Lecture Notes on Nonlinear Systems and Control

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ETH Zurich

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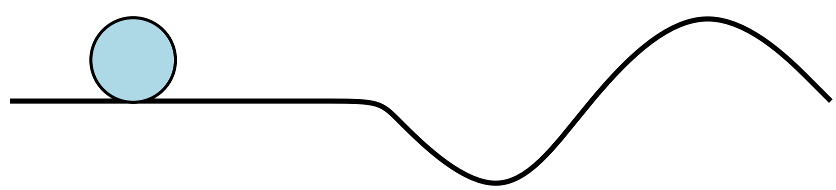

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Preface

The objective of this course is to provide the students with an introduction to nonlinear systems and the various methods of controlling them.

Part I of the course introduces the students to the notions of nonlinearities and the various ways of analyzing existence and uniqueness of solutions to ordinary differential equations, as well as understanding various notions of stability and their characterizations.

Part II of the course arms the students with a variety of control methods that are suitable for nonlinear systems. This part designed in such a way as to put the student in the position to deploy nonlinear control techniques in real applications.

All chapters are combined with exercises that are geared towards attaining better understanding of the pros and the cons of the different concepts.

Part I

Analysis

Chapter 1

Introduction

1.1 Main Concepts

When engineers analyze and design nonlinear dynamical systems in electrical circuits, mechanical systems, control systems, and other engineering disciplines, they need to be able to use a wide range of nonlinear analysis tools. Despite the fact that these tools have developed rapidly since the mid 1990s, nonlinear control is still largely a tough challenge.

In this course, we will present basic results for the analysis of nonlinear systems, emphasizing the differences to linear systems, and we will introduce the most important nonlinear feedback control tools with the goal of giving an overview of the main possibilities available. Additionally, the lectures will aim to give the context on which each of these tools are to be used. Table 1.1 contains an overview of the topics to be considered in this course.

Requirements	Challenges	Theoretical Results
Modeling and Simulation	Well posedness	ODE Theory Bifurcations
Disturbance Rejection Stabilization Tracking	Sensors Uncertainty Nonlinearities	Observers Lyapunov Feedback Linearization Sliding Mode
Economic Optimization	Control Effort Constraints	Optimal Control Model Predictive Control

Table 1.1: Course Content

1.1.1 Nonlinear Models and Nonlinear Phenomena

We will deal with systems of the form:

$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2, .., x_n, u_1, u_2, \ldots, u_m) \\
\dot{x}_2 &= f_2(x_1, x_2, .., x_n, u_1, u_2, \ldots, u_m) \\
&\cdots \\
\dot{x}_n &= f_n(x_1, x_2, .., x_n, u_1, u_2, \ldots, u_m)
\end{aligned}
$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

Often, we will neglect the time-varying aspect. In the analysis phase, external inputs are also often neglected, leaving system

$$\dot{x} = f(x). \tag{1.1}$$

Working with an unforced state equation does not necessarily mean that the input to the system is zero. It rather means that the input has been specified as a given function of the state $u = u(x)$.

Definition 1.1 *A system is said to be autonomous or time invariant if the function f does not depend explicitly on t, that is, $\dot{x} = f(x)$.*

Definition 1.2 *A point $\bar{x}$ is called equilibrium point of $\dot{x} = f(x)$ if $x(\tau) = \bar{x}$ for some τ implies $x(t) = \bar{x}$ for $t \geq \tau$.*

For an autonomous system the set of equilibrium points is equal to the set of real solutions of the equation $f(x) = 0$.

- $\dot{x} = x^2$: isolated equilibrium point
- $\dot{x} = \sin(x)$: infinitely many equilibrium points
- $\dot{x} = \sin(1/x)$: infinitely many equilibrium points in a finite region

Linear Systems satisfy the following 2 properties:

1. Homogeneity: $f(\alpha x) = \alpha f(x)$, $\forall \alpha \in \mathbb{R}$
2. Superposition: $f(x + y) = f(x) + f(y)$, $\forall x, y \in \mathbb{R}^n$

For example, consider the system given by the linear differential equation:

$$\dot{x} = Ax + Bu \tag{1.2}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$.

Then the solution is given by

$$x(t) = \exp^{At} x_0 + \int_0^t \exp^{A(t-\tau)} Bu(\tau)d\tau. \tag{1.3}$$

Note that the expression for $x(t)$ is linear in the initial condition x_0 and in the control function $u(\cdot)$. Nonlinear systems are those systems that do not satisfy these nice properties.

As we move from linear to nonlinear systems, we shall face a more difficult situation. The superposition principle no longer holds, and analysis tools necessarily involve more advanced mathematics. Most importantly, as the superposition principle does not hold, we cannot assume that an analysis of the behavior of the system either analytically or via simulation may be scaled up or down to tell us about the behavior at large or small scales. These must be checked separately.

The first step when analyzing a nonlinear system is usually to linearize it about some nominal operating point and analyze the resulting linear model. However, it is clear that linearization alone will not be sufficient. We must develop tools for the analysis of nonlinear systems. There are two basic limitation of linearization. First, since linearization is an approximation in the neighborhood of an operating point, it can only predict the local behavior of the nonlinear system in the vicinity of that point. Secondly, the dynamics of a nonlinear system are much richer than the dynamics of a linear system. There are essentially nonlinear phenomena that can take place only in the presence of nonlinearity; hence they cannot be described or predicted by linear models. The following are examples of nonlinear phenomena:

- **Finite escape time:** The state of an unstable linear system can go to infinity as time approaches infinity. A nonlinear system's state, however, can go to infinity in finite time.

- **Multiple isolated equilibrium points:** A linear system can have only one equilibrium point, and thus only one steady-state operating point that attracts or repels the state of the system irrespective of the initial state. A nonlinear system can have more than one equilibrium point.

- **Limit cycles:** A linear system can have a stable oscillation if it has a pair of eigenvalues on the imaginary axis. The amplitude of the oscillation will then depend on the initial conditions. A nonlinear system can exhibit an oscillation of fixed amplitude and frequency which appears independently of the initial conditions.
- **Chaos:** A nonlinear system can have a more complicated steady-state behavior that is not equilibrium or periodic oscillation. Some of these chaotic motions exhibit randomness, despite the deterministic nature of the system.
- **Multiple Modes of behaviour:** A nonlinear system may exhibit very different forms of behaviour depending on external parameter values, or may jump from one form of behaviour to another autonomously. These behaviours cannot be observed in linear systems, where the complete system behaviour is characterized by the eigenvalues of the system matrix A.

1.2 Typical Nonlinearities

In the following subsections, various nonlinearities which commonly occur in practice are presented.

1.2.1 Memoryless Nonlinearities

Most commonly found nonlinearities are:

- **Relay**, see Figure1.1. Relays appear when modelling mode changes.
- **Saturation**, see Figure 1.2. Saturations appear when modelling variables with hard limits, for instance actuators.
- **Dead Zone**, see Figure 1.3. Dead Zone appear in connection to actuator or sensor sensitivity.
- **Quantization**, see Figure 1.4. Quantization is used to model discrete valued variables, often actuators.

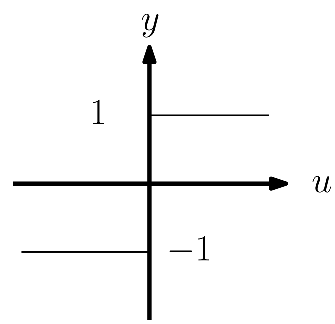


Figure 1.1: Relay

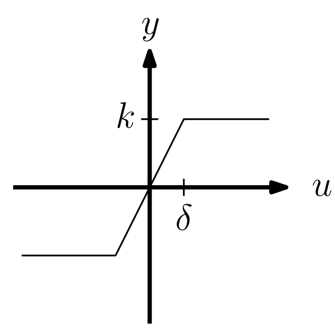


Figure 1.2: Saturation

This family of nonlinearities are called memoryless, zero memory or static because the output of the nonlinearity at any instant of time is determined uniquely by its input at that time instant; it does not depend on the history of the input.

1.2.2 Nonlinearities with Memory

Quite frequently though, we encounter nonlinear elements whose input-output characteristics have memory; that is, the output at any instant of time may depend on the recent or even entire, history of the input.

In the case of hysteresis one is confronted with a situation where the path forward is not the same as the path backward. This behavior is often observed when dealing with defective actuators such as valves.

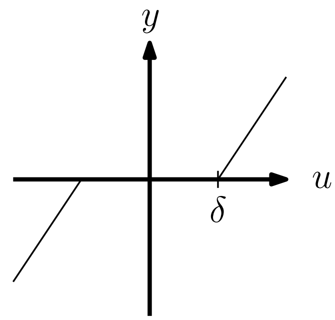


Figure 1.3: Dead Zone

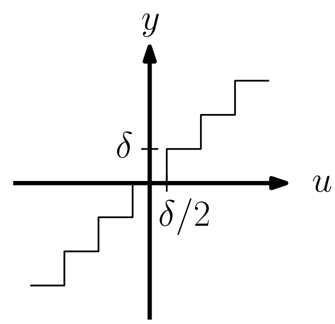


Figure 1.4: Quantization

1.3 Examples of Nonlinear Systems

In this section we present some examples of nonlinear systems which demonstrate how nonlinearities may be present, and how they are then represented in the model equations.

1.3.1 Chemical Reactor

This is an example of a strongly nonlinear system

$$[\dot{C_a}] = \frac{q}{V}([C_{af}] - [C_a]) - r[C_a]$$

The coefficient r is an exponential function of the temperature and the reagent concentration.

$$r = K \exp \frac{E}{RT}$$

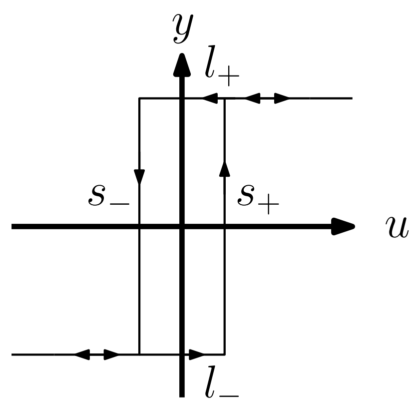


Figure 1.5: Relay with hysteresis

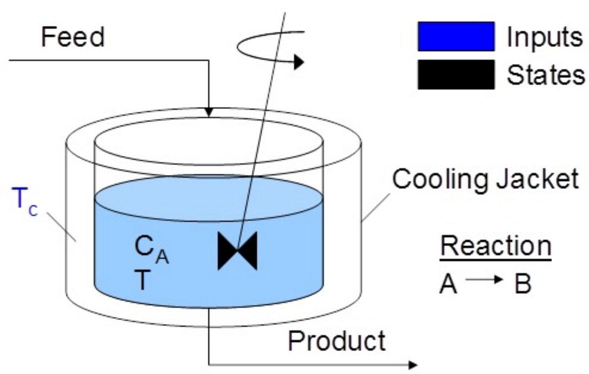


Figure 1.6: Chemical Reactor

while the temperature T is given by

$$\dot{T} = \frac{q}{V}(T_f - T) + K_r r[C_a] + K_c(T - T_c)$$

The model has 2 states: the concentration $[C_a]$ of A and the temperature T of the reaction vessel liquid. The manipulated variable is the jacket water temperature T_c. Depending upon the problem formulation,the feed temperature T_f and feed concentration $[C_{af}]$ can be considered either constant or as a disturbance. At a jacket temperature of 305K, the reactor model has an oscillatory response. The oscillations are characterized by reaction run-away with a temperature spike. When the concentration drops to a lower value, the reactor cools until the concentration builds and there is another run-away reaction.

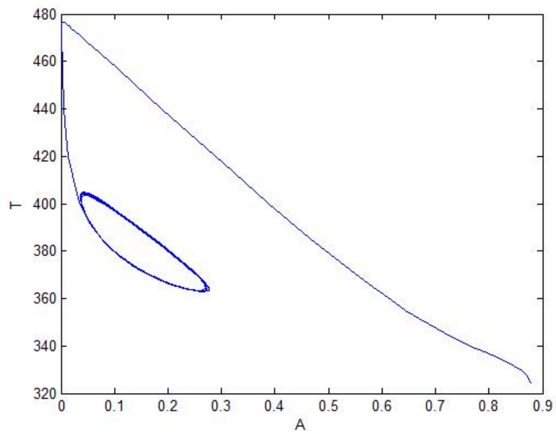


Figure 1.7: Chemical Reactor Phase Portrait

1.3.2 Diode

We assume a time invariant linear capacitor C, inductor L and resistor R.

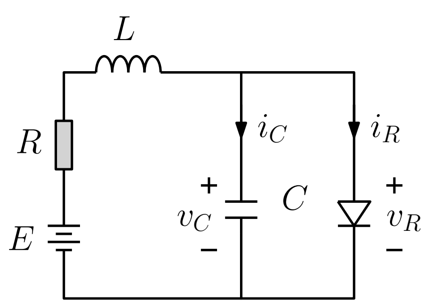


Figure 1.8: Diode

The tunnel diode characteristic curve $i_R = h(v_R)$ is plotted in the Figure 1.9.

Choosing $x_1 = v_C$ and $x_2 = i_L$ and $u = E$ we obtain the following dynamical system

$$
\begin{aligned}
\dot{x}_1 &= \frac{1}{C}(-h(x_1) + x_2) \\
\dot{x}_2 &= \frac{1}{L}(-x_1 - Rx_2 + E)
\end{aligned}
$$

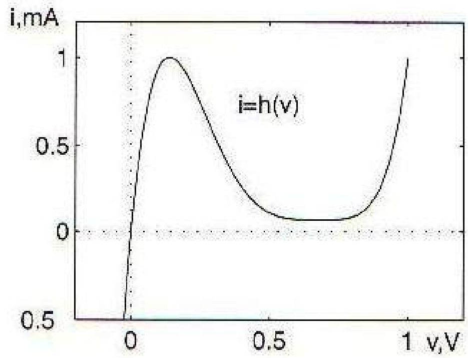


Figure 1.9: Diode Characteristic Curve

The equilibrium points of the system are determined by setting $\dot{x}_1 = \dot{x}_2 = 0$.

$$\begin{aligned} 0 &= \frac{1}{C}(-h(x_1) + x_2) \\ 0 &= \frac{1}{L}(-x_1 - Rx_2 + E) \end{aligned}$$

Therefore, the equilibrium points corresponds to the roots of the equation

$$h(x_1) = \frac{E - x_1}{R}$$

The next figure shows graphically that, for certain values of E and R, this equation has three isolated roots which correspond to three isolated equilibrium points of the system. The number of equilibrium points might change as the values of E and R change. For example, if we increase E for the same value of R, we will reach a point beyond which only Q_3 will exist.

As we will see in the next chapter, the phase portrait in this case has two stable equilibrium point and 1 unstable equilibrium point

The tunnel diode characteristic curve $i_R = h(v_R)$ is plotted in the next figure.

1.4 Second Order Systems

Second-order autonomous systems occupy an important place in the study of nonlinear systems because solution trajectories can be represented in the

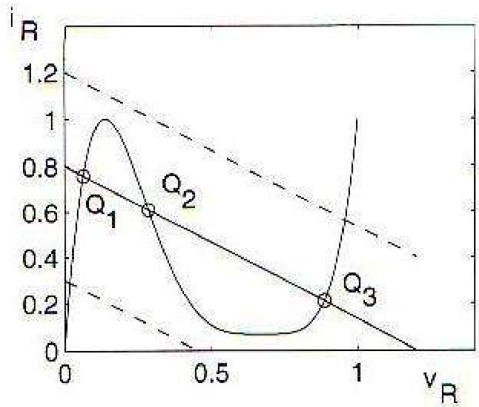


Figure 1.10: Diode Equilibria

2D-plane. This allows for easy visualization of the qualitative behavior of the system.

In the sequel we consider the following aspects of second order systems:

1. Behavior near equilibrium points
2. Nonlinear oscillations
3. Bifurcations

A second-order autonomous system is represented by two scalar differential equations

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2), \quad x_1(0) = x_{10} \\ \dot{x}_2 &= f_2(x_1, x_2), \quad x_2(0) = x_{20} \end{aligned}$$

or in vector notation

$$\dot{x} = f(x), \quad x(0) = x_0, \quad x, x_0 \in \mathbb{R}^2$$

The locus in the (x_1, x_2) plane of the solution $x(t)$ for all $t \geq 0$ is a curve that passes through the point x_0. This plane is usually called state plane or phase plane. The vector f gives the tangent vector to the curve $x(\cdot)$.

Definition 1.3 *We obtain a vector field diagram by assigning the vector* $(f_1(x), f_2(x)$ *to every point* (x_1, x_2) *in a grid covering the plane.*

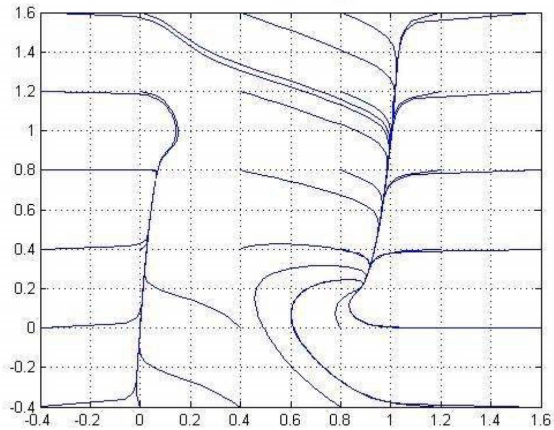


Figure 1.11: Diode Phase Portrait

For example, if $f(x) = (2x_1^2, x_2)$, then at $x = (1,1)$, we draw an arrow pointing from $(1,1)$ to $(1,1) + (2,1) = (3,2)$.

