper and lower reactive power limits are often not given as numeric values but as functions of the active generator power:

$$\begin{aligned} Q_{Generator_i}^{min} &= f_{min_i}(P_{Generator_i})) \\ Q_{Generator_i}^{max} &= f_{max_i}(P_{Generator_i})) \end{aligned} \tag{19}$$

This is graphically shown in Fig. 2.

Other operational constraints

Many other operational constraints must be formulated which includes minimum generator active power spinning reserve, transformer tap limits, upper and lower limits on total active power of a given set of branches (transfer limits), upper and lower voltage angle difference between Kirchhoff nodes of the network in order to obtain a stable network state (no instability in the network).

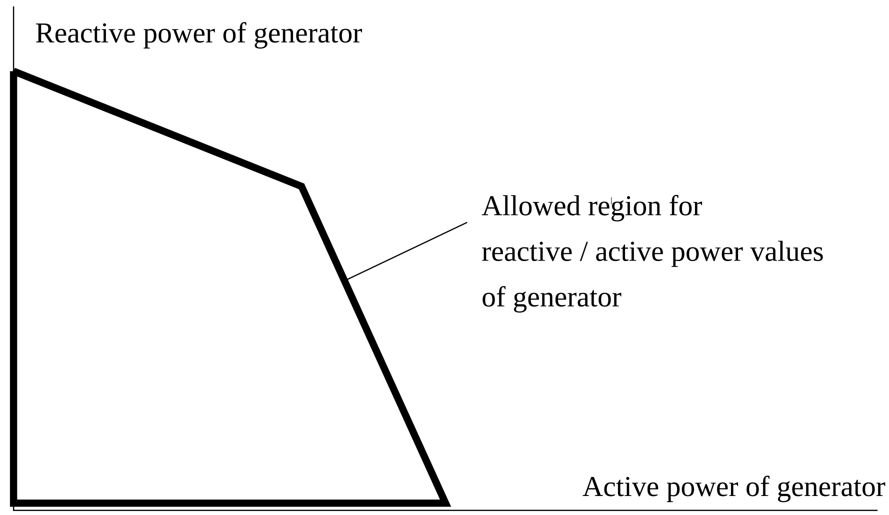


Figure 2: Allowed range of active/reactive power of a generator

2.2.3 Control variable constraints

In a power system the operator can only control certain quantities. A control always has an associated control apparatus which allows to influence the related value quickly.

These quantities are called controls and represent a physical apparatus to implement any operationally feasible and acceptable network state. Each control has an associated control variable. The most important control variables are the following:

- Generator active power control $P_{Generator}$: This is an explicit unknown variable in the chosen OPF optimization formulation.
- Generator reactive power control $Q_{Generator}$: This is an explicit unknown variable in the chosen OPF optimization formulation.
- Generator voltage magnitude control $U_{Generator}$: This is an explicit unknown variable in the chosen OPF optimization formulation.
- Many other controls exist: Phase shifter transformer tap position control, In-phase transformer tap position control, Shunt value control, Active power interchange transaction control. If they are to be used as explicit unknowns in the OPF and they are not yet modelled in the equality constraint set, an explicit function of these new unknown variables must be (conceptually) given as equality constraints.

The above mentioned variables are called control variables and represent quantities which can be directly influenced by the power system operator. The mathematical representation of control variables as used in this text is **u**.

All other unknown variables are denoted with the variable **x** and are called state variables[1].

[1] Note, however, that throughout this text, the unknown variables x can also refer to the complete set of unknowns, i.e. also including the unknown control variables.

Controls always have a lower and an upper limit due to physical reasons. Also, it is important to understand that the power system cannot realize the state of the desired control values immediately at the time t when the control variables and the associated physical controls are changed by the operator: Due to the dynamic nature of the power system there is always a time delay from changing the control setpoint values to the time when the power system actually shows the desired values. Also, it does not make sense to change the controls to values which cannot be realized in the power system at all or only after quite a long time.

Thus there is an additional set of inequalities, representing the maximum difference between actual power system state and the state where the power system can be moved from this state within the time frame for which the OPF result is valid. This time frame corresponds often to the discrete time interval in which one OPF calculation is done.

These additional lower and upper control variable constraints are formulated as follows in general form:

$$\mathbf{u}^{min} \leq \mathbf{u} \leq \mathbf{u}^{max} \tag{20}$$

In (20), the limit vectors are derived from the physical controller limits, from the values of the actual network state and the ability to move controls within a given time period. These limit values $\mathbf{u}^{min}, \mathbf{u}^{max}$ can be assumed to be numerically given.

2.3 Contingency constraints

In the preceding subsection all equality and inequality constraints refer to the actual network state. This means that the OPF transmission constraints are based on a given network structure with known branch, shunt, and generator statuses. It has been assumed that all these elements are in a 'switched-in' status. The network with 'switched-in' element-status is called the 'actual network'.

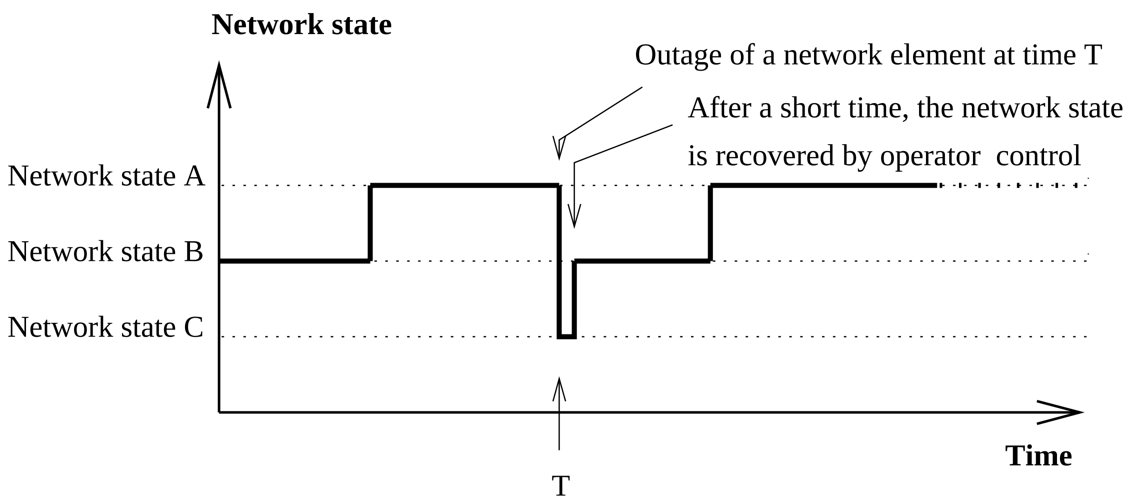
In addition to satisfying the transmission constraints for this 'actual network', law or economic reasons force utilities not only to satisfy all transmission constraints of the 'actual network', but also to satisfy the so-called 'n-1-security' based constraints which can be defined as follows:

The state of the 'actual network' is defined as 'n-1-secure' with post-contingency rescheduling at any time T if

1. all operational constraints of the 'actual network' are satisfied, i.e. the **transmission constraints and if**

2. any one of the branches, shunts or generators is outaged (representing a **contingency**), the new network state must be such that within a given short time period τ_{short}, the 'new actual network' can be moved to a transmission constrained state by control means. Since this control variable movement occurs immediately after the contingency happens it is called **post-contingency control movement**.

Figure 3 shows typical network state changes caused by the outage on an element:



Comments:

network state changing over time

Network state A: "n-1" corrective secure network state

Network state B: Transmission-constrained network state

Network state C: State with violated transmission constraints

Figure 3: Network state changes due to the outage of a network element at time t

Many utilities choose the time for recovering the network state after the contingency has occured to a transmission constrained network state, i.e. τ_{short} as zero. This means that the utility wants to operate its actual network in such a state that if any contingency occurs the new network state after the contingency is immediately such that no transmission constraints are violated.

It is important to note that it can be assumed that the control variables $\mathbf{u}$ do not change from the network state immediately before to immediately after the contingency occurs. Only the OPF variables $\underline{\mathbf{x}}$ change immediately due to the physical behavior of the power system. Thus a contingency can immediately lead to operational constraint violations which must be corrected by moving the control variables $\mathbf{u}$ within a short time τ_{short}.

A dynamic simulation not being the objective, the recovery time effect is translated into a maximum control variable move after the outage occurs (the so-called post-contingeny control variable movement).

It is known how long it takes to move a certain power system control by a certain amount, e.g. the active power of a generator from a generation of 100 MW to 130 MW. Thus it is assumed that the maximum possible and operationally acceptable movements of all control variables are known. The resulting new inequality constraint set representing the maximum allowable upward and downward movements of all control variables per outage is as follows:

First the maximum movements for all control variables are computed numerically. Note that different values can be chosen per contingency i:

$$\Delta\mathbf{u}^{(i)^{max}} = \tau_{short} \cdot \Delta\mathbf{u}/\mathbf{Min.}^{(i)^{max}} \tag{21}$$

The scalar value τ_{short} is given (in Minutes) and the vector $\Delta\mathbf{u}/\mathbf{Min.}^{(i)^{max}}$ represents the numerically given maximum possible movements per Minute for each OPF control variable.

$$\begin{aligned} \mathbf{u} - \Delta\mathbf{u}^{(i)^{max}} \quad \leq \mathbf{u}^{(\mathbf{i})} \leq \mathbf{u} + \Delta\mathbf{u}^{(i)^{max}} \\ \text{i: all possible contingencies} \end{aligned} \tag{22}$$

In (22) the vector $\mathbf{u}^{(\mathbf{i})}$ refers to the control variable set valid at time $t + \tau_{short}$, assuming that the outage of network element i has occured at time t. I.e. for every possibly outage network element i such a vector is created, representing the new state at time $t + \tau_{short}$. The size of the vector $\mathbf{u}^{(\mathbf{i})}$ per outage element i is the same as the size of $\mathbf{u}$, with the exception of a control variable related to an outaged network element.

Since the new network state after the outage must satisfy the transmission constraints the model must satisfy the same equality and inequality constraints like in the 'actual network' state. Thus for each outage case i the following equality and inequality constraints have to be formulated with the new outage state related variable set as defined above:

$$\mathbf{g}^{(\mathbf{i})}(\underline{\mathbf{x}}^{(\mathbf{i})}, \mathbf{u}^{(\mathbf{i})}) = \mathbf{0} \;\; ; \text{i: all possible contingencies} \tag{23}$$

$$\mathbf{h}^{(\mathbf{i})}(\underline{\mathbf{x}}^{(\mathbf{i})}, \mathbf{u}^{(\mathbf{i})}) \leq \mathbf{0} \;\; ; \text{i: all possible contingencies} \tag{24}$$

In summary, by formulating contingency constraints the following points are significant:

- The number of variables is increased to (n = number of possible contingencies) n times the number of variables of the 'actual network', i.e. the problem is tremendously increased.
- The absolute values for the difference of control variables of the 'actual network' and corresponding control variables of the outage cases must be less or equal to a numerically given value.
- A network state valid after the contingency has occured, must satisfy all operational equality and inequality constraints. The difference in the symbolic formulation to the actual network equations are as follows:
  - A new set of variables is used for each contingency.
  - The outaged network element must not be modelled in the equality constraints
  - If there is an inequality constraint for the outaged element this inequality constraint must not be formulated for the contingency network state.

Term	Description
Actual network	Network with given branch, shunt and generator statuses. I.e. it is known which of these elements are switched in and which ones are switched out.
Contingency i network	This is the network where one element i is outaged as compared to the 'actual network'. If the outage is lasting for a long time this new network will be a new 'actual network'.
Base case network state	This is any network state (i.e. voltages, currents, powers) computed or measured at an 'actual network'. Only the power flow equality constraints of the 'actual network' are satisfied. There can be violations on all kinds of operational constraints. It is the most general state from which an OPF optimization can be started. One of the goals of the operator is to shift the network from this network state via the 'transmission constrained network state' to the 'n-1-secure network state'.
Transmission constrained network state	An 'actual network' is in this state if all transmission constraints are satisfied (some of the contingency constraints are violated).
n-1-secure network state	This is a computed state of the 'actual network' where all operational constraints including those for contingencies are satisfied.

Table 1: State and network naming definitions

2.4 Overview: Network type, network state and constraint set related term definitions

Important goals of the power system control are the reliable and economic operation. These goals are translated into OPF problem parts: Reliability is translated into the operational constraints, economy into the objective function (discussed later in this paper). These operational constraints can be grouped and important terms have been introduced in this section. They are summarized in Table 1.

2.5 OPF objectives and objective functions

2.5.1 Introduction

An important part in any mathematical optimization problem is the objective function which allows to make a distinct and often unique, optimal selection out of the solution region defined by the equality and inequality constraints. Also, the objective function is needed to drive a mathematical optimization process towards an optimal solution.

As already discussed before utilities can have different goals or power system operation objectives. Some goals are clearly defined like minimum active power losses in the resistive parts of the transmission system branches or minimum total cost for the active power generation of the generators. These objectives are of economical nature and can thus easily be justified.

On the other side less clear goals exist which can be of the following types and often depend on operational utility policies: If the power system is monitored and if the power system is found to be in a state where either transmission or contingency constraints are violated, operate the system in such a way that the violations are eliminated as quickly as possibly.

In this text only the classical objective functions are discussed in more detail: The active power losses and the total operating cost.