Example 1.4 *Pendulum without friction.*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -10 \sin x_1 \end{aligned}$$

See Figure 1.12.

Definition 2.2: The family of all trajectories of a dynamical system is called the *phase portrait*.

As we will see later, the behavior of nonlinear systems near equilibrium points is well described by the behavior of the linearization. We approximate the nonlinear system by its linearization:

$$\begin{aligned} \dot{x}_1 &= \left.\frac{\partial f_1}{\partial x_1}\right|_{x=\bar{x}} x_1 + \left.\frac{\partial f_1}{\partial x_2}\right|_{x=\bar{x}} x_2 \\ \dot{x}_2 &= \left.\frac{\partial f_2}{\partial x_1}\right|_{x=\bar{x}} x_1 + \left.\frac{\partial f_2}{\partial x_2}\right|_{x=\bar{x}} x_2 \end{aligned}$$

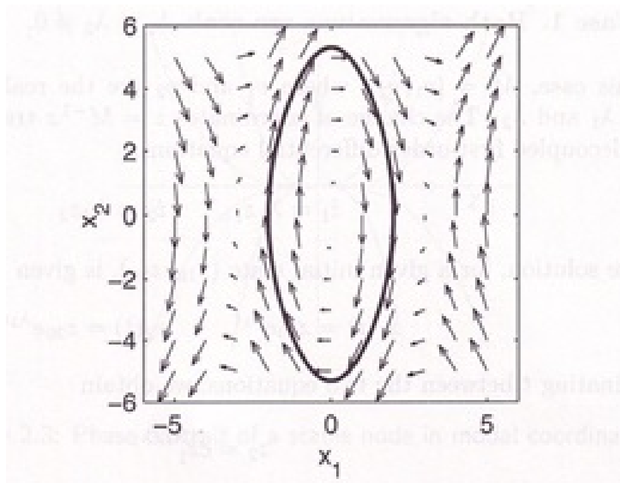


Figure 1.12: Frictionless Pendulum.

1.4.1 Qualitative Behavior of 2^{nd} Order Systems Near Equilibrium Points

Consider the linear time-invariant system $\dot{x} = Ax$ where A is a 2×2 real matrix. The solution of the equation for a given state x_0 is given by

$$x\left(t\right) = M \exp\left(J_r t\right) M^{-1} x_0,$$

where J_r is the real Jordan form of A and M is a real nonsingular matrix such that $M^{-1}AM = J_r$. Depending on the properties of A, the real Jordan form may take one of three forms:

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix}, \quad \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

where k is either 0 or 1. The first form corresponds to the case when the eigenvalues λ_1 and λ_2 are real and distinct, the second form corresponds to the case when the eigenvalues are real and equal, and the third form corresponds to the case of complex eigenvalues $\lambda_{1,2} = \alpha \pm j\beta$.

Real Distinct Eigenvalues: In this case λ_1 and λ_2 are different from zero and $M = [v_1, v_2]$, where v_1 and v_2 are the real eigenvectors associated with λ_1 and λ_2. The change of coordinates $z = M^{-1}x$ transforms the system into two decoupled first-order differential equations,

$$\begin{aligned} \dot{z}_1 &= \lambda_1 z_1 \\ \dot{z}_2 &= \lambda_2 z_2 \end{aligned}$$

whose solution, for a given initial state (z_{10}, z_{20}), is given by

$$\begin{aligned} z_1 &= z_{10} e^{\lambda_1 t} \\ z_2 &= z_{20} e^{\lambda_2 t}. \end{aligned}$$

Eliminating t between the two equations, we obtain

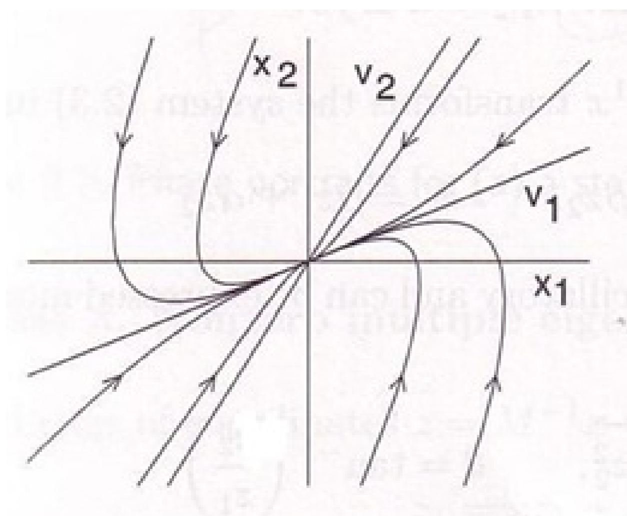


Figure 1.13: Stable Node.

$$z_2 = c z_1^{\lambda_2/\lambda_1}$$

where

$$c = z_{20} / \left(z_{10}\right)^{\lambda_2/\lambda_1}. \tag{1.4}$$

There are then three cases:

- **Stable node**: Both eigenvalues are negative, Figure 1.13.
- **Unstable node**: Both eigenvalues are positive, Figure 1.14.
- **Saddle point**: Eigenvalues have different sign, Figure 1.15

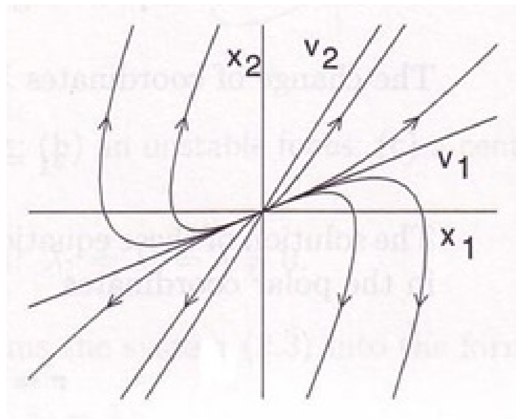


Figure 1.14: Unstable Node.

Complex Eigenvalues: The change of coordinates $z = M^{-1}x$ transforms the system into the form

$$\begin{aligned}\dot{z}_1 &= \alpha z_1 - \beta z_2 \\ \dot{z}_2 &= \beta z_1 + \alpha z_2\end{aligned}$$

The solution of these equations is oscillatory and can be expressed more conveniently in polar coordinates.

$$\begin{aligned}r &= \sqrt{z_1^2 + z_2^2} \\ \theta &= \tan^{-1}\left(\frac{z_2}{z_1}\right)\end{aligned}$$

where we have two uncoupled first-order differential equation:

$$\begin{aligned}\dot{r} &= \alpha r \\ \dot{\theta} &= \beta\end{aligned}$$

The solution for a given initial state (r_0, θ_0) is given by

$$\begin{aligned}r(t) &= r_0 e^{\alpha t} \\ \theta(t) &= \theta_0 + \beta t\end{aligned}$$

We now have three behaviors:

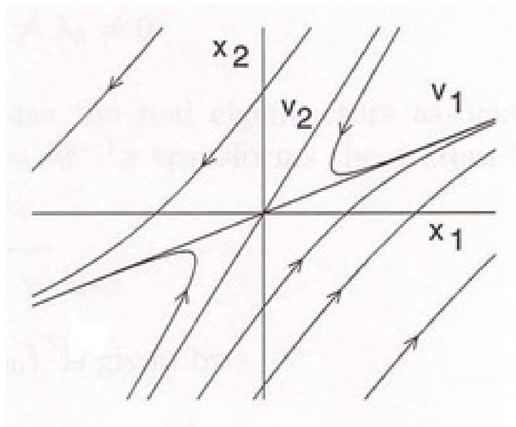


Figure 1.15: Saddle Point.

- **Stable Focus**, when $\alpha < 0$, the spiral converges to the origin, see Figure 1.16.
- **Unstable Focus**, when $\alpha > 0$, it diverges away from the origin, see Figure 1.17.
- **Circle**, when $\alpha = 0$, the trajectory is a circle of radius r_0, see Figure 1.18.

Nonzero Multiple Eigenvalues

The change of coordinates $z = M^{-1}x$ transforms the system into the form

$$\begin{aligned} \dot{z}_1 &= \lambda z_1 + k z_2 \\ \dot{z}_2 &= \lambda z_2 \end{aligned}$$

whose solution, for a given initial state (z_{10}, z_{20}), is given by

$$\begin{aligned} z_1(t) &= (z_{10} + k z_{20} t)\, e^{\lambda t} \\ z_2(t) &= e^{\lambda t} z_{20}. \end{aligned}$$

Eliminating t, we obtain the trajectory equation

$$z_1 = z_2 \left[\frac{z_{10}}{z_{20}} + \frac{k}{\lambda} \ln\left(\frac{z_2}{z_{20}} \right) \right]$$

Here there are only two cases:

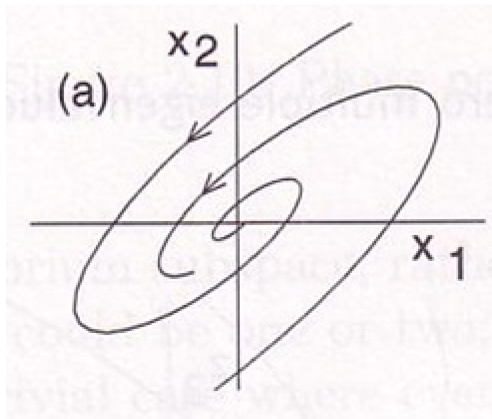


Figure 1.16: Stable Focus.

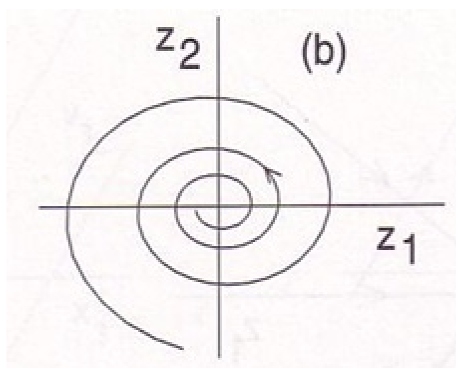


Figure 1.17: Unstable Focus.

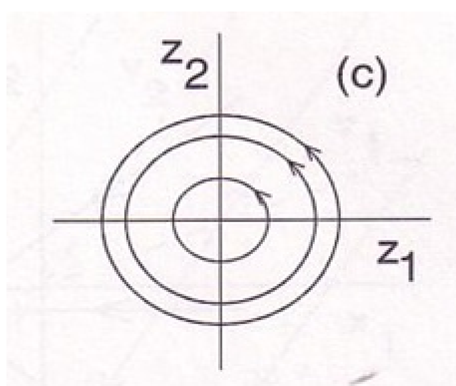


Figure 1.18: Circle.

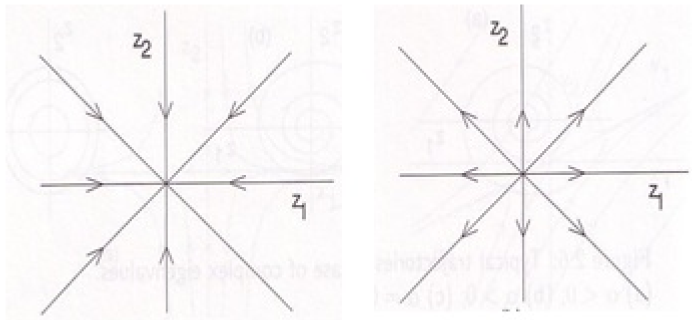


Figure 1.19: As many Eigenvectors as Eigenvalues.

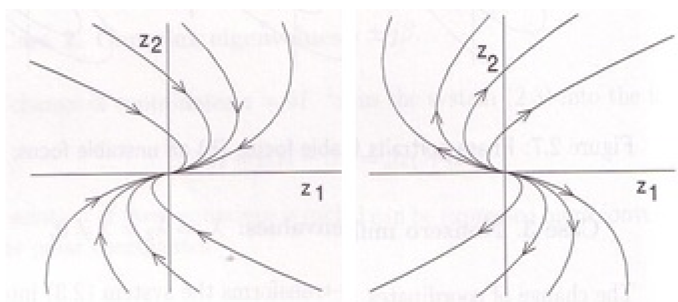


Figure 1.20: Fewer Eigenvectors than Eigenvalues.

- As many eigenvectors as eigenvalues: $k = 0$ ($\lambda < 0, \lambda > 0$), see Figure 1.19.
- Fewer eigenvectors than eigenvalues: $k = 1$ ($\lambda < 0, \lambda > 0$), see Figure 1.20.

One or more Eigenvalues are zero When one or both eigenvalues of A are zero, the phase portrait is in some sense degenerate. Here, the matrix A has a nontrivial null space. Any vector in the null space of A is an equilibrium point for the system; that is, the system has an equilibrium subspace, rather than an equilibrium point.

The dimension of the null space could be one or two; if it is two, the matrix A will be the zero matrix. When the dimension of the null space is one, the shape of the Jordan form of A will depend on the multiplicity of

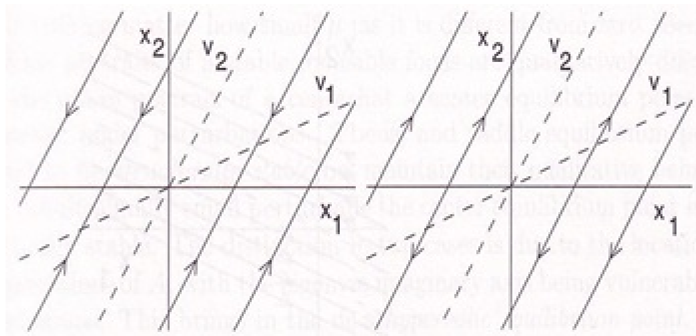


Figure 1.21: One Zero Eigenvalue.

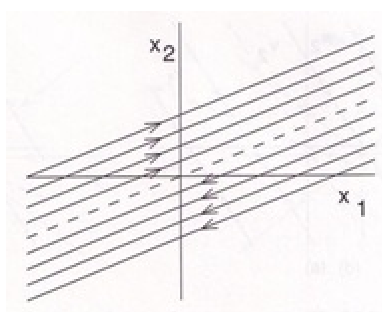


Figure 1.22: Zero Eigenvalues.

the zero eigenvalue. When $\lambda_1 = 0$ and $\lambda_2 \neq 0$, the matrix M is given by $M = [v_1, v_2]$ where v_1 and v_2 are the associated eigenvectors.

Here there are two cases:

- $\lambda_1 = 0$ and $(\lambda_2 < 0, \lambda_2 > 0)$, see Figure 1.21.
- $\lambda_1 = 0\lambda_2 = 0$, see Figure 1.22.

1.4.2 Limit Cycles and Bifurcations

Oscillations and Limit Cycles Oscillations are one of the most important phenomena that occur in dynamical systems. A system

$$\dot{x} = f(x)$$

oscillates when it has a nontrivial periodic solution, i.e.

$$x(t+T) = x(t), \forall t \geq 0$$

for some $T > 0$. The word nontrivial is used to exclude equilibrium point. In a phase portrait an oscillation or periodic solution is given by a closed trajectory, usually called a *closed* or *periodic orbit.* The trajectory resembles a closed curve in the phase plane. The trivial example is a *center.*

Example 1.5 *Van der Pol Oscillator*

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \varepsilon\left(1 - x_1^2\right)x_2\end{aligned}$$

In the case $\varepsilon = 0$ we have a continuum of periodic solutions, while in the case $\varepsilon \neq 0$ there is only one.

Definition 1.6 *An isolated periodic orbit is called a limit cycle.*

Figure 1.23 depicts a stable and an unstable limit cycle. Limit cycles are themselves special cases of *limit sets.* However the study of general limit sets is outside the scope of this course.

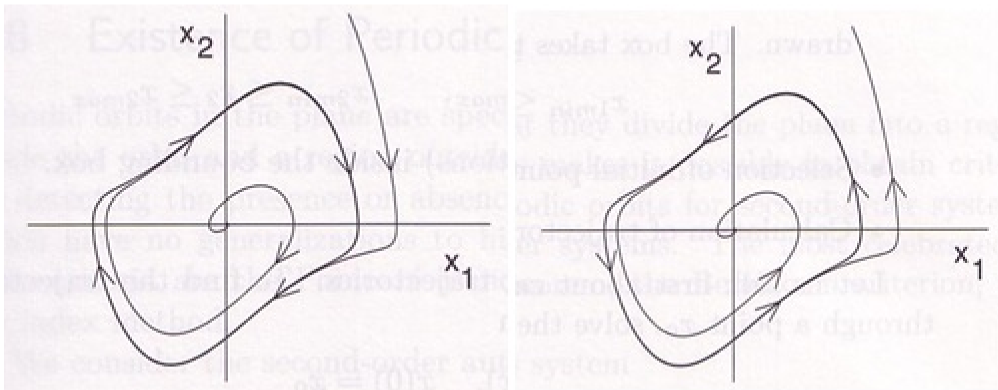


Figure 1.23: Limit Cycles.

Existence of Periodic Orbits Periodic orbits in the plane are special in that they divide the plane into a region inside the orbit and a region outside it. This makes it possible to obtain criteria for detecting the presence or absence of periodic orbits for second-order systems, which have no generalizations to higher order systems.