2.5.2 Mathematical formulation of various OPF objective functions

Objective: Minimum active power losses

The active power losses (called 'losses' from now on) represent a quantity whose minimization can easily be justified by an electric utility if the effort to actually operate in a minimum loss mode is not too expensive. Losses are generated in the resistances of the transmission lines and are a measure of the difference of the generated total active power to the total active load at any time.

Two different loss-related cases exist: If the network model comprises only network parts of the own utility controlled area the losses are computed as follows:
Loss case A: Total network losses (TL):

$$\text{Minimize } P_{Loss} = 1^T \cdot P_{Generator} + P^o_{Loss} \tag{25}$$

where 1^T is a transposed vector of only 1's, where P^o_{Loss} is a constant and computed as follows:

$$P^o_{Loss} = -1^T \cdot P^0_{load} \tag{26}$$

Note that in this case the objective function is a function of control variables only, i.e.

$$P_{Loss} = f_{TL}(\mathbf{u}) \tag{27}$$

Loss case B: Partial (area) network losses (PL):

Here the modelled network comprises network parts only. For this case B the losses are the sum of the active powers of each branch of the desired area. For each branch the active power flow from node i to node j must be added to the branch flow from node j to node i. This results in the active power losses per branch. Thus the losses for a defined area 'a' are computed as follows:

$$\text{Minimize } P_{Loss}^{\text{area 'a'}} = \sum_{i \in \text{area 'a'}} P_{Nport_i} \tag{28}$$

Note that in this case the objective function is a function of state variables $\mathbf{x}$ only, i.e.

$$P_{Loss}^{\text{area 'a'}} = f_{PL}(\mathbf{x}) \tag{29}$$

Objective: Minimum total active power operating cost

Most utilities have control over generation of different kind, e.g. hydro, hydro-thermal or thermal power generation. Each type of generation has cost associated with it: Hydro power is, if available, usually the cheapest power (water does not cost much in mountain areas), however it has very high fixed capital cost. Thermal power generation is more expensive (often oil, gas or nuclear material is used as primary energy resource. These resources have to be bought at market prices when storage place is limited).

In addition to own power resources the utility can buy or sell power from or to neighbouring utilitities. Sometimes it can be cheaper to buy power than to produce it within the power system control area.

It is in the interest of the utilities and also of the paying power consumers to minimize the cost associated with active power generation. It can be assumed that the cost of each generator i, i.e. C_i can be represented as a distinct curve of relating cost to the active power which the generator delivers. Usually this curve is given for the full range of the operating capability of the generator. The general type of a cost curve can be written as follows:

$$C_i = c_i(P_{Generator_i}) \tag{30}$$

where c_i is a general function of the active power of the generator i. A typical cost curve is shown in Fig. 4.

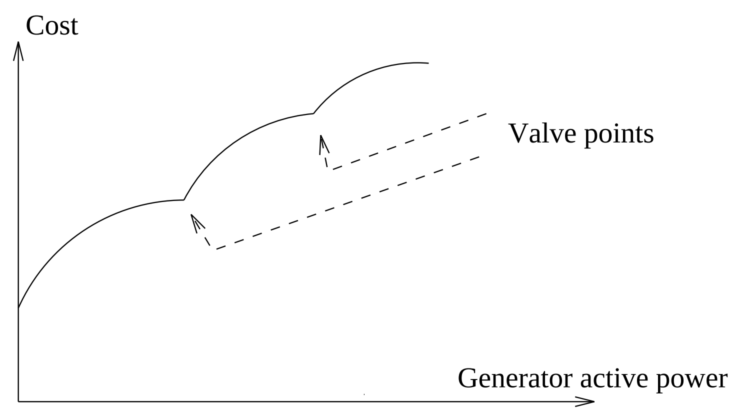


Figure 4: Typical true cost curve for hydro-thermal power generation

As seen in Fig. 4 the general cost curve type is very complex: Non-convex cost curves are possible. Also, it is important to note that the cost curves can be assumed to be separable with respect to the active power generation of the generatorsm, i.e. the cost of each generator is only dependent on the cost of its own active power generation and not on the cost of another generator power.

The problem is the shape of the cost curves. Optimization algorithm usually cannot deal with cost curve shapes as shown in Fig. 4. Thus, the cost curves are usually modified and a convex, smooth cost curve as shown in Fig. 5 can be assumed to be the cost curve model for the OPF.

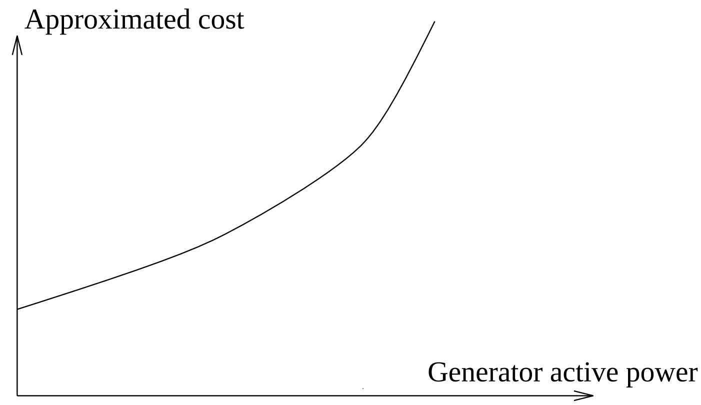


Figure 5: Corrected smooth and convex power generation cost curve for OPF model

Often these cost curves are assumed to be piecewise quadratic with smooth transition at the cost curve break points. Also, exponential cost curve representations are seen in OPF models. Usually the approximated smooth and convex cost curves (one per active power generation and interchange power) is given as the exact reference along which the OPF optimum solution must be found. However, it should be kept in mind that an inaccurate cost curve model can prevent an accurate optimization result, even if the remaining model parts are model with very high accuracy or in other words, the optimization result is not more accurate than the chosen accuracy of the model components.

The total cost of all generators and transaction interchange active power can be represented as objective function as follows:

$$\text{Minimize } C \quad = \sum_{i=(l+1)}^{N} c_{approximated_i}(P_{Generator_i}) \tag{31}$$

It is obvious that the objective function is a function of control variables only, i.e.

$$\text{Cost} = f_c(\mathbf{u}) \tag{32}$$

2.6 The complete OPF formulation

2.6.1 Mathematical OPF formulation - Summary

The OPF formulation can be split in three main parts: The equality constraints, the inequality constraints and the objective function. Each of these three parts has its own properties, as discussed in the preceding sections. The following three tables summarize the OPF problem:

Equality constraints:

Description	Mathematical formulation	Equations
Actual network state	$\mathbf{g}(\mathbf{x},\mathbf{u}) = 0$	(1_{OPT}) ... (9_{OPT}), see page 4
Contingency network state	$\mathbf{g}_{CCi}(\mathbf{x_{CCi}},\mathbf{u_{CCi}}) = 0$	(1_{OPT}) ... (9_{OPT}), see page 4