Theorem 1.7 ***Poincaré-Bendixson Criterion*** *Consider the system on the plane $\dot{x} = f(x)$ and let M be a closed bounded subset of the plane such that:*

1. *M contains no equilibrium points, or contains only one equilibrium point such that the Jacobian matrix $[df/dx]$ at this point has eigenvalues with positive real parts.*

2. *Every trajectory starting in M remains in M for all future time.*

Then M contains a periodic orbit of $\dot{x} = f(x)$.

Sketch of Proof: A trajectory in M is bounded, and so it must either converge to an equilibrium point or approach a periodic solution as time tends to infinity. As there are no stable equilibrium points, it must converge to a periodic orbit.

Theorem 1.8 ***Negative Pointcaré-Bendixson Criterion*** *If, on a simply connected region D of the plane, the expression $\partial f_1/\partial x_1 + \partial f_2/\partial x_2$ is not identically zero and does not change sign, then the system $\dot{x} = f(x)$ has no periodic orbits lying entirely in D.*

Proof: On any orbit $\dot{x} = f(x)$, we have

$$\frac{dx_2}{dx_1} = \frac{f_2}{f_1}.$$

Therefore, on any closed orbit γ, we have

$$\oint_{\Gamma} f_2(x_1, x_2)\, dx_1 - f_1(x_1, x_2)\, dx_2 = 0$$

This implies, by Green's theorem, that

$$\iint_S \left(\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) dx_1 dx_2 = 0$$

where S is the interior of γ. Now, if

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \neq 0 \ \ on \ \mathcal{D},$$

then we cannot find a region $S \subset \mathcal{D}$ such that the last equality holds. Hence, there can be no closed orbits entirely in $\mathcal{D}$.

1.4.3 Bifurcations

The qualitative behavior of a second-order system is determined by the pattern of its equilibrium points and periodic orbits, as well as by their stability properties. One issue of practical importance is whether the system maintains its qualitative behavior under infinitesimally small perturbations.

For a nonlinear autonomous dynamical system $\dot{x} = f(x)$, the set of equilibria is given by $\{x : f(x) = 0\}$. The eigenvalues λ of $\frac{\partial f}{\partial x}\big|_{x=x^*}$ the Jacobian of f at an equilibrium x^* determine the local stability properties for the system for $Re(\lambda) \neq 0$. For $Re(\lambda) = 0$ further analysis is required.

Example 1.9 *The systems*

$$\dot{x} = -x^3, \quad \dot{x} = x^3$$

have the same linearization but quite different stability properties.

In general, for a dynamical system dependent on a parameter μ,

$$\dot{x} = f(x, \mu)$$

the system is called *structurally stable* if small perturbations of the parameter μ do not cause the equilibria to change their stability properties, and if there are no additional equilibria created. Since the eigenvalues usually depend continuously on μ, this will usually occur only when $Re(\lambda(\mu)) = 0$.

Definition 1.10 *The creation of a new set of equilibria is called a bifurcation.*

We are interested in perturbations that will change the equilibrium points or periodic orbits of the system or change their stability properties. The points in parameter space at which the system behavior change are called bifurcation points. A bifurcation can include:

1. The creation of a stable-unstable pair
2. Vanishing of a stable-unstable pair
3. Pitchfork bifurcations
   (a) A stable equilibrium becomes 2 stable and one unstable equilibrium
   (b) An unstable equilibrium becomes 2 unstable and one stable equilibrium

Example 1.11 *We consider the system*

$$\begin{aligned} \dot{x}_1 &= \mu - x_1^2 \\ \dot{x}_2 &= -x_2. \end{aligned}$$

1. *If $\mu > 0$ we have two equilibrium points $Q_{1,2} = \left(\pm\sqrt{\mu}, 0\right)$, of which one is a saddle and the other one is a stable node.*
2. *If $\mu = 0$ we have the degenerated case $(0, 0)$. We have an equilibrium point at the origin, where the Jacobian matrix has one zero eigenvalue. This is the bifurcation point where saddle and stable node collide.*

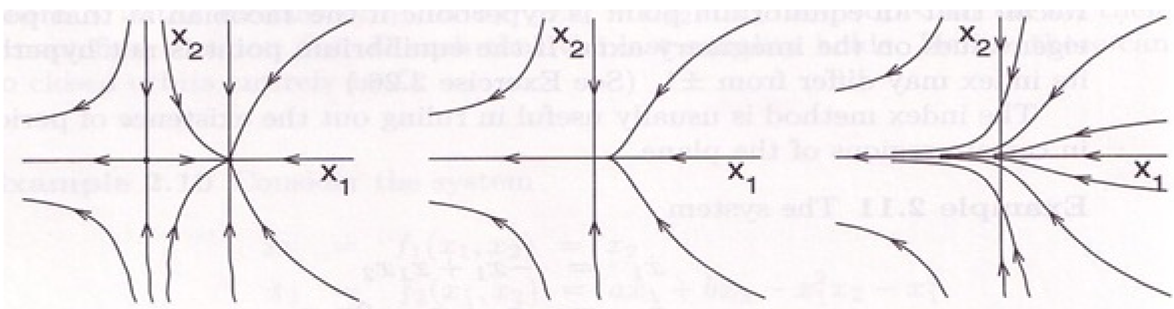


Figure 1.24: Bifurcations: phase portrait of the saddle-node bifurcation example for $\mu > 0$ (left), $\mu = 0$ (center), $\mu < 0$ (right).

3. *If $\mu < 0$ there are no equilibrium points anymore and both saddle and stable node disappeared.*

See Figure 1.24.

Theorem 1.12 ***Hopf Bifurcations*** *Consider the system*

$$\dot{x} = f_\mu(x_1, x_2),$$

where μ is a parameter. Let $(0,0)$ be an equilibrium point for any value of the parameter μ. Let μ_c be such that

$$eig\left[\frac{df}{dx}(0,0)\right] = \pm j\beta.$$

If the real parts of the eigenvalues $\lambda_{1,2}$ of $\frac{df}{dx}(0,0)$ are such that $\frac{d}{d\mu}Re\left(\lambda_{1,2}\right) > 0$ and the origin is stable for $\mu = \mu_c$ then

1. *there is a μ_l such that the origin remains stable for $\mu \in [\mu_l, \mu_c]$*

2. *there is a μ_L such that the origin is unstable surrounded by a stable limit cycle for $\mu_c < \mu < \mu_L$.*

Thus a Hopf Bifurcation is a bifurcation where a stable point bifurcates into a stable limit cycle which surrounds an unstable point.

Example 1.13 *We consider the system*

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + (\mu - x_1^2)x_2\end{aligned}$$

We realize that

$$\lambda_{1,2} = 0.5\mu \pm \sqrt{0.25\mu^2 - 1}.$$

Thus, it is easily checked that, as for $\mu \in [-0.2; 0.2]$, the derivative $\frac{d}{d\mu} Re\,(\lambda_{1,2}.) =$ $0.5 > 0$. The real parts become zero at $\mu = 0$. In the next three figures we depict the behavior of the system as μ goes from a negative to a positive value.

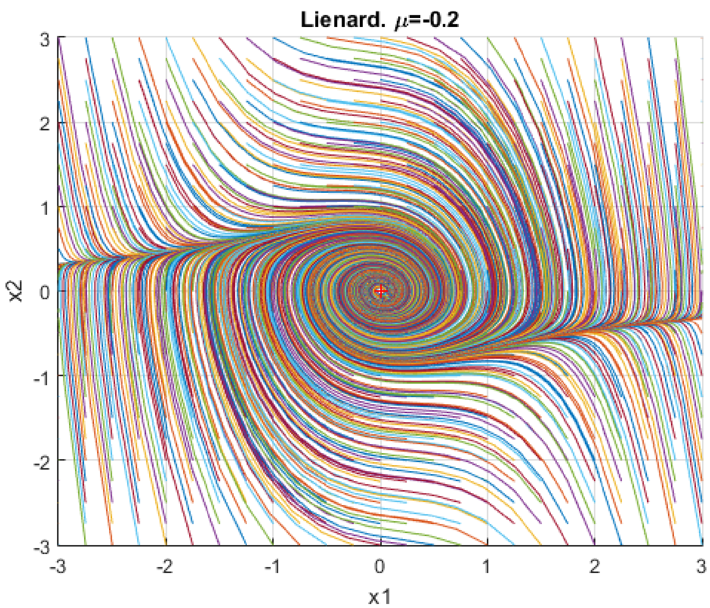


Figure 1.25: Hopf Bifurcation Example for $\mu = -1$.

Example 1.14 *Tunnel-Diode Circuit Revisiting Example 1.3.2 we see that E and R are bifurcations parameters for the system.*

Remark 1.15 *A cascade of bifurcations is usually a precursor to chaotic behavior. Chaotic behavior is observed when the effect of arbitrarily small differences in initial conditions are sufficient to make long term prediction of the trajectory of a system unpredictable. A further consideration of these phenomena is outside the scope of this nonlinear control course, but the reader is encouraged to read more on the subject of chaos of dynamical systems.*

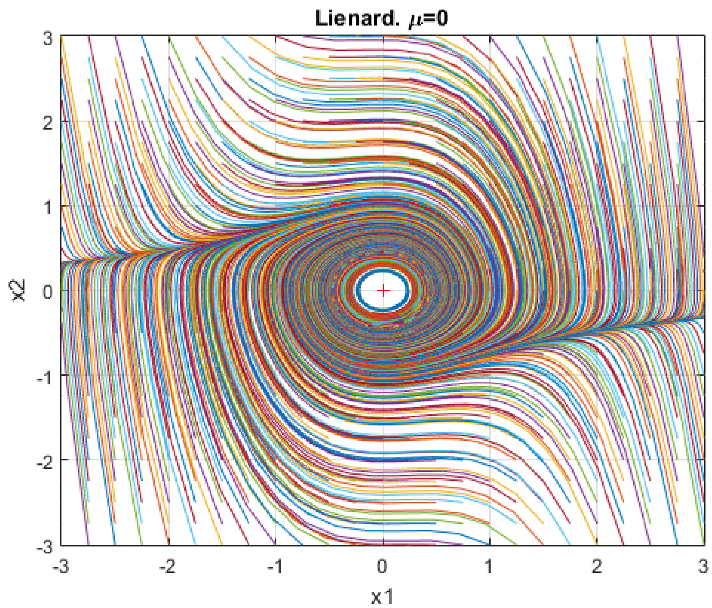


Figure 1.26: Hopf Bifurcation Example for $\mu = 0$.

1.5 Exercises

1. Consider the nonlinear system

$$\dot{x}_1 = \mu - x_1(x_1^2 - 1)$$
$$\dot{x}_2 = -x_2$$

   What is the behavior of the nonlinear system as the parameter μ varies?

2. Consider the nonlinear system

$$\dot{x}_1 = g(x_2) + 4x_1x_2^2$$
$$\dot{x}_2 = h(x_1) + 4x_1^2x_2$$

   Are there any limit cycles?

3. A pitchfork bifurcation example is given by

$$\dot{x}_1 = \mu x_1 - x_1^3$$
$$\dot{x}_2 = -x_2$$

   What is the behavior of the nonlinear system as the parameter μ varies?

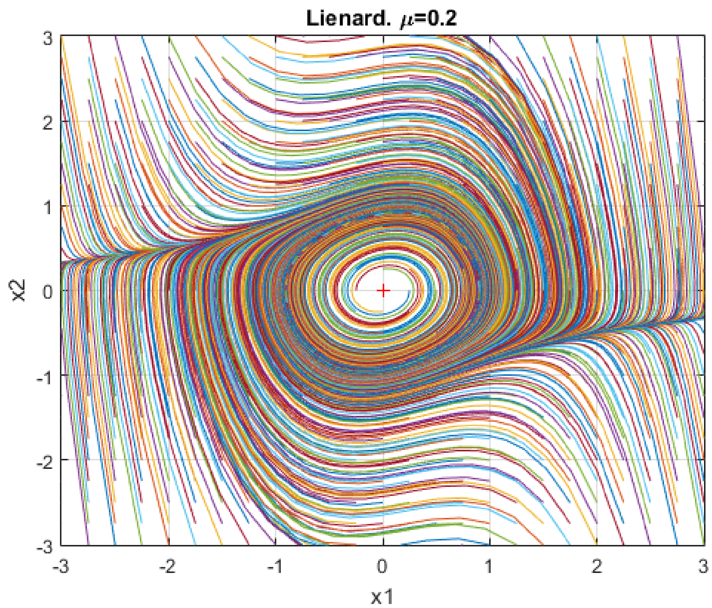


Figure 1.27: Hopf Bifurcation Example for $\mu = 1$.

4. Consider again the Tunnel Diode Circuit dynamics that are given by

$$\dot{x}_1 = \frac{1}{C}\left[-h(x_1) + x_2\right]$$
$$\dot{x}_2 = \frac{1}{L}\left[-x_1 - Rx_2 + u\right]$$

where we take $C = 2$, $L = 5$ and $R = 1.5$. Suppose that $h(\cdot)$ is given by

$$h(x_1) = 17.76x_1 103.79x_1^2 + 229.62x_1^3 226.31x_1^4 + 83.72x_1^5$$

- Find the equilibrium points of the system for $u = 1.2$V ? What is the nature of each equilibrium point?

Chapter 2

Ordinary Differential Equations

2.1 Existence and Uniqueness

Theorem 2.1 ***Local Existence and Uniqueness:*** *Let $f(t,x)$ be a piecewise continuous function in t and satisfy the Lipschitz condition*

$$\|f(t,x) - f(t,y)\| \leq L\|x-y\|$$

$\forall x, y \in \mathcal{B}(x_0, r) = \{x \in \mathbb{R}^n | \, \|x - x_0\| \leq r\}$, $\forall t \in [t_0, t_1]$. *Then there exists some $\delta > 0$ such that the state equation*

$$\dot{x} = f(t,x)$$

with $x(t_0) = x_0$ has a unique solution over $[t_0, t_0 + \delta]$.

Proof: We start by noting that if $x(t)$ is a solution of

$$\dot{x} = f(t,x), x(t_0) = x_0$$

then, by integration, we have

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds \tag{2.1}$$

It follows that the study of existence and uniqueness of the solution of the differential equation is equivalent to the study of existence and uniqueness of the solution of the integral equation above.

Let us introduce some convenient notation. Let

$$(Px)(t) := x(t_0) + \int_{t_0}^{t} f\left(s, x\left(s\right)\right) ds, \quad t \in [t_0, t_1],$$

be a mapping of a continuous function

$$x : [t_0, t_1] \to \mathbb{R}^n.$$

in to $\mathbb{R}$.

Then, we realize that we can rewrite the Equation (2.1) as

$$x\left(t\right) = \left(Px\right)\left(t\right)$$

Note that $\left(Px\right)\left(t\right)$ is continuous in t. A solution of $x\left(t\right) = \left(Px\right)\left(t\right)$ can be established by using the contraction mapping theorem, see Appendix 2.3.

Applying this result to our problem requires defining appropriate

1. Banach space $\mathcal{X}$ and
2. closed set $S \subset \mathcal{X}$

such that P maps S into S and is a contraction over S. Let

$$\mathcal{X} = C\left[t_0, t_0 + \delta\right]$$

with norm

$$\|x\|_C = \max_{t \in [t_0, t_0+\delta]} \|x\left(t\right)\|$$

and

$$S = \{x \in \mathcal{X} \mid \|x - x_0\|_C \leq r\}$$

where r is the radius of the ball $\mathcal{B}\left(x_0, r\right)$ and δ is a positive constant to be chosen. We will restrict the choice of δ to satisfy $\delta \leq t_1 - t_0$ so that $[t_0, t_0 + \delta] \subset [t_0, t_1]$. Note that $\|x\left(t\right)\|$ denotes a norm on $\mathbb{R}^n$, while $\|x\|_C$ denotes a norm on $\mathcal{X}$. Also, $\mathcal{B}\left(x_0, r\right)$ is a ball in $\mathbb{R}^n$, while S is a ball in $\mathcal{X}$.

By definition, P maps $\mathcal{X}$ into $\mathcal{X}$. To show that it maps S into S write

$$\left(Px\right)\left(t\right) - x_0 = \int_{t_0}^{t} f\left(s, x\left(s\right)\right) ds = \int_{t_0}^{t} \left[f\left(s, x\left(s\right)\right) - f\left(s, x_0\right) + f\left(s, x_0\right)\right] ds$$

By piecewise continuity of f, we know that $f\left(s, x_0\right)$ is bounded on $[t_0, t_1]$. Let

$$h = \max_{t \in [t_0, t_1]} \|f(t, x_0)\|$$

Using the Lipschitz condition and the fact that for each $x \in S$

$$\|x(t) - x_0\| \le r, \forall t \in [t_0, t_0 + \delta]$$

we obtain

$$\begin{aligned}
\|(Px)(t) - x_0\| &\le \int_{t_0}^{t} \left[\|f(s, x(s)) - f(s, x_0)\| + \|f(s, x_0)\|\right] ds \\
&\le \int_{t_0}^{t} \left[L\|x(s) - x_0\| + h\right] ds \\
&\le \int_{t_0}^{t} (Lr + h)\, ds \\
&\le (t - t_0)(Lr + h) \\
&\le \delta (Lr + h)
\end{aligned}$$

and

$$\|Px - x_0\|_C = \max_{t \in [t_0, t_0 + \delta]} \|(Px)(t) - x_0\| \le \delta (Lr + h)$$

Hence, choosing $\delta \le r/(Lr + h)$ ensures that P maps S into S.

To show that P is a contraction mapping over S, let x and $y \in S$ and consider

$$\begin{aligned}
\|(Px)(t) - (Py)(t)\| &\le \left\|\int_{t_0}^{t} \left[f(s, x(s)) - f(s, y(s))\right] ds\right\| \\
&\le \int_{t_0}^{t} \|\left[f(s, x(s)) - f(s, y(s))\right]\| ds \\
&\le \int_{t_0}^{t} L\|x(s) - y(s)\| ds \\
&\le \int_{t_0}^{t} L\|x - y\|_C\, ds
\end{aligned}$$

Therefore,

$$\|Px - Py\|_C \le L\delta \|x - y\|_C \le \rho \|x - y\|_C,$$

for $\delta < {}^{\rho}/_{L}$ with $\rho < 1$. Thus, choosing $\rho < 1$ and $\delta \le \rho/L$ ensures that P is a contraction mapping over S.

By the contraction mapping theorem, we can conclude that if δ is chosen to satisfy

$$\delta \leq \min\left\{t_1 - t_0, \frac{r}{Lr+h}, \frac{\rho}{L}\right\}, \quad \textit{for } \rho < 1,$$

then the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

will have a unique solution in S.

This is not the end of the proof though because we are interested in establishing uniqueness of the solution among all continuous functions $x(t)$, that is, uniqueness in $\mathcal{X}$. It turns out that any solution of

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

in $\mathcal{X}$ will lie in S. To see this, note that since $x(t_0) = x_0$ is inside the ball $\mathcal{B}(x_0, r)$, any continuous solution $x(t)$ must lie inside $\mathcal{B}(x_0, r)$ for some interval of time. Suppose that $x(t)$ leaves the ball $\mathcal{B}(x_0, r)$ and let $t_0 + \mu$ be the first time $x(t)$ intersects the boundary of $\mathcal{B}(x_0, r)$. Then,

$$\|x(t_0 + \mu) - x_0\| = r$$

On the other hand, for all $t \leq t_0 + \mu$,

$$\begin{aligned}
\|x(t) - x_0\| &\leq \int_{t_0}^{t} \left[\|f(s, x(s)) - f(s, x_0)\| + \|f(s, x_0)\|\right] ds \\
&\leq \int_{t_0}^{t} \left[L\|x(s) - x_0\| + h\right] ds \\
&\leq \int_{t_0}^{t} (Lr + h)\, ds
\end{aligned}$$

Therefore,

$$r = \|x(t_0 + \mu) - x_0\| \leq (Lr + h)\mu \Rightarrow \mu \geq \frac{r}{Lr + h} \geq \delta$$

Hence, the solution $x(t)$ cannot leave the set $\mathcal{B}(x_0, r)$ within the time interval $[t_0, t_0 + \delta]$, which implies that any solution in $\mathcal{X}$ lies in S. Consequently, uniqueness of the solution in S implies uniqueness in $\mathcal{X}$.

□

Remark 2.2 *Some remarks about the Lipschitz condition:*

1. *Lipschitz is stronger than continuity*
2. *There exist continuous functions which are not Lipschitz. For example*

$$f(x) = x^{\frac{1}{3}}$$

is not Lipschitz. In particular, the gradient becomes infinite at x=0. Accordingly, the differential equation

$$\dot{x} = x^{\frac{1}{3}}, \quad x(0) = 0$$

does not have a unique solution. There exists an entire family of solutions:

$$x(t) = \begin{cases} 0 & t \in [0, c) \\ \left(\frac{2}{3}\right)^{3/2} (t-c)^{3/2} & t \in [c, \infty) \end{cases}$$

for any $c > 0$.

3. *The norm for which the Lipschitz condition holds is not relevant.*
4. *Locally bounded $\frac{\partial f}{\partial x} \Rightarrow$ locally Lipschitz*
5. *Lipschitz $\neq$ differentiability. E.g.*

$$f(x) = \begin{bmatrix} x_2 \\ -sat(x_1 + x_2) \end{bmatrix}$$

is Lipschitz but not differentiable.

Theorem 2.3 ***Global Existence and Uniqueness:*** *Let $f(t, x)$ be piecewise continuous in t over the interval $[t_0, t_1]$ and globally Lipschitz in x. Then*

$$\dot{x} = f(t, x), \quad x(t_0) = x_0$$

has a unique solution over $[t_0, t_1)$.

Example 2.4 *Consider the function*

$$f(x) = x^2 \Rightarrow f'(x) = 2x$$

The derivative is not bounded therefore $f(x)$ is not globally Lipschitz. Accordingly we cannot expect the solution to the differential equation $\dot{x} = f(x)$ to have a unique solution over any time interval. Indeed, the differential equation:

$$\dot{x} = x^2, \quad x(0) = c > 0$$

has the solution

$$x(t) = \left(\frac{1}{c} - t\right)^{-1}$$

which has a vertical asymptote at $t = \frac{1}{c}$, which means that the solution exhibits finite escape time.

Nevertheless, it may be that a function $f(t, x)$ is not globally Lipschitz, and yet the differential does have a solution which is defined over the entire time interval. Consider the following example:

Example 2.5 *Consider the function $f(x) = -x^3$. $\frac{\partial f}{\partial x}$ is not bounded therefore the function f isn't globally Lipschitz. However*

$$x(t) = sign(x_0)\sqrt{\frac{x_0^2}{1 + 2x_0^2(t - t_0)}}$$

is the solution of $f(x) = -x^3$ for any x_0 and $t > t_0$.

To develop this further, recall the mapping P from the proof of Theorem 2.1. Then:

Theorem 2.6 *Let f be piecewise continuous in t, locally Lipschitz in x for $x \in \mathcal{D}$. Let w be a compact subset of $\mathcal{D}$. If $x(t) = (Px)(t) \in w$, $\forall t > t_0$, i.e., w is invariant, then the solution x(t) exists $\forall t > t_0$.*

Example 2.7 *Van der Pol Oscillator.*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \varepsilon\left(1 - x_1^2\right)x_2 \end{aligned}$$

Then

$$\frac{\partial f}{\partial x} = \begin{pmatrix} 0 & 1 \\ -1 - 2\varepsilon x_1 x_2 & \varepsilon\left(1 - x_1^2\right) \end{pmatrix}$$

The Jacobian $\frac{\partial f}{\partial x}$ *is bounded in a bounded domain, thus* f *is locally Lipschitz and the solution of the differential equation exists locally. However, global existence has not been established due to unboundedness of* $\frac{\partial f}{\partial x}$ *over the entire state space.*

2.2 Continuity and Differentiability

For the solution of the state space equation

$$\dot{x} = f\left(t, x\right)$$

with $x\left(t_0\right) = x_0$ to be of interest in real applications, it must depend continuously on the initial state x_0, the initial time t_0, and the right-hand side function $f\left(t, x\right)$. Continuous dependence on the initial time t_0 is obvious from the integral expression

$$x\left(t\right) = x_0 + \int_{t_0}^{t} f\left(s, x\left(s\right)\right) ds$$

Let $y\left(t\right)$ be a solution of the equation that starts at $y\left(t_0\right) = y_0$ and is defined on the compact time interval $[t_0, t_1]$. The solution depends continuously on y_0 if solutions starting at nearby points are defined on the same time interval and remain close to each other in that interval.
Let $x\left(t, \lambda_0\right)$ with nominal parameters λ_0 be a solution of $\dot{x} = f\left(t, x, \lambda_0\right)$ defined on $[t_0, t_1]$, with $x\left(t_0, \lambda_0\right) = x_0$. The solution is said to be dependent continuously on λ if for any $\varepsilon > 0$, there is $\delta > 0$ such that for all λ in the ball $\{\lambda \in \mathbb{R}^p | \, \|\lambda - \lambda_0\| < \delta\}$, the equation $\dot{x} = f\left(t, x, \lambda\right)$ has a unique solution $x\left(t, \lambda\right)$ defined on $[t_0, t_1]$, with $x\left(t_0, \lambda\right) = x_0$, and satisfies $\|x\left(t, \lambda\right) - x\left(t, \lambda_0\right)\| < \varepsilon$ for all $t \in [t_0, t_1]$.

Theorem 2.8 *Let* $f\left(t, x, \lambda\right)$ *be continuous in* $\left(t, x, \lambda\right)$ *and locally Lipschitz in* x *(uniformly in* t *and* λ*) on* $[t_0, t_1] \times \mathcal{D} \times \{\|\lambda - \lambda_0\| \leq c\}$, *where* $\mathcal{D} \subset \mathbb{R}^n$ *is an open connected set.*

Let $y(t,\lambda_0)$ be a solution of $\dot{x} = f(t,x,\lambda_0)$ with $y(t_0,\lambda_0) = y_0 \in \mathcal{D}$. Suppose $y(t,\lambda_0)$ is defined and belongs to $\mathcal{D}$ for all $t \in [t_0,t_1]$. Then, given $\varepsilon > 0$, there is $\delta > 0$ such that if

$$\|z_0 - y_0\| < \delta, \quad \|\lambda - \lambda_0\| < \delta$$

Then there is a unique solution $z(t,\lambda)$ of $\dot{x} = f(t,x,\lambda)$ defined on $[t_0,t_1]$, with $z(t_0,\lambda) = z_0$, and $z(t,\lambda)$ satisfies

$$\|z(t,\lambda) - y(t,\lambda_0)\| < \varepsilon, \quad \forall t \in [t_0,t_1].$$

Proof: By continuity of $y(t,\lambda_0)$ in t and the compactness of $[t_0,t_1]$, we know that $y(t,\lambda_0)$ is bounded on $[t_0,t_1]$. Define a tube U around the solution $y(t,\lambda_0)$ by

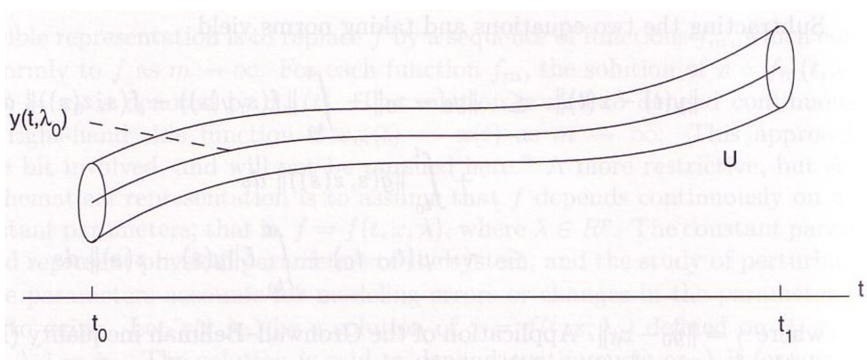


Figure 2.1: Tube constructed around solution $y(t,\lambda_0)$

$$U = \{(t,x) \in [t_0,t_1] \times \mathbb{R}^n | \, \|x - y(t,\lambda_0)\| \le \varepsilon\}$$

We can estimate

$$\|y(t) - z(t)\| \le \|y_0 - z_0\| \exp[L(t-t_0)] + \frac{\mu}{L}\{\exp[L(t-t_0) - 1]\}$$

where L is the Lipschitz constant and μ is a constant related to the size of the perturbation of the parameters.

□

Remark 2.9 *Note that the proof does not estimate the radius of the "tube" in Figure 2.1.*

2.3 Differentiability and Sensitivity Equation

The continuous differentiability of f with respect to x and λ implies the additional property that the solution $x(t,\lambda)$ is differentiable with respect to λ near λ_0. To see this, write

$$x(t,\lambda) = x_0 + \int_{t_0}^{t} f(s, x(s,\lambda), \lambda)\, ds$$

Taking partial derivative with respect to λ yields

$$\frac{\partial x}{\partial \lambda} = \int_{t_0}^{t} \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial \lambda} + \frac{\partial f}{\partial \lambda} \right] ds$$

Differentiating with respect to t, it can be seen that

$$\frac{\partial x}{\partial \lambda}(t_0,\lambda) = 0$$

$$\frac{\partial}{\partial t}\frac{\partial x}{\partial \lambda} = A(t,\lambda)\left(\frac{\partial x}{\partial \lambda}\right) + B(t,\lambda)$$

where

$$\begin{aligned} A(t,\lambda) &= \frac{\partial f}{\partial x}(t, x(t,\lambda), \lambda) \\ B(t,\lambda) &= = \frac{\partial f}{\partial \lambda}(t, x(t,\lambda), \lambda) \end{aligned}$$

Now we see that $A(t,\lambda)$, $B(t,\lambda)$ are continuous functions. Let

$$S(t) = \left.\frac{\partial x}{\partial \lambda}\right|_{\lambda=\lambda_0}$$

Then $S(t)$ is the unique solution of the equation

$$\begin{aligned} \dot{S}(t) &= A(t,\lambda_0) S(t) + B(t,\lambda_0) \\ S(t_0) &= 0 \end{aligned}$$

The function $S(t)$ is called the sensitivity function. Sensitivity functions provide a first-order estimate of the effect of parameter variations on the solutions of the differential equations.

They can also be used to approximate the solution when λ is sufficiently close to it nominal value λ_0. Indeed, for small $\|\lambda - \lambda_0\|$, $x(t, \lambda)$ can be expanded in a Taylor series about the nominal solution $x(t, \lambda_0)$. Neglecting the higher-order terms, the solution $x(t, \lambda)$ can be approximated by

$$x(t, \lambda) \approx x(t, \lambda_0) + S(t)(\lambda - \lambda_0)$$

Example 2.10 *Van der Pol Oscillator. Let us consider (again) the following system.*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \varepsilon\left(1 - x_1^2\right) x_2 \end{aligned}$$

First, we plot the nominal trajectories of the nonlinear system for $\varepsilon = 1$ and the corresponding nominal phase portrait.

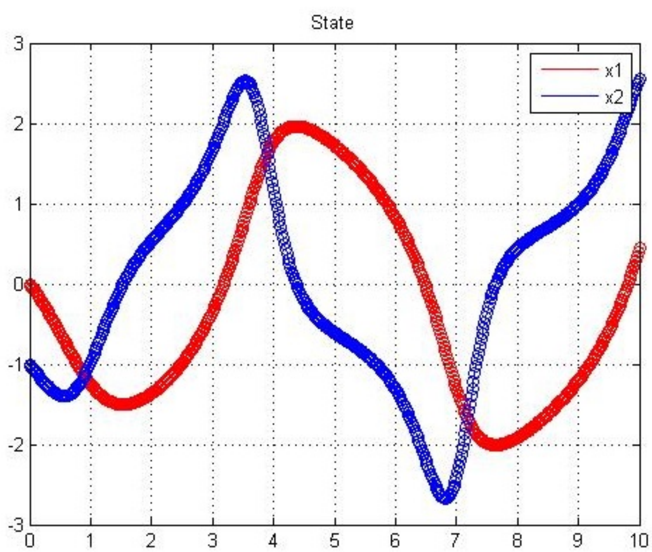


Figure 2.2: Oscillator Nominal Trajectories

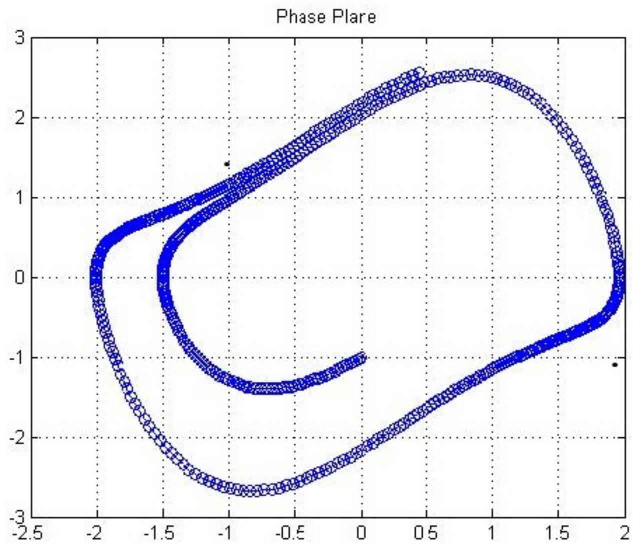


Figure 2.3: Oscillator Nominal Phase Portrait

The expressions for the sensitivity equations are

$$A(t,\lambda) = \frac{\delta f}{\delta x}\Big|_{\lambda=\lambda_0} = \begin{pmatrix} 0 & 1 \\ -1-2\varepsilon x_1 x_2 & \varepsilon(1-x_1^2) \end{pmatrix}$$
$$B(t,\lambda) = \frac{\delta f}{\delta \lambda}\Big|_{\lambda=\lambda_0} = \begin{pmatrix} 0 \\ (1-x_1^2)\,x_2 \end{pmatrix}$$

The sensitivity equations can now be solved along the nominal trajectory. The results can be inspected in Figure 2.4. Examining the shape of these curves, we conclude that the solutions are more sensitive to parameter disturbances at $t = 4$ and $t = 7$, which correspond to the points $x = (2,2)$ and $x = (1,-1)$ in the phase portrait'Figure 2.3. In Figure 2.5 we plot the approximated trajectories together with the nominal phase portrait.

$$x(t,\varepsilon) = x_{nominal}(t,\varepsilon_0) + S(t)(\varepsilon-\varepsilon_0), \quad \varepsilon = 1.75$$

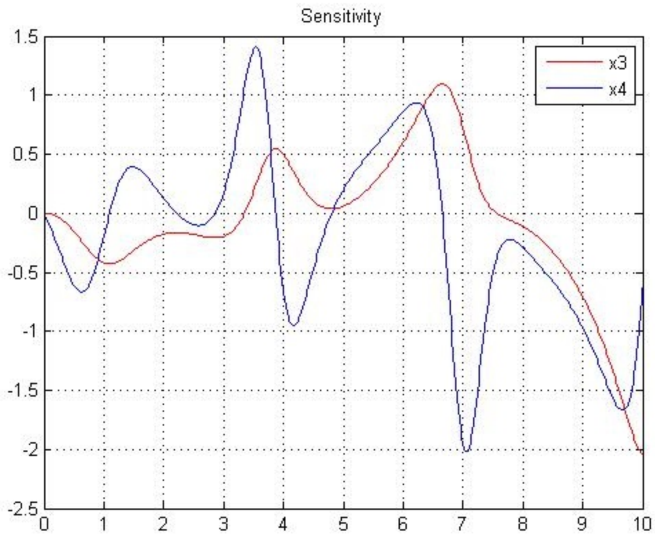


Figure 2.4: Oscillator Nominal Sensitivity

Appendix: Contraction Mappings and Fixed Point Theorem

Let $(\mathcal{X}, \|\cdot\|)$ be a non-empty complete normed space (Banach) and $S \subset \mathcal{X}$ a closed subset. Let $P : \mathcal{X} \to \mathcal{X}$ be a mapping that

1. it is a *contraction mapping* on S, i.e: there exists $\rho \in [0, 1)$ such that

$$\|Px - Py\| \leq \rho \|x - y\|, \forall x, y \in S$$

2. and $P(S) \subset S$, i.e., $Px \in S$, for all $x \in S$,

then the map P admits one and only one fixed point x^* in S (this means $Px^* = x^*$). Furthermore, this fixed point can be found as follows: start with an arbitrary element x_0 in S and define an iterative sequence by $x[n] = P(x[n-1])$ for $n = 1, 2, 3, \cdots$ This sequence converges, and its limit is x^*. When using the theorem in practice, the most difficult part is typically to define S properly so that P actually maps elements from S to S, i.e. that Px is always an element of S.