Inequality constraints:

Description	Mathematical formulation	Equations
Transmission constraints - Actual network state	$\mathbf{h}(\mathbf{x},\mathbf{u}) \leq 0$	(11) ... (19)
Control variables contraints - Actual network state	$\mathbf{u} \leq u_{max}$ $\mathbf{u} \geq u_{min}$	(20)
Contingency constraints - Contingency network state	$\mathbf{h}_{CCi}(\mathbf{x_{CCi}},\mathbf{u_{CCi}}) \leq 0$	(11) ... (19)
Control variables contraints - Contingency network state	$\mathbf{u}_{CCi} \leq u_{max}$ $\mathbf{u}_{CCi} \geq u_{min}$	(20)
Control variables contraints - Contingency network state	$\lvert\mathbf{u}_{BC} - \mathbf{u}_{CCi}\rvert \leq \Delta u_{max}$	

Typical OPF objective functions:

Description	Equations
Total active power losses	$f_{TL}(\mathbf{u}) \longrightarrow$ Min.
Partial (area) active power losses	$f_{PL}(\mathbf{x}) \longrightarrow$ Min.
Total active power operating cost	$f_c(\mathbf{u}) \longrightarrow$ Min.

Note that only one of the possible objective functions together with all equality and inequality constraints can be chosen and can be solved for one optimal solution variable set.

Here another problem with the present OPF formulations and solutions becomes obvious: Although a utility would like to satisfy more than one objective at the same time, the classical mathematical optimization algorithms allow only one objective function at a time.

2.6.2 OPF problem size and OPF solution discussion

The OPF formulation as shown in the preceding sections leads to huge non-linear optimization problems.

- The OPF formulation comprises a huge number of variables and equality / inequality constraints.
- The equations are usually non-linear. The OPF problem can be characterized as an almost quadratically constrained optimization problem.
- The equality constraint set describes the power flow relationships. Thus the underlying network characteristics are modelled there. This implies that the linearized equality constraints will be based on sparse linear system matrices with a very high number of zero matrix elements. This must be considered in any solution algorithm related to OPF problems.
- Since the OPF has no characteristics of well known mathematical problems such as LP, QP (solved by Simplex or Interior Point) or equality constrained optimization (solved by Newton's method), the problem has to be solved by using a combination of these well known techniques. Approximations of the OPF problem will be needed.

At this point of time there is no single best algorithm which solves the OPF problem. It is the power system engineer who must understand

- the power flow with its underlying characteristics,
- the Newton-Raphson solution process to get a solution for a well-determined non-linear system of equations,
- the sparsity aspects of the solution of a sparse linear system of equations,
- the solution of unconstrained and equality constrained optimization problems,
- the application of the Karush-Kuhn-Tucker (KKT) optimality conditions to a general non-linear optimization problem,
- the established solution methods for LP and QP problems.
- the handling of contingency and operational constraints which lead to extremely large and mathematically difficult problems.

Putting all knowhow together leads to different solution approaches of the OPF problem. They are discussed in the next subsection.

2.7 Classification of algorithms to solve the non-linear OPF problem

The separation of OPF algorithms into classes is mainly governed by the fact that very powerful methods exist for the ordinary power flow which provide an easy access to intermediate solutions in the course of an iterative process. Further, it can be observed that the optimum solution is usually near an existing power flow solution and hence sensitivity relations lead the way to the optimum. Hence, one class exists which relies on a solved power flow and on tools provided by the power flow.

The second class originates from a rigorous formulation of the OPF problem, employs the exact optimality conditions and uses techniques to fulfill the latter. In this case a solved load flow is not a prerequiste.

There are advantages and disadvantages in both methods which have a certain bearing depending on the objective, the size of the problem and the envisaged application.

Hence, optimal power flow algorithms can be put in two classes:

- **Class A**: Methods whereby the optimization starts from a solved power flow. The Jacobian and other sensitivity relations are used in the optimizing process. The process as a whole is iterative. After each iteration the power flow is solved anew.
- **Class B**: Methods relying on the exact optimality conditions whereby the power flow relations are attached as equality constraints. There is no prior knowledge of a power flow solution. The process is iterative and each intermediate solution approaches the optimality conditions which include the power flow equations.

3 OPF class B: Integrated iterative solution of KKT-optimality conditions

3.1 Introduction

In this section the OPF formulation is solved by an integrated method as compared to the OPF formulation of the Class A where the power flow is separated from the optimization part.

In this text one approach is discussed. It is the so-called non-linear Interior Point (IP) approach which is based on an efficient solution of the non-linear KKT optimality conditions using a combination of Newton-Raphson, step-length control and barrier function parameter decrease during all iterations.

Other algorithms are found in the literature in this OPF class B. All class B algorithms have in common that an iterative solution of possibly transformed non-linear Karush-Kuhn-Tucker (KKT) optimality conditions will be achieved.

3.2 Solution of KKT by Interior Point algorithm

3.2.1 KKT optimality conditions

The problem is defined as follows:

$$\begin{aligned} \text{Minimize} \quad & F(\mathbf{x}) \qquad (33) \\ \text{subject to} \quad & \mathbf{g}(\mathbf{x}) = \mathbf{0} \\ \text{and} \quad & \mathbf{h}(\mathbf{x}) \leq \mathbf{0} \end{aligned}$$

Note that only variables $\mathbf{x}$ are used in the class B approach. This is slightly different from the class A approach where a distinction between the control and state variables is advantageous.

The optimality conditions can be derived by formulating the Lagrange function L:

$$L = F(\mathbf{x}) + \lambda^T \mathbf{g}(\mathbf{x}) + \mu^T \mathbf{h}(\mathbf{x}) \qquad (34)$$

The first order necessary KKT optimality conditions are as follows:

$$\begin{array}{rcrcl} \dfrac{\partial L}{\partial \mathbf{x}} & = & \dfrac{\partial}{\partial \mathbf{x}} \left(F(\mathbf{x}) + \widehat{\lambda}^T \mathbf{g}(\mathbf{x}) + \widehat{\mu}^T \mathbf{h}(\mathbf{x}) \right) |_{\widehat{\mathbf{x}}} & = & \mathbf{0} \\ \dfrac{\partial L}{\partial \lambda} & = & \mathbf{g}(\widehat{\mathbf{x}}) & = & \mathbf{0} \\ \mathbf{Diag}\{\widehat{\mu}_\mathbf{i}\} \dfrac{\partial L}{\partial \mu} & = & \mathbf{Diag}\{\widehat{\mu}_\mathbf{i}\}\, \mathbf{h}(\widehat{\mathbf{x}}) & = & \mathbf{0} \\ \dfrac{\partial L}{\partial \mu} & = & \mathbf{h}(\widehat{\mathbf{x}}) & \leq & \mathbf{0} \\ & & \widehat{\mu} & \geq & \mathbf{0} \end{array} \qquad (35)$$

The third constraint set together with the last set indicates that an inequality constraint is only active (i.e. being limited) when $\mu_i > 0$, i.e. $h_i(x) = 0$.