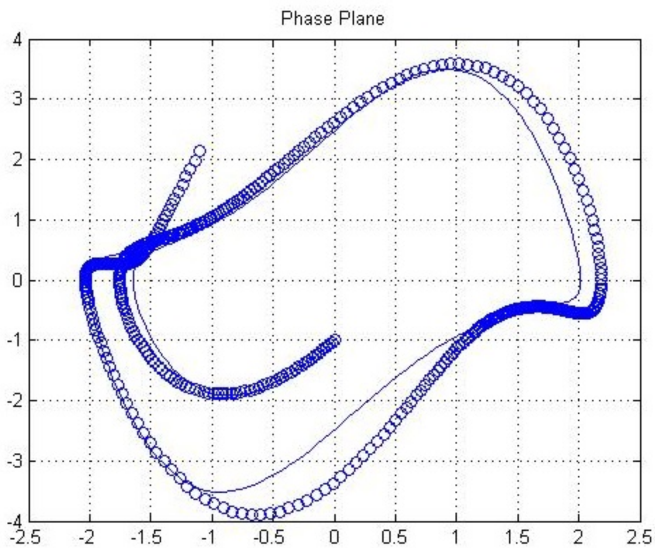


Figure 2.5: Phase Portrait after Perturbation

2.4 Exercises

1. Derive the sensitivity equations for the system

$$\begin{aligned} \dot{x}_1 &= \tan^{-1}(ax_1) - x_1x_2 \\ \dot{x}_2 &= bx_1^2 - cx_2 \end{aligned} \tag{2.2}$$

as the parameters a, b, c vary from their nominal values $a_0 = 1$, $b_0 = 0$, and $c_0 = 1$.

2. Determine whether the following differential equations have a unique solution for $t \in [0, \infty)$:

   (a) $\dot{x}(t) = 3x(t)$

   (b) $\dot{x}(t) = e^{x(t)} - 1 - x$ Hint: expand e^x in a Taylor's series

   (c) $\dot{x}(t) = e^{-|x(t)|} - 1$

   (d) $\begin{cases} \dot{x}_1(t) = -x_1^3(t) \\ \dot{x}_2(t) = x_1(t) - x_2(t) - x_1^3(t) \end{cases}$

3. Consider the following Cauchy problem

$$\begin{cases} \dot{x} = \sqrt{x} \\ x(0) = 1 \end{cases} .$$

Is there a unique solution to it? Why? If so, find its analytical expression.
What if the initial condition becomes $x(0) = 0$? Can you write down at least two solutions to the problem? How many solution are there? Motivate your answer.

4. The functions $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz continuous . Prove that the following functions are locally Lipschitz continuous:

   (a) $f_1 + f_2 : x \to f_1(x) + f_2(x)$
   (b) $f_1 \cdot f_2 :\to f_1(x) f_2(x)$
   (c) f_1 o $\mathrm{f}_2 : \mathrm{x} \to \mathrm{f}_2\left(\mathrm{f}_1\left(\mathrm{x}\right)\right)$

5. Given a square matrix $A \in \mathbb{R}^{n \times n}$, consider the linear mapping $P : \mathbb{R}^n \mapsto \mathbb{R}^n$

$$P(x) = Ax\,.$$

If A is symmetric, under which conditions is $P(x)$ a contraction from $\mathbb{R}^n$ into itself? What about the general case, i.e. when A is not symmetric?
Hint: recall that a mapping $P : \mathbb{R}^n \mapsto \mathbb{R}^n$ is contractive if $||P(x) - P(y)|| < \rho||x - y||\ \forall x, y \in \mathbb{R}^n$ with $\rho < 1$.

Chapter 3

Lyapunov Stability Theory

3.1 Introduction

In this lecture we consider the stability of equilibrium points of nonlinear systems, both in continuous and discrete time. Lyapunov stability theory is a standard tool and one of the most important tools in the analysis of nonlinear systems. It may be utilized relatively easily to provide a strategy for constructing stabilizing feedback controllers.

3.2 Stability of Autonomous Systems

Consider the nonlinear autonomous (no forcing input) system

$$\dot{x} = f(x) \tag{3.1}$$

where $f : \mathcal{D} \longrightarrow \mathbb{R}^n$ is a locally Lipschitz map from the domain $\mathcal{D} \subseteq \mathbb{R}^n$ to $\mathbb{R}^n$. Suppose that the system (3.1) has an equilibrium point $\bar{x} \in \mathcal{D}$, i.e., $f(\bar{x}) = 0$. We would like to characterize if the equilibrium point $\bar{x}$ is stable. In the sequel, we assume that $\bar{x}$ is the origin of state space. This can be done without any loss of generality since we can always apply a change of variables to $\xi = x - \bar{x}$ to obtain

$$\dot{\xi} = f(\xi + \bar{x}) \triangleq g(\xi) \tag{3.2}$$

and then study the stability of the new system with respect to $\xi = 0$, the origin.

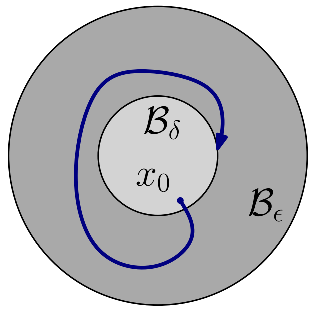


Figure 3.1: Illustration of the trajectory of a stable system

Definition 3.1 *The equilibrium $x = 0$ of (3.1) is*

1. ***stable***, *if for each $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t > t_0 \tag{3.3}$$

2. ***asymptotically stable*** *if it is stable and in addition δ can be chosen such that*

$$\|x(t_0)\| < \delta \Rightarrow \lim_{t\to\infty} \|x(t)\| = 0. \tag{3.4}$$

The following section provides characterizations of stability.

3.2.1 Lyapunov's Direct Method

Let $V : \mathcal{D} \longrightarrow \mathbb{R}$ be a continuously differentiable function defined on the domain $\mathcal{D} \subset \mathbb{R}^n$ that contains the origin. The rate of change of V along the trajectories of (3.1) is given by

$$\begin{aligned}\dot{V}(x) \triangleq \frac{d}{dt}V(x) &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i}\frac{d}{dt}x_i \\ &= \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} & \cdots & \frac{\partial V}{\partial x_n}\end{bmatrix} \dot{x} \quad = \frac{\partial V}{\partial x} f(x)\end{aligned} \tag{3.5}$$

The main idea of Lyapunov's theory is that if $\dot{V}(x)$ is negative along the trajectories of the system, then $V(x)$ will decrease as time goes forward. Moreover, we do not really need to *solve* the nonlinear ODE (3.1) for every initial condition; we only need some information about the drift $f(x)$.

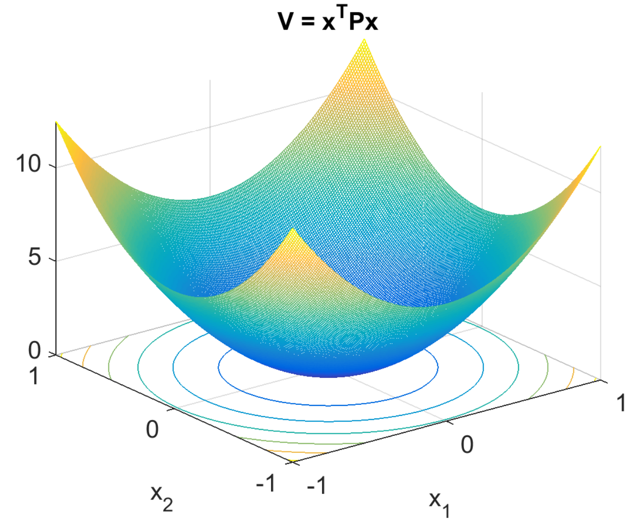


Figure 3.2: Lyapunov function in two states $V = x_1^2 + 1.5x_2^2$. The level sets are shown in the x_1x_2-plane.

Example 3.2 *Consider the nonlinear system*

$$\dot{x} = f(x) = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix} = \begin{bmatrix} -x_1 + 2x_1^2 x_2 \\ -x_2 \end{bmatrix}$$

and the candidate Lyapunov function

$$V(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2$$

with λ_1, $\lambda_2 > 0$. If we plot the function $V(x)$ for some choice of λ's we obtain the result in Figure 3.2. This function has a unique minimum over all the state space at the origin. Moreover, $V(x) \to \infty$ as $\|x\| \to \infty$.

Calculate the derivative of V along the trajectories of the system

$$\dot{V}(x) = 2\lambda_1 x_1(-x_1 + 2x_1^2 x_2) + 2\lambda_2 x_2(-x_2) = -2\lambda_1 x_1^2 - 2\lambda_2 x_2^2 + 4\lambda_1 x_1^3 x_2$$

If $\dot{V}(x)$ is negative, V will decrease along the solution of $\dot{x} = f(x)$.

We are now ready to state Lyapunov's stability theorem.

Theorem 3.3 (Direct Method) *Let the origin $x = 0 \in \mathcal{D} \subset \mathbb{R}^n$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that*

$$\begin{aligned} &V(0) = 0 \text{ and } V(x) > 0, \quad \forall x \in D \setminus \{0\} \\ &\dot{V}(x) \leq 0, \quad \forall x \in D \end{aligned} \tag{3.6}$$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0, \quad \forall x \in D \setminus \{0\}$$

then $x = 0$ is asymptotically stable

Remark 3.4 *If $V(x) > 0$, $\forall x \in D \setminus \{0\}$, then V is called locally positive definite. If $V(x) \geq 0$, $\forall x \in D \setminus \{0\}$, then V is locally positive semi-definite. If the conditions (3.6) are met, then V is called a Lyapunov function for the system $\dot{x} = f(x)$.*

Proof: Given any $\varepsilon > 0$, choose $r \in (0, \varepsilon]$ such that $B_r = \{x \in \mathbb{R}^n, \|x\| \leq r\} \subset D$. Let $\alpha = \min_{\|x\|=r} V(x)$. Choose $\beta \in (0, \alpha)$ and define $\Omega_\beta = \{x \in B_r, V(x) \leq \beta\}$.

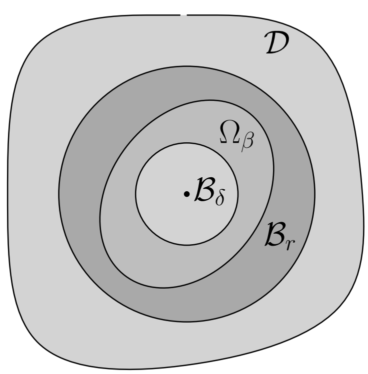


Figure 3.3: Various domains in the proof of Theorem 3.3

It holds that if $x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \ \forall t$ because

$$\dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta$$

Further $\exists\delta > 0$ such that $\|x\| < \delta \Rightarrow V(x) < \beta$. Therefore, we have that

$$B_\delta \subset \Omega_\beta \subset B_r$$

and furthermore

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

Finally, it follows that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \leq \varepsilon,\ \forall t > 0$$

In order to show asymptotic stability, we need to to show that $x(t) \to 0$ as $t \to \infty$. In this case, it turns out that it is sufficient to show that $V(x(t)) \to 0$ as $t \to \infty$. Since V is monotonically decreasing and bounded from below by 0, then

$$V(x) \to c \geq 0,\ \text{ as } t \to \infty$$

Finally, it can be further shown by contradiction that the limit c is actually equal to 0, using the following estimate of the decay rate

$$V(t) \leq V(0) + \int_0^t \dot{V}(x(\tau))d\tau - \gamma t$$

where $-\gamma = \max_{d \leq \|x\| \leq r} \dot{V}(x)$ is the slowest rate of decay of $V(x)$ over a compact set. γ exists since the function $\dot{V}(x)$ is continuous.

□

Example 3.5 *Recall Example 3.2. The derivative of the Lyapunov function candidate was given by*

$$\dot{V}(x) = -2\lambda_1 x_1^2 - 2\lambda_2 x_2^2 + 4\lambda_1 x_1^3 x_2$$

For simplicity, assume that $\lambda_1 = \lambda_2 = 1$. Then

$$\dot{V}(x) = -2x_2^2 - 2x_1^2 g(x)$$

where $g(x) \triangleq 1 - 2x_1 x_2$. Then the derivative of V is guaranteed to be negative whenever $g(x) > 0$. The level sets of V, where $\dot{V} < 0$ will be invariant, or equivalently when $g(x) > 0$, i.e., when $x_1 x_2 < 1/2$. So we conclude that the origin is locally asymptotically stable.

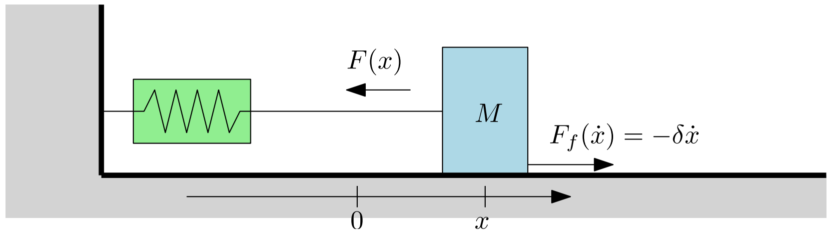


Figure 3.4: Mass spring system

Example 3.6 *Consider a mass M connected to a spring, as shown in Figure 4.2, where $x = 0$ is defined as the equilibrium, the point where there is no force exerted by the spring, i.e.,*

$$F(x)x > 0,\ \forall x \neq 0, \quad F(x) = 0 \Leftrightarrow x = 0$$

The dynamics of this system are given by

$$M\ddot{x} = -F(x) - \delta\dot{x} \tag{3.7}$$

Let $x_1 \triangleq x$, $x_2 \triangleq \dot{x}$ and $M = 1$, then

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -F(x_1) - \delta x_2 \end{bmatrix} = f(x) \tag{3.8}$$

Consider the Lyapunov function candidate

$$V(x) = \int_0^{x_1} F(s)\, ds + \frac{1}{2}x_2^2$$

Then $\frac{\partial V(x)}{\partial x} = \begin{bmatrix} F(x_1), & x_2 \end{bmatrix}$ and[1]

$$\dot{V}(x) = F(x_1)\, x_2 + x_2\left(-F(x_1) - \delta x_2\right) = -\delta x_2^2 \leq 0$$

Therefore, the system is stable but we cannot prove asymptotic stability because $\dot{V}\left(\begin{bmatrix} x_1 \\ 0 \end{bmatrix}\right) = 0, \quad \forall x_1$. However, we shall see that using LaSalle's invariance principle we can conclude that the system is in fact asymptotically stable.

[1]Here we use the Leibniz rule for differentiating integrals, i.e.,

$$\frac{d}{d\phi}\left(\int\limits_{a(\phi)}^{b(\phi)} f(\xi,\phi)\right) d\xi = \int\limits_{a(\phi)}^{b(\phi)} \frac{\partial}{\partial\phi} f(\xi,\phi) d\xi + f(b(\phi),\phi)\frac{d}{d\phi}b(\phi) - f(a(\phi),\phi)\frac{d}{d\phi}a(\phi)$$

The result in Theorem 3.3 can be extended to become a global result.

Theorem 3.7 *Let $x = 0$ be an equilibrium point of the system $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0 \text{ and } V(x) > 0, \quad \forall x \neq 0 \tag{3.9}$$

$$\|x\| \to \infty \Rightarrow V(x) \to \infty \tag{3.10}$$

$$\dot{V}(x) < 0, \quad \forall x \neq 0 \tag{3.11}$$

then the origin is globally asymptotically stable.

Remark 3.8 *If the function V satisfies the condition (3.10), then it is said to be radially unbounded.*

Example 3.9 *Consider the system*

$$\dot{x} = \begin{bmatrix} x_2 \\ -h(x_1) - x_2 \end{bmatrix}$$

where the function h is locally Lipschitz with $h(0) = 0$ and $x_1 h(x_1) > 0, \ \forall x_1 \neq 0$. Take the Lyapunov function candidate

$$V(x) = \frac{1}{2} x^T \begin{bmatrix} k & k \\ k & 1 \end{bmatrix} x + \int_0^{x_1} h(s) ds$$

The function V is positive definite for all $x \in \mathbb{R}^2$ and is radially unbounded. The derivative of V along the trajectories of the system is given by

$$\dot{V}(x) = -(1 - k)x_2^2 - k x_1 h(x_1) < 0$$

Hence the derivative of V is negative definite for all $x \in \mathbb{R}^2$, since $0 < k < 1$ (otherwise V is not positive definite! - convince yourself). Therefore, the origin is globally asymptotically stable.

3.2.2 Lyapunov's Indirect Method

In this section we prove stability of the system by considering the properties of the linearization of the system around the origin. Before proving the main result, we require an intermediate result.

Definition 3.10 *A matrix $A \in \mathbb{R}^{n \times n}$ is called Hurwitz or asymptotically stable, if and only if*

$$Re(\lambda_i) < 0, \forall i = 1, 2, \cdots, n$$

where λ_i's are the eigenvalues of the matrix A.

Consider the system $\dot{x} = Ax$. We look for a quadratic function

$$V(x) = x^T P x$$

where $P = P^T > 0$. Then

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x} = x^T \left(A^T P + PA \right) x = -x^T Q x$$

If there exists $Q = Q^T > 0$ such that

$$A^T P + PA = -Q,$$

then V is a Lyapunov function and $x = 0$ is globally stable. This equation is called the *Matrix Lyapunov equation*.

We formulate this as a matrix problem: Given Q positive definite, symmetric, how can we find out if there exists $P = P^T > 0$ satisfying the Matrix Lyapunov equation.

The following result gives existence of a solution to the Lyapunov matrix equation for any given Q.

Theorem 3.11 *For $A \in \mathbb{R}^{n \times n}$ the following statements are equivalent:*

1. *A is Hurwitz*
2. *For all $Q = Q^T > 0$ there exists a unique $P = P^T > 0$ satisfying the Lyapunov equation*
$$A^T P + PA = -Q$$

Proof: We make a constructive proof of 1. $\Rightarrow$ 2. For a given $Q = Q^T > 0$, consider the following candidate solution for P:

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt$$

That $P = P^T > 0$ follows from the properties of Q. Note that the integral will converge if and only if A is a Hurwitz matrix. We now show that P satisfies the matrix Lyapunov equation:

$$\begin{aligned} A^T P + PA &= \int_0^\infty \left[A^T e^{A^T t} Q e^{At} + e^{A^T t} Q e^{At} A\right] dt \\ &= \int_0^\infty \frac{d}{dt}\left[e^{A^T t} Q e^{At}\right] dt \\ &= e^{A^T t} Q e^{At}\Big|_0^\infty = -Q \end{aligned}$$

Thus P satisfies the matrix Lyapunov equation. In order to show uniqueness, assume that here exists another matrix $\bar{P} = \bar{P}^T > 0$ that solves the Lyapunov equation and $\bar{P} \neq P$. Then,

$$A^T(P - \bar{P}) + (P - \bar{P})A = 0$$

From which it follows that

$$0 = e^{A^T t}\left[A^T(P - \bar{P}) + (P - \bar{P})A\right] e^{At} = \frac{d}{dt}\left[e^{A^T t}(P - \bar{P})e^{At}\right]$$

Therefore,

$$e^{A^T t}(P - \bar{P})e^{At} = a, \forall t$$

where a is some constant matrix. Now, this also holds for $t = 0$, i.e.,

$$e^{A^T 0}(P - \bar{P})e^{A0} = (P - \bar{P}) = e^{A^T t}(P - \bar{P})e^{At} \to 0, \text{ as } t \to \infty$$

Where the last limit follows from the fact that A is Hurwitz. Therefore, $P = \bar{P}$.

The fact that 2. $\Rightarrow$ 1. follows from taking $V(x) = x^T P x$.

$\square$

Theorem 3.11 has an interesting interpretation in terms of the energy available to a system. If we say that the energy dissipated at a particular point in phase space x is given by $q(x) = x^T Q x$ - meaning that a trajectory passing through x is losing $q(x)$units of energy per unit time, then the equation $V(x) = x^T P x$, where P satisfies the matrix Lyapunov equation gives the total amount of energy that the system will dissipate before reaching the origin. Thus $V(x) = x^T P x$ measures the energy stored in the state x. We shall revisit this concept in the next chapter.

Theorem 3.12 (Indirect Method) *Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ where $f : \mathcal{D} \to \mathbb{R}^n$ is a continuously differentiable and $\mathcal{D}$ is a neighborhood of the origin. Let*

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=0} \tag{3.12}$$

then

1. *The origin is asymptotically stable if $Re(\lambda_i) < 0$ for all eigenvalues of A*
2. *The origin is unstable if $Re(\lambda_i) > 0$ for one or more of the eigenvalues of A*

Proof: If A is Hurwitz, then there exists $P = P^T > 0$ so that $V(x) = x^T P x$ is a Lyapunov function of the linearized system. Let us use V as a candidate Lyapunov function for the nonlinear system

$$\dot{x} = f(x) = Ax + (f(x) - Ax) \triangleq Ax + g(x)$$

The derivative of V is given by

$$\dot{V}(x) = x^T P f(x) + f(x)^T P x = x^T P A x + x^T A^T P x + 2x^T P g(x) \tag{3.13}$$
$$= -x^T Q x + 2x^T P g(x) \tag{3.14}$$

The function $g(x)$ satisfies

$$\frac{\|g(x)\|_2}{\|x\|_2} \to 0 \text{ as } \|x\|_2 \to 0$$

Therefore, for any $\gamma > 0$, there exists $r > 0$ such that

$$\|g(x)\|_2 < \gamma \|x\|_2, \quad \forall \|x\|_2 < r$$

As such,

$$\dot{V}(x) < -x^T Q x + 2\gamma \|P\|_2 \|x\|_2^2, \quad \forall \|x\|_2 < r$$

But we have the following fact $x^T Q x \geq \lambda_{\min}(Q) \|x\|_2^2$, where $\lambda_{\min}$ indicates the minimal eigenvalue of Q. Therefore,

$$\dot{V}(x) < -(\lambda_{\min}(Q) - 2\gamma \|P\|_2) \|x\|_2^2, \quad \forall \|x\|_2 < r$$

and choosing $\gamma < \frac{\lambda_{\min}(Q)}{2\|P\|_2}$ renders $\dot{V}(x)$ negative definite (locally). Hence, the origin of the nonlinear system is asymptotically stable. This proves point 1.

The proof of point 2 shall be omitted; however, it relies on the Instability results in Section 3.4. More details can be found in [4, Ch. 4].

□

Theorem 3.12 does not say anything when $Re\,(\lambda_i) \leq 0\ \forall i$ with $Re\,(\lambda_i) = 0$ for some i. In this case linearization fails to determine the stability of the equilibrium point, and further analysis is necessary. The multi-dimensional result which is relevant here is the *Center Manifold Theorem*. This theorem is beyond the scope of this course.

Example 3.13 *The system $\dot{x} = ax^3$, $a > 0$ has an unstable equilibrium point at $x = 0$. The same system is asymptotically stable at the origin for $a < 0$. In both cases the linearized system is given by $\dot{x} = 0$, and we cannot conclude anything from the linearization or Lyapunov's indirect method. However, if we use Lyapunov's direct method with $V(x) = \frac{1}{4}x^4$, then $\dot{V}(x) = ax^6$ and if $a < 0$ then the system is globally asymptotically stable.*

3.3 The Invariance Principle

Definition 3.14 *A domain $\mathcal{D} \subseteq \mathbb{R}^n$ is called invariant for the system $\dot{x} = f\,(x)$, if*

$$\forall x\,(t_0) \in D \Rightarrow x\,(t) \in D,\ \forall t \in \mathbb{R}$$

A domain $\mathcal{D} \subseteq \mathbb{R}^n$ is called positively invariant for the system $\dot{x} = f\,(x)$, if

$$\forall x\,(t_0) \in D \Rightarrow x\,(t) \in D,\ \forall t \geq t_0$$

Theorem 3.15 (LaSalle's Invariance Principle) *Let $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f\,(x)$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}\,(x) \leq 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}\,(x) = 0$. Let M be the largest invariant set in E. Then every solution starting in Ω approaches M as $t \to \infty$.*

Remark 3.16 *Theorem 3.15 result can also be used to find limit cycles.*

An important result that follows from Theorem 3.15 is the following corollary.

Corollary 3.17 *Let $x = 0 \in \mathcal{D}$ be an equilibrium point of the system $\dot{x} = f(x)$. Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable positive definite function on the domain $\mathcal{D}$, such that $\dot{V}(x) \leq 0,\ \forall x \in \mathcal{D}$. Let $S \triangleq \{x \in \mathcal{D} \mid \dot{V}(x) = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) \equiv 0$. Then, the origin is asymptotically stable.*

Example 3.18 *Consider again the mass-spring example 3.6. Apply LaSalles invariance principle by noting that*

$$\dot{V}(x) = 0 \Leftrightarrow x_2 = 0$$

As such $E = \{x \mid x_2 = 0\}$. Now, if $x_2 = 0$, then from the dynamics we obtain $F(x_1) = 0$ and since $F(x_1) = 0 \Leftrightarrow x_1 = 0$, we conclude that $M = \{0\}$. Thus, trajectories must converge to the origin, and we have proven that the system is asymptotically stable.

3.4 Instability Theorem

Theorem 3.19 *Let $x = 0$ be an equilibrium point of the system $\dot{x} = f(x)$. Let $V : \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that*

$$V(0) = 0, \text{ and } V(x_0) > 0$$

for some x_0 with arbitrary small $\|x_0\|$. Define the set

$$U = \{x \in \mathcal{B}_r \mid V(x) > 0\}$$

and suppose that $\dot{V}(x) > 0$ in U. Then $x = 0$ is unstable.

Sketch of proof: First note that x_0 belongs to the interior of U. Moreover, we can show that the trajectory starting at x_0 must leave the set U and that the trajectory leaves the set U through the surface $\|x\|_2 = r$. Since this can happen for arbitrarily small $\|x_0\|$, thus the origin is unstable.

□

Note that the requirements on $V(x)$ in Theorem 3.19 are not as strict as the requirements on a Lyapunov function that proves stability.

Example 3.20 *The set U for $V(x) = \frac{1}{2}(x_1^2 - x_2^2)$ is shown in Figure 3.5. Consider the system*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}$$

then $\dot{V}(x) = x_1^2 + x_2^2 > 0$ for $x \in U$, proving that the system is unstable.

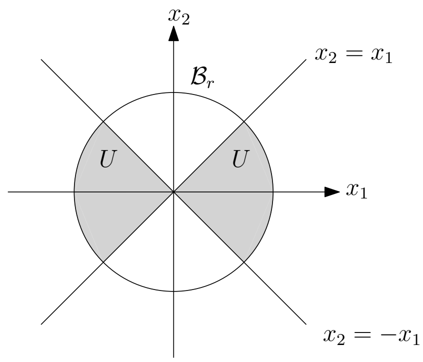


Figure 3.5: Instability regions

3.5 Comparison Functions

Before we move on to the analysis of nonautonomous systems, we need to introduce few definitions of classes of functions that will aid us in extending the analysis we have done so far to the nonautonomous case.

Definition 3.21 *A continuous function $\alpha_1 : [0, a) \to [0, \infty)$ belongs to class $\mathcal{K}$ if it is strictly increasing and $\alpha_1(0) = 0$. Moreover, α_2 belongs to class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha_2(r) \to \infty$ as $r \to \infty$.*

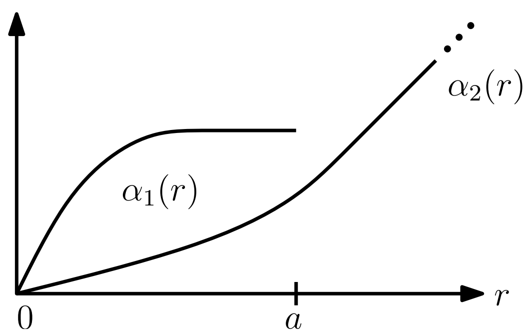


Figure 3.6: Illustration of $\alpha_1 \in \mathcal{K}$ and $\alpha_2 \in \mathcal{K}_\infty$)

For example $\alpha(r) = \tan^{-1}(r)$ belongs to class $\mathcal{K}$ but not to class $\mathcal{K}_\infty$, while the functions $\alpha(r) = r^2$ and $\alpha(r) = \min\{r, r^4\}$ belong to class $\mathcal{K}_\infty$ (why?).

Definition 3.22 *A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ belongs to class $\mathcal{KL}$ if for each fixed s, the mapping $\beta(r, s)$ belongs to class $\mathcal{K}$ with* *respect to r, and for each fixed r, $\beta(r,s)$ is decreasing with respect to s and $\beta(r,s) \to 0$ as $s \to \infty$.*

The functions $\beta(r,s) = r^2 e^{-s}$ and $\beta(r,s) = \frac{r}{sr+1}$ belong to class $\mathcal{K}L$. The

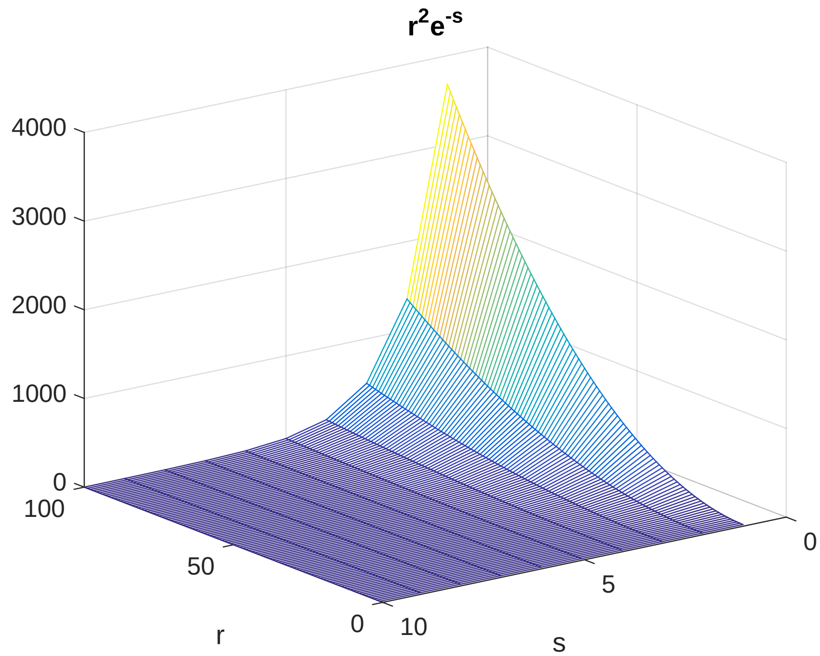


Figure 3.7: Illustration of class $\mathcal{K}L$ function: $\beta(r,s) = r^2 e^{-s}$

following result is quite useful for analysis.

Lemma 3.23 *Let α_1, α_2 be class $\mathcal{K}$ functions on $[0,a)$, a_3, a_4 be class $\mathcal{K}_\infty$ functions, and β be class $\mathcal{K}L$ function. Then,*

- α_1^{-1} *is defined on* $[0, \alpha_1(a))$ *and belongs to class* $\mathcal{K}$
- α_3^{-1} *is defined on* $[0, \infty)$ *and belongs to class* $\mathcal{K}_\infty$
- $\alpha_1 \circ \alpha_2$ *belongs to class* $\mathcal{K}$
- $\alpha_3 \circ \alpha_4$ *belongs to class* $\mathcal{K}_\infty$
- $\gamma(r,s) = \alpha_1(\beta(\alpha_2(r), s))$ *belongs to class* $\mathcal{K}L$

with the understanding that α_i^{-1} denotes the inverse of α_i and $\alpha_i \circ \alpha_j = \alpha_i(\alpha_j(r))$.

We can relate the foregoing definitions to our previous definitions of positive definite functions.

Lemma 3.24 *Let $V : \mathcal{D} \to \mathbb{R}$ be a continuous positive definite function defined on the domain $\mathcal{D} \subset \mathbb{R}^n$ that contains the origin. Let $\mathcal{B}_r \subset \mathcal{D}$ be a ball of radius $r > 0$. Then, there exist class $\mathcal{K}$ functions α_1 and α_2 defined on $[0, r)$ such that*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

Moreover, if $\mathcal{D} = \mathbb{R}^n$ and $V(x)$ is radially unbounded, then the last inequality holds for α_1 and α_2 in class $\mathcal{K}_\infty$.

For example, if $V(x) = x^T P x$ with P being symmetric positive definite, then it follows that

$$\alpha_1(\|x\|) \triangleq \lambda_{min}(P)\, \|x\|_2^2 \leq V(x) \leq \lambda_{max}(P)\, \|x\|_2^2 \triangleq \alpha_2(\|x\|)$$

3.6 Stability of Nonautonomous Systems

Consider the system

$$\dot{x} = f(t, x) \tag{3.15}$$

where $f : [0, \infty) \times \mathcal{D} \to \mathbb{R}^n$ is piecewise continuous in t and locally Lipschitz in x on the domain $[0, \infty) \times \mathcal{D}$, and $\mathcal{D} \subset \mathbb{R}^n$ contains the origin. We say that the origin is an equilibrium point of (3.15) if $f(t, 0) = 0, \quad \forall t \geq 0$. Henceforth, we shall assume that the origin is an equibrium point of the system (3.15), since we can apply a similar analysis to the case of autonomous systems to shift any equilibrium point to the origin.