For the case where the inequality i is not active at its limit, $h_i(x) < 0$ and $\mu_i = 0$.
(35) can be reformulated:

$$
\begin{array}{rcrcl}
\frac{\partial L}{\partial \mathbf{x}} & = & \frac{\partial F\ (\mathbf{x})}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}}\right)^T \lambda + \left(\frac{\partial \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}}\right)^T \mu & = & \mathbf{0} \\
\frac{\partial L}{\partial \lambda} & = & \mathbf{g}(\mathbf{x}) & = & \mathbf{0} \\
\mathbf{Diag}\{\mu_\mathbf{i}\}\frac{\partial L}{\partial \mu} & = & \mathbf{Diag}\{\mu_\mathbf{i}\}\ \mathbf{h}(\mathbf{x}) & = & \mathbf{0} \\
\frac{\partial L}{\partial \mu} & = & \mathbf{h}(\mathbf{x}) & \leq & \mathbf{0} \\
 & & \mu & \geq & \mathbf{0}
\end{array}
\tag{36}
$$

In this formulation the Jacobian matrices of both equality constraints $(\mathbf{g}(\mathbf{x})$ and inequality constraints $(\mathbf{h}(\mathbf{x})$ appear in transposed form.

For the case of a contingency constrained OPF, the following substructures can be formulated (BC: Base case network, *CCi*: Contingency case i network):

$$\mathbf{x} = [x_{BC}, x_{CC1}, x_{CC2}, \ldots]^T \tag{37}$$

$$\mathbf{g}(\mathbf{x}) = [g_{BC}(x_{BC}), g_{CC1}(x_{CC1}), g_{CC2}(x_{CC2}), \ldots]^T \tag{38}$$

$$\mathbf{h}(\mathbf{x}) = [h_1, h_2] \tag{39}$$

where two different types of inequality constraints are indicated: h_1 indicates inequality constraints which are either valid for the base case only or for each individual contingency case *CCi*.

$$\mathbf{h}_1(\mathbf{x}) = [h_{1_{BC}}(x_{BC}), h_{1_{CC1}}(x_{CC1}), h_{1_{CC2}}(x_{CC2}), \ldots]^T \tag{40}$$

h_2 indicates the coupling inequality constraints between the base case and each individual contingency case *CCi*:

$$\mathbf{h}_2(\mathbf{x}) = [-\Delta x^{max} \leq x_{BC} - x_{CC1} \leq \Delta x^{max}, -\Delta x^{max} \leq x_{BC} - x_{CC2} \leq \Delta x^{max}, -\Delta x^{max} \leq x_{BC} - x_{CC3} \leq \Delta x^{max}, \ldots]^T \tag{41}$$

Note that the objective function usually depends only on base case variables x_{BC}:

$$f(\mathbf{x}) = f_{BC}(x_{BC}) \tag{42}$$

It is the goal of the optimization algorithms to find a solution point $\widehat{\mathbf{x}}$ and corresponding vector $\widehat{\lambda}, \widehat{\mu}$ which satisfy the above conditions (36).

The main problem in reaching the solution of the optimality conditions lies in the handling of inequality constraints.

3.2.2 Interior Point (IP) OPF

The idea behind a solution of the KKT optimality conditions is the formulation of a Newton-Raphson algorithm including the handling of inequality constraints. The Newton-Raphson algorithm for the solution of a system of equality constraints is simple in that only a Newton-Algorithm has to be applied to the Lagrangian of the equality constrained optimization problem. The key step is the solution of a linear system of equations where the matrix to be inverted (or better factorized) is the Jacobian matrix. This matrix is derived by making partial derivatives of all KKT equality constraints with respect to all variables. In this subsection the idea of the Newton-Raphson for equality constraints is extended to include also the inequality constraints in the formulation.

In order to understand the key points of the IP algorithm for a non-linear optimization problem, we must state the following:

The original optimization problem is reformulated as follows:

$$\begin{aligned} \text{Minimize} \quad & F(\mathbf{x}) - \zeta \textstyle\sum_{i=1}^{p} ln(z_i) \;\; (\zeta > 0) \\ \text{subject to} \quad & \mathbf{g}(\mathbf{x}) = \mathbf{0} \\ & \mathbf{h}(\mathbf{x}) + \mathbf{z} = \mathbf{0} \\ \text{and} \quad & \mathbf{z} > \mathbf{0} \end{aligned} \tag{43}$$

We note that a so-called barrier function $-\zeta \sum_{i=1}^{p} ln(z_i)$ has been added to the original objective function $F(\mathbf{x})$. This barrier function has several special properties:

- In order to get back the original optimization problem the positive barrier parameter ζ has to become almost zero. As will be seen later, we will gradually decrease the barrier parameter ζ towards zero during an iterative solution process.
- The barrier function $\zeta \sum_{i=1}^{p} ln(z_i)$ implicitely assumes that $\mathbf{z} > \mathbf{0}$, because the logarithmic function exists only for positive real numbers. This has a consequence on the IP problem formulation and iterative solution: We will have to force the variables z to remain positive during all iterations of the algorithm. This fact gives the algorithm the name “Interior”. The term barrier has its justification in that the “barrier function” cannot cross the border at zero. Thus, zero is the functional barrier value.

.

We can formulate now the KKT conditions of this new optimization problem assuming implicitely that $z > 0$ (i.e. omitting $z > 0$):

$$L_{IP} = F(\mathbf{x}) - \zeta \sum_{i=1}^{p} ln(z_i) + \lambda^T \mathbf{g}(\mathbf{x}) + \mu^T (\mathbf{h}(\mathbf{x}) + \mathbf{z}) \tag{44}$$

From this special IP-oriented Lagrangian we derive the KKT-conditions:

$$\begin{aligned}
\frac{\partial L_{IP}}{\partial \mathbf{x}} &= \frac{\partial F(\mathbf{x})}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}}\right)^T \lambda + \left(\frac{\partial \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}}\right)^T \mu &= \mathbf{0} \\
\frac{\partial L_{IP}}{\partial \lambda} &= \mathbf{g}(\mathbf{x}) &= \mathbf{0} \\
\frac{\partial L_{IP}}{\partial \mathbf{z}} &= \mu - \zeta Diag\left(\frac{1}{z_i}\right)\mathbf{e} &= \mathbf{0} \\
\frac{\partial L_{IP}}{\partial \mu} &= \mathbf{h}(\mathbf{x}) + \mathbf{z} &= \mathbf{0} \\
& \mu &\geq \mathbf{0} \\
& \mathbf{z} &> \mathbf{0}
\end{aligned} \tag{45}$$

Note that the last inequality was added because we have to assume this inequality in order to have a valid augmented objective function (augmented by barrier terms).