We saw in the stability analysis of autonomous systems that the solution depends on the difference $(t - t_0)$, where t_0 is the initial time instant. For nonautonomous systems, the solution generally depends on both t and t_0. As such, we expect a modification of the definition of stability as seen next.

Definition 3.25 *The equilibrium point $x = 0$ of the system (3.15) is*

1. *stable, if for each $\epsilon > 0$, there is a $\delta = \delta(\epsilon, t_0) > 0$ such that*

$$\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq t_0 \tag{3.16}$$

2. *uniformly stable, if for each* $\epsilon > 0$*, there is a* $\delta = \delta(\epsilon) > 0$ *(independent of* t_0*) such that (3.16) holds.*

3. *unstable, if it is not stable*

4. *asymptotically stable if it is stable and there is a positive constant* $c = c(t_0)$ *such that* $x(t) \to 0$ *as* $t \to \infty$*, for all* $\|x(t_0)\| < c$

5. *uniformly asymptotically stable, if it is uniformly stable and the constant* c *is independent of* t_0*, i.e., for each* $\eta > 0$ *there exists a time horizon* $T = T(\eta) > 0$ *such that*

$$\|x(t)\| < \eta, \quad \forall t \geq t_0 + T(\eta), \forall \|x(t_0)\| < c \tag{3.17}$$

6. *globally uniformly asymptotically stable, if it is uniformly asymptotoicaly stable,* $\delta(\epsilon)$ *can be chosen to satisfy* $\lim_{\epsilon\to\infty} \delta(\epsilon) = \infty$*, and, for each pair of positive numbers* η *and* c*, there is* $T = T(\eta, c) > 0$ *such that*

$$\|x(t)\| < \eta, \quad \forall t \geq t_0 + T(\eta, c), \forall \|x(t_0)\| < c \tag{3.18}$$

In terms of class $\mathcal{K}$, $\mathcal{K}_\infty$ and $\mathcal{K}L$ definitions above can be characterized as follows: The origin of (3.15) is:

- Uniformly Stable (US) $\Leftrightarrow$ there exist a function $\alpha \in \mathcal{K}$ and a constant $c > 0$ independent of t_0 such that

$$\|x(t)\| \leq \alpha\left(\|x(t_0)\|\right), \quad \forall t \geq t_0 \geq 0, \forall \|x(t_0)\| < c \tag{3.19}$$

- Uniformly asymptotically stable (UAS) $\Leftrightarrow$ there exist a function $\beta \in \mathcal{K}L$ and a constant $c > 0$ independent of t_0 such that

$$\|x(t)\| \leq \beta\left(\|x(t_0)\|, t - t_0\right), \quad \forall t \geq t_0 \geq 0, \forall \|x(t_0)\| < c \tag{3.20}$$

- Globally UAS (GUAS) $\Leftrightarrow$ there exist a function $\beta \in \mathcal{K}L$ such that

$$\|x(t)\| \leq \beta\left(\|x(t_0)\|, t - t_0\right), \quad \forall t \geq t_0 \geq 0, \forall x(t_0) \tag{3.21}$$

- Exponentially stable if there exist constants $c, k, \lambda > 0$ such that

$$\|x(t)\| \leq k \|x(t_0)\| e^{-\lambda(t-t_0)}, \quad \forall t \geq t_0 \geq 0, \forall \|x(t_0)\| < c \tag{3.22}$$

- Globally exponentially stable if the condition (3.22) is satisfied for any initial state $x(t_0)$.

$\longrightarrow$ *Can you write down the corresponding notions for the autonomous systems?*

Similarly to the autonomous systems case, we can write the Lyapunov characterizations of stability for nonautonomous systems.

Theorem 3.26 *Consider the system (3.15) and let $x = 0 \in \mathcal{D} \subset \mathbb{R}^n$ be an equilibrium point. Let $V : [0, \infty) \times \mathcal{D} \to \mathbb{R}$ be a continuously differentiable function such that*

$$W_1(x) \leq V(t, x) \leq W_2(x) \tag{3.23}$$
$$\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0 \tag{3.24}$$

$\forall t \geq 0$ and $\forall x \in \mathcal{D}$, where W_1 and W_2 are continuous positive definite functions on $\mathcal{D}$. Then, the origin is uniformly stable. If instead we have the stronger conditions

$$W_1(x) \leq V(t, x) \leq W_2(x) \tag{3.25}$$
$$\dot{V}(t, x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x) \tag{3.26}$$

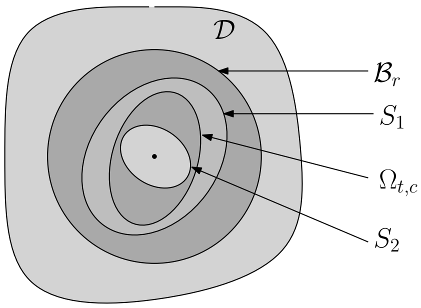
with all the previous assumptions holding and W_3 is continuous positive definite on $\mathcal{D}$, then the origin is UAS.

Moreover, if we choose some constants r and c such that $\mathcal{B}_r = \{\|x\| \leq r\} \subset \mathcal{D}$ and $c < \min_{\|x\|=r} W_1(x)$, then every trajectory starting in $\{x \in \mathcal{B}_r | W_2(x) \leq c\}$ satisfies the estimate

$$\|x\| \leq \beta(\|x_0\|, t - t_0) \quad \forall t \geq t_0 \geq 0 \tag{3.27}$$

for some $\beta \in \mathcal{KL}$. Finally, if $\mathcal{D} = \mathbb{R}^n$ and $W_1(x)$ is radially unbounded, then the origin is GUAS.

Sketch of Proof: The proof is similar to that of Theorem 3.3, however now we work with level sets that are formed by the functions W_i. Consider Figure 3.8. Choose some radius $r > 0$ and a constant $c > 0$, such that $\mathcal{B}_r \subset \mathcal{D}$ and $c < \min_{\|x\|=r} W_1(x) \Rightarrow S_1 := \{x \in \mathcal{B}_r | W_1(x) \leq c\} \subset \text{interior}(\mathcal{B}_r)$. Define



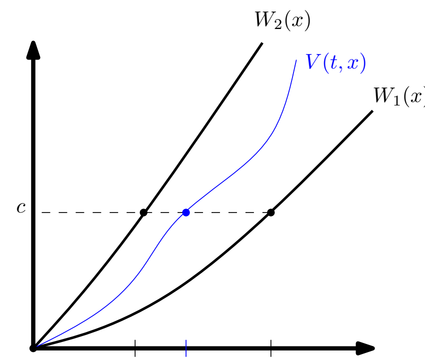


Figure 3.8: A sketch of the various domains of Lyapunov stability for nonautonomous systems (left) and the corresponding functions (right)

the set (time-independent) $\Omega_{t,c} := \{x \in \mathcal{B}_r | V(t,x) \leq c\}$. It follows from the relation (3.23) that

$$S_2 := \{x \in \mathcal{B}_r | W_2(x) \leq c\} \subset \Omega_{t,c} \subset S_1 \subset \mathcal{B}_r \subset \mathcal{D}$$

which shows along with (3.24) that any solution that starts in S_2 remains in S_1 for all time beyond t_0. Moreover, we have that

$$V(t, x(t)) \leq V(t_0, x(t_0)), \quad \forall t \geq t_0 \tag{3.28}$$

$$\alpha_1(\|x\|) \leq W_1(x) \leq V(t,x) \leq W_2(x) \leq \alpha_2(\|x\|) \tag{3.29}$$

for some $\alpha_1, \alpha_2 \in \mathcal{K}_{[0,r]}$, from which it follows that

$$\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\|x(t_0)\|)), \quad \forall t_0 \tag{3.30}$$

and hence the origin is uniformly stable.

Under the stronger condition (3.26), we have the following differential inequality

$$\dot{V} \leq -W_3(x) \leq -\alpha_3(\alpha_2^{-1}(V(x))) \tag{3.31}$$

where $\alpha_3 \in \mathcal{K}_{[0,r]}$ and $W_3(x) \geq \alpha_3(\|x\|)$. Using the comparison system

$$\dot{y} = -\alpha_3(\alpha_2^{-1}(y)), \quad y(t_0) = V(t_0, x(t_0)) \geq 0$$

we can show under some technical conditions that

$$V(t, x(t)) \leq \sigma(V(t_0, x(t_0)), t - t_0), \quad \forall V(t_0, x(t_0)) \in [0, c] \tag{3.32}$$

$$\Rightarrow \|x(t)\| \leq \alpha_1^{-1}(\sigma(\alpha_2(\|x(t_0)\|), t - t_0)) =: \beta(\|x(t_0)\|, t - t_0) \tag{3.33}$$

where $\sigma \in \mathcal{KL}_{[0,r]\times[0,\infty)}$. Hence the origin is UAS. Finally, if $\mathcal{D} = \mathbb{R}^n$, then all α_i's are defined on $[0, \infty)$ from which we can deduce that the system is GUAS.

$\square$

The following two examples illustrate the utilization of the result of Theorem 3.26.

Example 3.27 *Consider the scalar system*

$$\dot{x} = -(1 + g(t))x^3$$

where $g(t)$ is a continuous function with $g(t) \geq 0$, $\forall t$. Take the time-invariant candidate Lyapunov function $V(x) = \frac{x^2}{2}$. Then,

$$\dot{V}(x) = -(1 + g(t))x^4 \leq -x^4, \quad \forall x \in \mathbb{R}, t \geq 0$$

Hence, the system is GUAS.

Example 3.28 *Consider the second order system*

$$\begin{aligned}
\dot{x}_1 &= -x_1 - g(t)x_2 \\
\dot{x}_2 &= x_1 - x_2
\end{aligned}$$

where $g(t)$ is a continuously differentiable function, such that $0 \leq g(t) \leq k$ and $\dot{g}(t) \leq g(t)$, $\forall t$ and for some positive constant k. We now consider the time-varying Lyapunov function candidate $V(t, x) = x_1^2 + (1 + g(t))x_2^2$. First, we have that

$$\alpha_1(\|x\|) \triangleq x_1^2 + x_2^2 \leq V(t, x) \leq x_1^2 + (1 + k)x_2^2 \triangleq \alpha_2(\|x\|)$$

Hence, the function V is bounded above and below by class $\mathcal{K}_\infty$ functions.

We now take the derivative of V along the trajectories of the system:

$$\begin{aligned}
\dot{V}(t, x) &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}\dot{x} \\
&= x_2^2\dot{g}(t) + 2x_1\dot{x}_1 + 2(1 + g(t))x_2\dot{x}_2 \\
&= x_2^2\dot{g}(t) + 2x_1(-x_1 - g(t)x_2) + 2(1 + g(t))x_2(x_1 - x_2) \\
&= -2x_1^2 + 2x_1x_2 - (2 + 2g(t) - \dot{g}(t))x_2^2 \\
&\leq -2x_2^2 + 2x_1x_2 - 2x_2^2 = -\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}^T \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} < 0
\end{aligned}$$

since $Q = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} > 0$. *Hence, the system is GUAS.*

Moreover, since we have that

$$\rho_1 \|x\|^2 \leq V(t,x) \leq \rho_2 \|x\|^2, \text{ and } \dot{V}(t,x) \leq -\rho_3 \|x\|^2 \tag{3.34}$$

for a choice of $\rho_1 \triangleq 1$, $\rho_2 \triangleq 1+k$, *and* $\rho_3 \triangleq \lambda_{\min}(Q)$, *it follows that*

$$\dot{V}(t,x) \leq -\frac{\rho_3}{\rho_2} V(t,x)$$

From which it follows that (using the so-called Comparison Lemma in the Appendix)

$$V(t,x) \leq V(t_0, x(t_0)) e^{-\frac{\rho_3}{\rho_2}(t-t_0)} \tag{3.35}$$

Using (3.34) *again, we have that*

$$\begin{aligned} \|x(t)\|^2 &\leq \frac{1}{\rho_1} V(t,x) \leq \frac{1}{\rho_1} V(t_0, x(t_0)) e^{-\frac{\rho_3}{\rho_2}(t-t_0)} \\ &\leq \frac{\rho_2}{\rho_1} \|x(t_0)\|^2 e^{-\frac{\rho_3}{\rho_2}(t-t_0)} \end{aligned}$$

$\Rightarrow \|x(t)\| \leq \sqrt{\frac{\rho_2}{\rho_1}} \|x(t_0)\| e^{-\frac{\rho_3}{2\rho_2}(t-t_0)}$ *implying that the system is actually globally exponentially stable!*

Remark 3.29 *One can generalize the result of the last example on exponential stability if the bounds on* $V(t,x)$ *and* $\dot{V}(t.x)$ *satisfy the following*

$$\begin{aligned} \rho_1 \|x\|^\alpha &\leq V(t,x) \leq \rho_2 \|x\|^\alpha \\ \dot{V}(t,x) &\leq -\rho_3 \|x\|^\alpha \end{aligned}$$

for any positive constants $\rho_1, \rho_2, \rho_3, \alpha$.

3.7 Existence of Lyapunov Functions

Having gone through Lyapunov methods of showing stability, a crucial question remains regarding existence of a Lyapunov function: *assume that* $x = 0$ *is an asymptotically stable equilibrium of the system* (3.1), *does a Lyapunov function for the system exist?* These types of results are often referred to as *converse theorems*. Consider Autonomous systems for simplicity, we have the following result.

Theorem 3.30 *Let $x = 0$ be locally exponentially stable for the system $\dot{x} = f(x)$ on the domain $\mathcal{D}_0 \triangleq \{x \in \mathbb{R}^n \| \, \|x\| < r_0\}$. Then there exists a Lyapunov function $V : \mathcal{D} \to \mathbb{R}^+$ for the system such that*

$$c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2 \tag{3.36}$$

$$\dot{V}(x) \leq -c_3 \|x\|^2 \tag{3.37}$$

$$\left\| \frac{\partial V(x)}{\partial x} \right\| \leq c_4 \|x\| \tag{3.38}$$

where $c_1,\ c_2,\ c_3,\ c_4 > 0$.

Sketch of Proof: Analogously to the construction of a solution to the matrix Lyapunov equation, a Lyapunov Function is constructed by:

$$V(x) = \int_0^\infty \phi(t,x)^T \phi(t,x) dt$$

where $\phi(t,x)$ is the solution to the system differential equations, defining the trajectory starting at x at time $t = 0$. Due to local exponential stability, the integral may be shown to converge locally. By bounding the rate of growth of the integral away from 0, the properties may be proven.

□

The last theorem can be generalized for the non-autonomous systems case.

3.8 Input-to-State Stability

We end this chapter with a robustness-type result. Consider again the nonautonomous system

$$\dot{x} = f(t,x,d) \tag{3.39}$$

with the same assumptions as in the last section on t and x, and where $d(t)$ is a piecewise continuous function on $\mathbb{R}^m$ which is bounded for all $t \geq 0$. Assume that the unforced system $\dot{x} = f(t,x,0)$ enjoys some stability properties, what can we say in the presence of the input d?

Definition 3.31 *The system (3.39) is said to be input-to-state stable (ISS) if there exists a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for* *any initial state $x(t_0)$ and any bounded input $d(t)$, the solution $x(t)$ exists for all $t \geq t_0$ and*

$$\|x(t)\| \leq \beta\left(\|x(t_0)\|, t - t_0\right) + \gamma\left(\sup_{\tau \in [t_0, t]} \|d(\tau)\|\right) \tag{3.40}$$

The function β allows us to determine the response of the system in terms of the initial condition x_0 and the corresponding overshoot. The function γ determines the asymptotic behavior of the system with respect to a bounded disturbance input d. This concept generalizes what we already know about linear systems. More specifically, consider the linear system with A Hurwitz

$$\dot{x} = Ax + Bd, \quad x(0) = x_0 \tag{3.41}$$

The solution is given by

$$x(t) = e^{At}x_0 + \int_0^t e^{A(t-\eta)}Bd(\eta)d\eta \tag{3.42}$$

from which we can deduce the following inequality on the size of the state over time

$$\begin{aligned} \|x(t)\| &\leq \left\|e^{At}\right\| \|x_0\| + \left\|\int_0^t e^{A(t-\eta)}d\eta B\right\| \sup_{\tau \in [0,t]} \|d(\tau)\| \\ &\triangleq \beta(\|x_0\|, t) + \gamma \sup_{\tau \in [0,t]} \|d(\tau)\| \end{aligned} \tag{3.43}$$

Therefore, assuming the matrix A is Hurwitz, the response due to the initial condition asymptotically decays to 0 and we expect the state to asymptotically converge to a ball that is proportional to the size of the input d. We also have the following Lyapunov characterization of ISS.

Theorem 3.32 *Let $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that*

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|) \tag{3.44}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, d) \leq -W(x), \quad \forall \|x\| \geq \rho(\|d\|) > 0 \tag{3.45}$$

for all $(t, x, d) \in [0, \infty) \times \mathbb{R}^n \times \mathbb{R}^m$, where α_1, α_2 are class $\mathcal{K}_\infty$ functions, ρ is a class $\mathcal{K}$ function, and W is a continuous positive definite function on $\mathbb{R}^n$. Then, the system (3.39) is ISS with $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Note that if the system is autonomous, then the conditions of the result above become necessary and sufficient.