The vector **e** in (45) indicates a vector with 1's only.

The third set of KKT conditions in (45) (ie. the first order partial derivatives with respect to **z**) can be reformulated and leads to the following IP-KKT conditions:

$$\begin{aligned}
\frac{\partial F(\mathbf{x})}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}}\right)^T \lambda + \left(\frac{\partial \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}}\right)^T \mu &= \mathbf{0} \\
\mathbf{g}(\mathbf{x}) &= \mathbf{0} \\
Diag(\mu_\mathbf{i})\,\mathbf{z} - \zeta\mathbf{e} &= \mathbf{0} \\
\mathbf{h}(\mathbf{x}) + \mathbf{z} &= \mathbf{0} \\
\mu &\geq \mathbf{0} \\
\mathbf{z} &> \mathbf{0}
\end{aligned} \tag{46}$$

(46) represents IP-KKT first order conditions which must be valid in the optimum for any given barrier parameter $\zeta > 0$.

The principal idea behind the IP solution algorithm of (46) is as follows:

- Formulate a Newton-Raphson solution step for the **equality** constraints only. Note, that the number of equality constraints and the number of unknown variables are identical. The number of variables and equations is $n+m+2p$ where n is the number of unknowns $\mathbf{x}$, m is the number of equality constraints $\mathbf{g}(\mathbf{x})$ (and also unknown Lagrange multipliers λ) and p is the number of inequality constraints $\mathbf{h}(\mathbf{x})$ or unknowns $\mathbf{z}$ or Lagrange multipliers μ.

- Choose a starting point for the unknown variables in such a way that all variables which are limited get a positive value (i.e. $\mathbf{z}^{(0)} > 0, \mu^{(0)} > 0$). The choice for the starting values is quite sensitive for a convergence success of the algorithm. The most important rule is not to use values of zero for $\mathbf{z}^{(0)}$ and $\mu^{(0)}$.

  The Newton-Raphson step for the above equality constraints is as follows (The index $^{(k)}$ indicates the iteration counter k):

$$
\begin{aligned}
&\left(\frac{\partial^2 F(\mathbf{x})}{\partial \mathbf{x}^2} + \sum_{i=1}^{m}\left(\lambda_i \frac{\partial^2 g_i(\mathbf{x})}{\partial \mathbf{x}^2}\right) + \sum_{i=1}^{p}\left(\mu_i \frac{\partial^2 h_i(\mathbf{x})}{\partial \mathbf{x}^2}\right)\right)\Bigg|_{\mathbf{x}^{(k)},\lambda^{(k)},\mu^{(k)}} \Delta\mathbf{x} \\
&+ \left(\frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}}\right)^T\Bigg|_{x^{(k)}} \Delta\lambda + \left(\frac{\partial \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}}\right)^T\Bigg|_{x^{(k)}} \Delta\mu \\
&= -\left.\frac{\partial F(\mathbf{x})}{\partial \mathbf{x}}\right|_{x^{(k)}} - \left(\frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}}\right)^T\Bigg|_{x^{(k)}} \lambda^{(k)} - \left(\frac{\partial \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}}\right)^T\Bigg|_{x^{(k)}} \mu^{(k)} \\
&Diag\left(z_i^{(k)}\right)\Delta\mu + Diag\left(\mu_i^{(k)}\right)\Delta\mathbf{z} = -\left(Diag\left(z_i^{(k)}\right)\mu^{(k)} - \zeta\mathbf{e}\right) \\
&\left.\frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}}\right|_{x^{(k)}} \Delta\mathbf{x} = -\mathbf{g}(\mathbf{x}^{(k)}) \\
&\left.\frac{\partial \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}}\right|_{x^{(k)}} \Delta\mathbf{x} + \Delta\mathbf{z} = -\mathbf{h}(\mathbf{x}^{(k)}) - \mathbf{z}^{(k)} \\
&\mu^{(k+1)} \geq \mathbf{0} \\
&\mathbf{z}^{(k+1)} > \mathbf{0}
\end{aligned} \tag{47}
$$

With the following abbreviations (47) can be written in a simpler and clearer representation:

$$
\begin{aligned}
\mathbf{Q} &= \left(\frac{\partial^2 F(\mathbf{x})}{\partial \mathbf{x}^2} + \sum_{i=1}^{m}\left(\lambda_i \frac{\partial^2 g_i(\mathbf{x})}{\partial \mathbf{x}^2}\right) + \sum_{i=1}^{p}\left(\mu_i \frac{\partial^2 h_i(\mathbf{x})}{\partial \mathbf{x}^2}\right)\right)\Bigg|_{\mathbf{x}^{(k)},\lambda^{(k)},\mu^{(k)}} \\
\mathbf{c} &= \left.\frac{\partial F(\mathbf{x})}{\partial \mathbf{x}}\right|_{x^{(k)}} + \left(\left.\frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}}\right|_{x^{(k)}}\right)^T \lambda^{(k)} + \left(\left.\frac{\partial \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}}\right|_{x^{(k)}}\right)^T \mu^{(k)} \\
\mathbf{A}_1 &= \left.\frac{\partial \mathbf{g}(\mathbf{x})}{\partial \mathbf{x}}\right|_{x^{(k)}} \\
\mathbf{A}_2 &= \left.\frac{\partial \mathbf{h}(\mathbf{x})}{\partial \mathbf{x}}\right|_{x^{(k)}} \\
\mathbf{b}_1 &= -\mathbf{g}(\mathbf{x}^{(k)}) \\
\mathbf{b}_2 &= -\mathbf{h}(\mathbf{x}^{(k)}) - \mathbf{z}^{(k)}
\end{aligned} \tag{48}
$$

With these abbreviations the problem is as follows: The optimality conditions can be written in matrix form as follows:

$$
\begin{aligned}
\begin{bmatrix}
\mathbf{Q} & \mathbf{A}_1^T & \mathbf{A}_2^T & \mathbf{0} \\
\mathbf{A}_1 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\
\mathbf{A}_2 & \mathbf{0} & \mathbf{0} & \mathbf{U} \\
\mathbf{0} & \mathbf{0} & Diag(\mathbf{z_i^{(k)}}) & Diag(\mu_\mathbf{i}^{(\mathbf{k})})
\end{bmatrix}
\begin{bmatrix}
\Delta\mathbf{x}^{(k)} \\ \Delta\lambda^{(k)} \\ \Delta\mu^{(k)} \\ \Delta\mathbf{z}^{(k)}
\end{bmatrix}
&=
\begin{bmatrix}
-\mathbf{c} \\ \mathbf{b}_1 \\ \mathbf{b}_2 \\ -\left(Diag(z_i^{(k)}\mu^{(k)} - \zeta\mathbf{e}\right)
\end{bmatrix} \\
\mu^{(k+1)} &\geq 0 \\
\mathbf{z}^{(k+1)} &> 0
\end{aligned} \tag{49}
$$

In (49), the matrix $\mathbf{U}$ represents a unity (or identity) matrix.

- After computing one update step with the Newton-Raphson solution procedure, use a step- length control in such a way that all variables $\mathbf{z}$ and μ remain positive during all iterations of this iterative algorithm. This is done as follows:

$$
\begin{aligned}
\mathbf{x}^{(k+1)} &:= \mathbf{x}^{(k)} + \alpha_z\, \Delta\mathbf{x}^{(k)} \\
\lambda^{(k+1)} &:= \lambda^{(k)} + \alpha_\mu\, \Delta\lambda^{(k)} \\
\mu^{(k+1)} &:= \mu^{(k)} + \alpha_\mu\, \Delta\mu^{(k)} \\
\mathbf{z}^{(k+1)} &:= \mathbf{z}^{(k)} + \alpha_z\, \Delta\mathbf{z}^{(k)}
\end{aligned} \tag{50}
$$

The maximum step length for α must be chosen in such a way that all $\mu^{(k)} + \alpha_\mu \Delta\mu^{(k)}$ and $\mathbf{z}^{(k)} + \alpha_z \Delta\mathbf{z}^{(k)}$ will remain positiv. This is achieved with the following simple rule:

$$
\begin{aligned}
\alpha_\mu' &= \min_{\Delta\mu_i^{(k)}<0}\left\{\frac{-\mu_i^{(k)}}{\Delta\mu_i^{(k)}}\right\} \\
\alpha_\mu &= \min\{1, 0.9995\,\alpha_\mu'\}
\end{aligned} \tag{51}
$$

and

$$
\begin{aligned}
\alpha_z' &= \min_{\Delta z_i^{(k)}<0}\left\{\frac{-z_i^{(k)}}{\Delta z_i^{(k)}}\right\} \\
\alpha_z &= \min\{1, 0.9995\,\alpha_z'\}
\end{aligned} \tag{52}
$$

Using these two updating mechanisms for the two different global values α_z and α_μ forces all variables of $\mathbf{z}$ and μ to remain positive during all iterations. Note that the last factor of 0.9995 has the effect that the exact value of zero is never perfectly reached.

- Obviously the barrier parameter ζ must be very near to zero at the optimum. If this is not the case the optimization problem is not the same as originally formulated.

  The barrier parameter ζ is decreased during all iterations as follows: After each iteration of the Newton-Raphson main step (solution of linearized system of equations) a heuristic formula

$$
\zeta^{(k+1)} = f(\mu^{(k)}, \mathbf{z}^{(k)}) = \beta\frac{{\mu^{(k)}}^T \mathbf{z}^{(k)}}{2n} \tag{53}
$$

  is applied which will force ζ towards zero. In (53) the value of β depends on the problem. However, best values have been obtained with $\beta = 0.1 \ldots 0.5$.

The iterative IP solution process for the general optimization problem including both non-linear equality constraints $\mathbf{g}(\mathbf{x}) = \mathbf{0}$ and non-linear inequality constraints $\mathbf{h}(\mathbf{x}) \leq \mathbf{0}$ is summarized as follows:

1. Choose initial values for $\mathbf{x} = \mathbf{x}^{(\mathbf{0})}$; $\lambda = \lambda^{(\mathbf{0})}$, $\mu = \mu^{(0)}$ and $\mathbf{z} = \mathbf{z}^{(0)}$ and set index k = 0; Compute ζ according to (53) or set it to some suitable positive value.

2. Compute the optimal solution $\Delta\mathbf{x}_{opt}$, $\Delta\lambda_{opt}$, $\Delta\mu_{opt}$ and $\Delta\mathbf{z}_{opt}$ of the linear system of equations (49)

3. Update the vectors $\mathbf{x}$, λ, μ and $\mathbf{x}$ with (50) using (51) and (52).

4. If all absolute values of right hand side elements of the original optimality conditions (see right hand sides of (49)) are below a certain given small tolerance value ε, **stop** iterative process, **otherwise** continue.

5. Compute the new barrier parameter ζ using (53).

6. Increase the index k by 1: k = k + 1; go to step 2

4 Theoretical aspects of the Interior Point (IP) approaches to convex programming

4.1 Introduction

We consider the primal formulation of the convex programming problem:

$$\begin{aligned} &\text{Maximize} \quad f_0(y) \\ &\text{s.t.} \;\; f_i(y) \leq 0 \;\; i = 1,2,\ldots n \\ &y \in \Re^m \end{aligned} \tag{54}$$

The feasible region is denoted by F and the interior region by F^0. The following assumptions are made:

- The functions $-f_0(y)$ and $f_i(y)$, $1 \leq i \leq n$ are convex functions with continuous first and second order derivatives in F^0.
- F^0 is non-empty.
- F^0 is bounded.

The dual problem of (54) is according to Wolfe:

$$\begin{aligned} &\text{Minimize} \quad f_0(y) - \textstyle\sum_{i=1}^{n}(\mu_i f_i(y)) \\ &\text{s.t.} \;\; \textstyle\sum_{i=1}^{n} \mu_i \nabla f_i(y) = \nabla f_0(y) \\ &\mu \geq 0 \end{aligned} \tag{55}$$

(55) is not necessarily convex.

For the optimization expressed both in primal (54) and in dual (55) representation, two IP approaches can be applied: The logarithmic barrier function and the so-called center method.

4.2 The logarithmic barrier function

The logarithmic barrier function associated with the primal problem (54):

$$\phi_B(y,\zeta) = -\frac{f_0(y)}{\zeta} - \sum_{i=1}^{n} ln(-f_i(y)) \tag{56}$$

where $\zeta > 0$ is the barrier parameter.

The logarithmic barrier function associated with the dual problem (55) (and in addition a constant factor) is:

$$\phi_B^d(y,\mu,\zeta) = -\frac{f_0(y)}{\zeta} + \frac{1}{\zeta}\sum_{i=1}^{n}(\mu_i f_i(y)) + \sum_{i=1}^{n} ln(\mu_i) + n(1 - ln\zeta)) \tag{57}$$

The first $g(y,\zeta)$ and second $H(y,\zeta)$ order derivatives of ϕ_B are needed in later discussions:

$$g := g(y,\zeta) = \nabla\phi_B(y,\zeta) = -\frac{\nabla f_0(y)}{\zeta} + \sum_{i=1}^{n} \frac{\nabla f_i(y)}{-f_i(y)} \tag{58}$$

$$H := H(y,\zeta) = \nabla^2\phi_B(y,\zeta) = -\frac{\nabla^2 f_0(y)}{\zeta} + \sum_{i=1}^{n} \left[\frac{\nabla^2 f_i(y)}{-f_i(y)} + \frac{\nabla f_i(y) \nabla f_i(y)^T}{f_i(y)^2} \right] \tag{59}$$

One can prove that ϕ_B is strictly convex on its bounded domain F^0. This means that H is positive definite on its domain F^0. Also, this function achieves the minimal value in its domain (for fixed ζ) at a unique point which is denoted by $y(\zeta)$. This unique point is called the ζ-center.

The necessary and sufficient KKT conditions for $y(\zeta)$ are:

$$\begin{array}{ll} f_i(y) \leq 0 & 1 \leq i \leq n \\ \sum_{i=1}^{n} \mu_i \nabla f_i(y) = \nabla f_0(y) & \mu \geq 0 \\ -f_i(y)\mu_i = \zeta & 1 \leq i \leq n \end{array} \tag{60}$$

The so-called **central path** is defined as follows: The primal (dual) central path is defined as the set of centers $y(\zeta)$, (Dual: $\mu(\zeta), y(\zeta)$) where ζ goes from ∞ to 0.

The duality gap (the difference between the primal and dual objective function) satisfies

$$f_0(y(\zeta)) - \sum_{i=1} n\mu_i(\zeta) f_i(y(\zeta)) - f_0(y(\zeta)) = -\sum_{i=1}^{n} \mu_i(\zeta) f_i(y(\zeta)) = n\zeta \tag{61}$$

Since $\mu(\zeta))$ and $y(\zeta))$ are continuous in ζ, $y(\zeta)$ and $\mu(\zeta)$ will converge to optimal solutions of both the primal (54) and the dual (55), if ζ goes towards zero.

Three points are important:

- the method used to minimize $\phi_B(y,\zeta)$ (or: used to approximately minimize $\phi_B(y,\zeta)$).
- the criterion to terminate this approximate minimization.
- the updating scheme for the barrier parameter ζ.

The following lemma is valid for the primal and dual logarithmic barrier functions:

- The objective function $f_0(y(\zeta))$ of the primal problem (54) is monotonically increasing and the objective function of the dual problem (55) $f_0(y(\zeta)) - \sum_{i=1}^{n} \mu_i(\zeta) f_i(y(\zeta))$ is monotonically decreasing if ζ decreases. $\mu(\zeta)$ and $y(\zeta)$ are defined by (60).

Algorithm:

1. $y := y^{(0)}, \zeta := \zeta^{(0)}$
2. while $\zeta > \frac{\varepsilon}{4n}$, do
   (a) $\zeta := (1-\theta)\zeta, \;\; (0 < \theta < 1)$
   (b) Repeat next two steps until high inner solution accuracy reached
      i. $\tilde{\alpha} := arg \;\; min_{\alpha>0}\{\phi_D(y+\alpha p,\zeta) : y+\alpha p \in F^0\}$ where $p = -H^{-1}g$
      ii. $y := y + \tilde{\alpha} p$

4.3 The center method

We define a so-called **distance function** with the primal problem (54):

$$\phi_D(y,\kappa) = -q ln(f_0(y) - \kappa) - \sum_{i=1}^{n} ln(-f_i(y)) \tag{62}$$

where κ is a lower bound for the optimal valuea $f_0(y)$ and q is a given positive integer.
The necessary and sufficient KKT conditions for ϕ_D are:

$$\begin{aligned} & f_i(y) \leq 0 && 1 \leq i \leq n \\ & \textstyle\sum_{i=1}^{n} \mu_i \nabla f_i(y) = \nabla f_0(y) && \mu \geq 0 \\ & -f_i(y)\mu_i = \tfrac{f_0(y)-\kappa}{q} && 1 \leq i \leq n \end{aligned} \tag{63}$$

Comparing (63) with (60) one recognizes that they are identical if

$$\zeta = \frac{f_0(y) - \kappa}{q} \tag{64}$$

This can be interpreted in such a way that the logarithmic barrier function of the previous subsection and the distance function of this subsection yield two different parameterization possibilities of the same central path.

ϕ_D can be rewritten as:

$$\phi_D(y,\kappa) = -\sum_{i=1}^{n+q} ln(-f_i(y)) \tag{65}$$

where

$$-f_i(y) = f_0(y) - \kappa, \;\; n+1 \leq i \leq n+q \tag{66}$$

The first $g(y,\zeta)$ and second $H(y,\zeta)$ order derivatives of ϕ_D are needed in later discussions:

$$g := g(y,\kappa) = \nabla\phi_D(y,\kappa) = \sum_{i=1}^{n+q} \frac{\nabla f_i(y)}{-f_i(y)} \tag{67}$$

$$H := H(y,\kappa) = \nabla^2\phi_D(y,\kappa) = +\sum_{i=1}^{n+q} \left[\frac{\nabla^2 f_i(y)}{-f_i(y)} + \frac{\nabla f_i(y) \nabla f_i(y)^T}{f_i(y)^2} \right] \tag{68}$$

We define F_z:

$$F_z = \{ y \;\; : \;\; f_i(y) \leq 0, \;\; 1 \leq i \leq n+q \} \tag{69}$$

F_z^0 is the interior of F_z. The concept of the so-called analytic center of the bounded convex region F_z can be defined:

The analytic center of the bounded convex region F_z is the point which maximizes

$$e^{-\phi_D(y,\kappa)} = \prod_{i=1}^{m+q} (-f_i(y)) \tag{70}$$

It follows:

$$
\begin{array}{ll}
f_i(y) \le 0 & 1 \le i \le n \\
\sum_{i=1}^{n+q} \widetilde{\mu}_i \nabla f_i(y) = 0 & \widetilde{\mu} \ge 0 \\
-f_i(y)\widetilde{\mu}_i = 1 & 1 \le i \le n+q
\end{array}
\tag{71}
$$

Note:

- The analytic center is an analytic and not a geometric concept.
- The analytic center depends on the description of the feasible region. Replicating a constraint has an effect on the position of the analytic center.

The method of centers works as follows:

Given a lower bound κ for the optimal value of $f_0(y)$, one tries to reach the vicinity of $y(\kappa)$, i.e. the analytic center of the current region. Then, the lower bound is increased. This means that the q objective constraints are shifted. Next, the new vicinity of the new center is approached.

Algorithm:

1. $y := y^{(0)}, \kappa := \kappa^{(0)}; \Delta = 4(1+\frac{n}{q})$
2. while $(f_0(y) - \kappa > \frac{\varepsilon}{\Delta})$, do
   (a) $\kappa := \kappa + \theta(f_0(\kappa) - \kappa), \;\; (0 < \theta < 1)$
   (b) Repeat next two steps until high inner solution accuracy reached
      i. $\widetilde{\alpha} := arg \;\; min_{\alpha > 0}\{\phi_D(y + \alpha p, \kappa) : y + \alpha p \in F^0\}$ where $p = -H^{-1}g$
      ii. $y := y + \widetilde{\alpha} p$

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