Example 3.33 *Consider the system*

$$\dot{x} = f(x,u) = x + (x^2+1)u \tag{3.46}$$

We can simply design the control input $u = \frac{-2x}{x^2+1}$ *that renders the closed-loop system* $\dot{x} = -x$ *which is globaly asymptotically stable. However, the question remains if the resulting system is robust (in our context ISS). To see this, assume that we have a small disturbance that comes in with the input, i.e., our input is actually* $u = \frac{-2x}{x^2+1} + d$*. The resulting closed-loop system is given by*

$$\dot{x} = -x + (x^2+1)d$$

This latter system can be shown to go unstable, for example if $d \equiv 1$ *the solution diverges to* ∞ *in finite time. Therefore, our feedback does not render the system robust even to small disturbances! Alternatively, consider now the new input* $u = \frac{-2x}{x^2+1} - x$*, which gives the following closed-loop system under the same type of disturbance as before*

$$\dot{x} = -2x - x^3 + (x^2+1)d$$

Let us now show that this last system is actually ISS. Take the following Lyapunov function candidate $V = \frac{1}{2}x^2$*, then*

$$\begin{aligned}
\dot{V} &= x\left(-2x - x^3 + (x^2+1)d\right) && (3.47)\\
&\le -x^2 - \frac{x^4}{2} - x^2\left(1 - \frac{|d|}{|x|}\right) - \frac{x^4}{2}\left(1 - \frac{2|d|}{|x|}\right) && (3.48)\\
&\le -x^2 - \frac{x^4}{2}, \quad \forall |x| > 2|d| && (3.49)
\end{aligned}$$

Therefore, by Theorem 3.32, the closed-loop system is ISS.

Lemma 3.34 *Consider the cascade system*

$$\dot{x}_1 = f_1(x_1, x_2) \tag{3.50}$$

$$\dot{x}_2 = f_2(x_2, u) \tag{3.51}$$

where the system (3.50) is ISS with respect to x_2*, (3.51) is ISS with respect to* u*. Then the origin of the cascade system (3.50)-(3.51) is ISS.*

Sketch of Proof: Since each of the subsystems is ISS, then each admits an ISS Lyapunov function, which can be shown to satisfy (after some rescaling) the following inequalities

$$\begin{aligned}\dot{V}_1(x_1, x_2) &\leq -\alpha_1(\|x_1\|) + \rho_1(\|x_2\|)\\ \dot{V}_2(x_1, u) &\leq -2\rho_1(\|x_2\|) + \rho_2(\|u\|)\end{aligned}$$

from which it follows that $V := V_1 + V_2$ is an ISS Lyapunov function that satisfies

$$\dot{V}(x_1, x_2, u) \leq -\alpha_1(\|x_1\|) - \rho_1(\|x_2\|) + \rho_2(\|u\|)$$

□

Example 3.35 *Consider the system*

$$\begin{aligned}\dot{x}_1 &= -x_1 - x_2 + u_1\\ \dot{x}_2 &= x_1 - x_2 + u_2\end{aligned}$$

and the quadratic function $V(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$. The derivative of V along the trajectories of the system is given by

$$\begin{aligned}\dot{V} &= x_1(-x_1 - x_2 + u_1) + x_2(x_1 - x_2 + u_2) = -x_1^2 - x_2^2 + x_1 u_1 + x_2 u_2\\ &\leq -(x_1^2 + x_2^2)/2, \quad \forall |x_i| > 2|u_i|\end{aligned}$$

which shows that the system is ISS. Of course this is not surprising since we know that the system is a stable linear one and hence is robust with respect to bounded disturbances.

3.9 Stability of Discrete-Time Systems

One remaining question for this chapter is stability characterization of discrete-time systems. Consider the discrete-time nonlinear system

$$x(k+1) = f(x(k)) \tag{3.52}$$

where $f : \mathcal{D} \to \mathbb{R}^n$ is a nonlinear map, and we use the shorthand notation of indices k instead of the more precise one of kT_s, with T_s being the sampling period. We shall assume that the system has a equilibrium at the origin, i.e., $f(0) = 0$.

Remark 3.36 *For the discrete-time system (3.52), the equilibrium point is characterized by the fixed point condition* $x^* = f(x^*)$.

Analogously to the continuous-time systems case, we can state the following global stability theorem.

Theorem 3.37 *Let the origin* $x = 0 \in \mathbb{R}^n$ *be an equilibrium point for the system (3.52). Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a continuous function such that*

$$\begin{aligned} V(0) = 0 \ \textit{and} \ V(x) > 0, \forall x \neq 0 \\ \Delta V(x(k)) \triangleq V(x(k)) - V(x(k-1)) < 0, \forall x(k) \in \mathcal{D} \\ \|x\| \to \infty \Rightarrow V(x) \to \infty \end{aligned} \tag{3.53}$$

then the origin is globally asymptotically stable.

Let us interpret the result in the previous theorem. For continuous time systems, we require that the derivative of the Lyapunov function is negative along the trajectories. For discrete-time systems, we require that the difference in the Lyapunov function is negative along the trajectories. Also, quite importantly we do not require that V is continuously differentiable, but to be only continuous.

If we now consider a linear discrete-time system given by

$$x(k+1) = Fx(k) \tag{3.54}$$

where $F \in \mathbb{R}^{n\times n}$. The asymptotic stability of such a system is characterized by the eigenvalues being strictly inside the unit circle in the complex plane.

Definition 3.38 *The matrix* F *is called Schur or asymptotically stable, if and only if*

$$|\lambda_i| < 1, \forall i = 1, \cdots, n$$

where λ_i*'s are the eigenvalues of the matrix* F.

Theorem 3.39 *Consider the linear discrete-time system (3.54), the following conditions are equivalent:*

1. *The matrix* F *is Schur stable*
2. *Given any matrix* $Q = Q^T > 0$ *there exists a positive definite matrix* $P = P^T$ *satisfying the discrete-time matrix Lyapunov equation*

$$F^T PF - P = -Q \tag{3.55}$$

Proof: Let's first show that 1. $\Rightarrow$ 2. Let F be Schur stable and take any matrix $Q = Q^T > 0$. Take the matrix $P = \sum_{i=0}^{\infty}(F^T)^i Q F^i$, which is well defined by the asymptotic stability of F, and $P = P^T > 0$ by definition. Now, substitute P into (3.55)

$$\begin{aligned} F^T P F - P &= F^T \left(\sum_{i=0}^{\infty} (F^T)^i Q F^i \right) F - \sum_{i=0}^{\infty} (F^T)^i Q F^i \\ &= \sum_{i=1}^{\infty} (F^T)^i Q F^i - \sum_{i=0}^{\infty} (F^T)^i Q F^i = -Q \end{aligned}$$

In order to show uniqueness, suppose that there is another matrix $\bar{P}$ that satisfies the Lyapunov equation. After some reccurssions, we can show that if both P and $\bar{P}$ satisfy the Lyapunov equation then

$$(F^T)^N (P - \bar{P}) F^N = P - \bar{P}$$

Letting $N \to \infty$ yields the result.

In order to show that 2. $\Rightarrow$ 1., consider the Lyapunov function $V(x) = x^T P x$, and fix an initial state $x(0)$. We have that (by applying the Lyapunov equation recursively and summing up the steps)

$$V(x(N)) - V(x(0)) = -\sum_{i=0}^{N-1} x(i)^T Q x(i) \leq -\lambda_{min}(Q) \sum_{i=0}^{N-1} \|x(i)\|_2^2$$

Therefore, the sequence $[V(x(k))]_{k\in\mathbb{N}}$ is strictly decreasing and bounded from below, hence it attains a non-negative limit. We can further show by contradiction that this limit is actually 0, or equivalently $\lim_{i\to\infty} \|x(i)\| = 0$, since this holds for any choice of $x(0)$, it follows that F is Schur stable.

$\square$

Appendix: Comparison Lemma

Lemma 3.40 *Consider the scalar system* $\dot{u} = f(t, u)$, $u(t_0) = u_0$, *where* $f(t, u)$ *is continuous in* t *and locally Liptschitz in* u, $\forall t \geq 0$ *and* $\forall u \in D \subset \mathbb{R}$. *Let* $[t_0, T)$ *be the maximal interval of existence of the solution* $u(t)$, *and* *suppose that* $u(t) \in D$, $\forall t \in [t_0, T)$. *Let* $v(t)$ *be a continuously differentiable function*[2] *whose derivative satisfies*

$$\dot{v}(t) \leq f(t, v(t)), \quad v(t_0) \leq u_0$$

with $v(t) \in D$, $\forall t \in [t_0, T)$. *Then, the solution* $v(t)$ *satisfies*

$$v(t) \leq u(t), \quad \forall t \in [t_0, T)$$

$\square$

[2] This condition can be relaxed to simply having $v(t)$ being continuous with an upper right-hand derivative existing

3.10 Exercises

1. Show that the origin of the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1^3 - x_2^3 \end{cases}$$

is globally asymptotically stable, using a suitable Lyapunov function.

2. Show that the origin of the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1 - \text{sat}(2x_1 + x_2) \end{cases}$$

is locally but not globally asymptotically stable. Recall that the saturation function is defined as

$$\text{sat}(s) = \begin{cases} s\,, & \text{if } |s| \leq 1 \\ \text{sgn}(s)\,, & \text{if } |s| \leq 1 \end{cases}.$$

3. Use a quadratic Lyapunov function in order to design an appropriate (feedback) input function $u(x)$ that renders the following system stable

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_1^2 + x_1 x_2 + u(x) \end{cases}.$$

4. [**Krasovskii's Method**] Consider the system

$$\dot{x} = f(x), \quad f(0) = 0, \quad x \in \mathbb{R}^n$$

Assume that the Jackobian satisfies

$$P\left[\frac{\partial f}{\partial x}(x)\right] + \left[\frac{\partial f}{\partial x}(x)\right]^T P \leq -I, \quad \forall x \in \mathbb{R}^n$$

where $P = P^T > 0$.

- Show using the representation $f(x) = \int_0^1 \frac{\partial f(\sigma x)}{\partial x} x d\sigma$ that

$$x^T P f(x) + f(x)^T P x \leq -x^T x, \quad \forall x \in \mathbb{R}^n$$

- Show that the function $V(x) = f(x)^T P f(x)$ is positive definite and radially unbounded.
- Show that the origin is globally asymptotically stable.

5. Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= h(t)x_2 - g(t)x_1^3 \\ \dot{x}_2 &= -h(t)x_1 - g(t)x_2^3\end{aligned}$$

where $h(t)$ and $g(t)$ are bounded continously differentiable functions, with $0 \leq g_0 \leq g(t)$.

- Is the equilibrium $x = 0$ uniformly asymptotically stable (UAS)? Is it globally UAS?
- Is it (globally/locally) exponentially stable?

6. Consider the nonlinear system

$$\begin{cases}\dot{x}_1 = -x_1 - e^{-2t}x_2 \\ \dot{x}_2 = x_1 - x_2\end{cases}.$$

Determine whether the equilibrium point at 0 is stable or not?
Hint: use the following Lyapunov function $V(x,t) = x_1^2 + (1 + e^{-2t})x_2^2$

7. Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -\phi(t)x_1 + a\phi(t)x_2 \\ \dot{x}_2 &= b\phi(t)x_1 - ab\phi(t)x_2 - c\psi(t)x_2^3\end{aligned}$$

where $a, b, c > 0$ are constant and $0 \leq \phi_0 \leq \phi(t)$ and $0 \leq \psi_0 \leq \psi(t)$, $\forall t \geq 0$ are bounded functions.

Show that the origin is globally uniformly asymptotically stable. Is it exponentially stable?

8. Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2^2 \\ \dot{x}_2 &= -x_2 + u\end{aligned}$$

- Show that the unforced system (setting $u \equiv 0$) is globally asymptotically stable.
- Now, let $u \neq 0$ and show that the resulting system is input-to-state stable (ISS).

Hint: For both parts use the candidate function $V(x) = (1/2)x_1^2 + (1/4)ax_2^4$ *for a proper choice of* $a > 0$.

Chapter 4

Dissipative Systems

4.1 Introduction

We shall study a special class of systems called *dissipative systems*. Intuitively, we can think of systems interacting with the surrounding via some input / output ports, exchanging power with the surrounding, storing some energy and dissipating some (Figure 4.1).

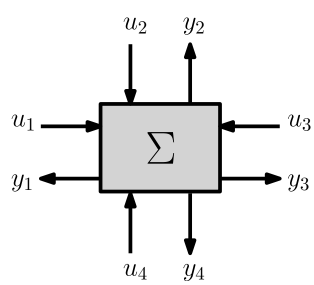


Figure 4.1: A system Σ interacting with the outside world via inputs and outputs

For example, electrical systems exchange power with the surrounding via an inner product between the current and voltage, i.e., $\mathcal{P}_e = v^T i$. Energy in these systems may be stored in capacitors and/or inductors in the form of voltage or current, respectively. Another example is mechanical systems; these systems may by supplied with linear and/or rotational power, i.e., $\mathcal{P}_m = \omega^T T$ or $\mathcal{P}_m = v^T F$, where v is the velocity, F is the applied force, ω is the rotational speed, and T is the applied torque. This supply may be stored in the form of potential and/or kinetic energy. In what follows, we shall provide a solid foundation for such supply and storage concepts that allows us to describe systems from an input-output perspective.

Example 4.1 *Consider again the mass-spring system from the lecture on Lyapunov Stability Theory, as shown in Figure 4.2, with an extra input force* u*. Define* $x_1 = x$ *and* $x_2 = \dot{x}$*, the position and speed, respectively, let* $M = 1$*, and assume that we measure the output* $y = \dot{x} = x_2$*. Then, we can write the dynamical equations of the system as*

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -F(x_1) - \delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = x_2 \tag{4.1}$$

where $F(x_1)x_1 > 0$, $\forall x_1 \neq 0$*. How can we make a statement about the*

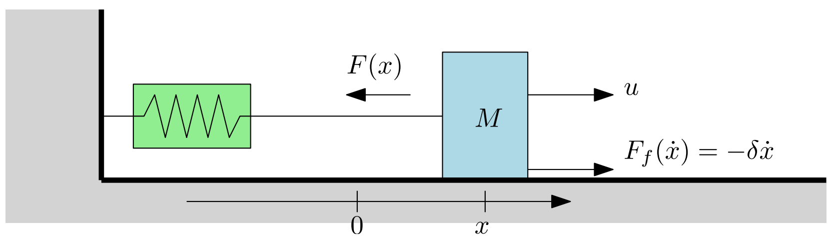


Figure 4.2: Mass spring system

stability of the system? We consider the energy that is stored in the system

$$S(x) = \int_0^{x_1} F(\xi)\, d\xi + \frac{1}{2}x_2^2 \tag{4.2}$$

The derivative of $S(x)$ *along the trajectories of* (4.1) *is given by*

$$\dot{S}(x) = -\delta x_2^2 + x_2 u = -\delta y^2 + yu \tag{4.3}$$

Assume that at $t = 0$, $x_1 = x_2 = 0$*. Then*

$$\begin{aligned} S(t) = \int_0^t \dot{S}(\tau)\, d\tau &= \int_0^t \left(-\delta y^2(\tau) + y(\tau)u(\tau)\right) d\tau \\ &\leq \int_0^t y(\tau)\, u(\tau)\, d\tau \leq \int_0^t |y(\tau)|\, |u(\tau)|\, d\tau \end{aligned}$$

Therefore, we can see that if u *and* y *are bounded signals (in some sense), then* S *is also bounded. Due to the properties of the function* S*, we can then limit the state. This reasoning helps us to go from an input-output boundedness property to an internal state boundedness property.*

4.2 Dissipative Systems

Consider the nonlinear state-space system, given by

$$\begin{aligned} \dot{x} &= f(x,u) \\ y &= h(x,u) \end{aligned} \tag{4.4}$$

where $x \in \mathcal{X} \subseteq \mathbb{R}^n$, $u \in \mathcal{U} \subseteq \mathbb{R}^m$, and $y \in \mathcal{Y} \subseteq \mathbb{R}^p$. Associated with (4.4), we have the following *supply rate* function

$$w(u,y) : \mathcal{U} \times \mathcal{Y} \to \mathbb{R} \tag{4.5}$$

Definition 4.2 *The state-space system (4.4) is said to be dissipative with respect to the supply rate $w(u,y)$, if there exists a function $S : \mathcal{X} \to \mathbb{R}_{\geq 0}$, called the storage function, such that $\forall x_0 \in \mathcal{X}$, $\forall t_1 > t_0$, and all input functions u the following dissipation inequality holds*

$$S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} w(u(t), y(t))\, dt \tag{4.6}$$

where $x(t_0) = x_0$, and $x(t_1)$ is the state system (4.4) at time t_1 resulting from initial condition x_0 and the input function $u(t)$.

The dissipation inequality (4.6) expresses the concept that the stored energy $S(x(t_1))$ of the system (4.4) at any time t_1 is at most equal to the sum of the stored energy $S(x(t_0))$ present at the time t_0 and the total energy $\int_{t_0}^{t_1} w(u(t), y(t))\, dt$ which is supplied externally during the time interval $[t_0, t_1]$. Hence, as the name suggests, dissipative systems cannot internally create energy, but can rather either store it or dissipate it.

We shall study next a special type of storage functions.

Theorem 4.3 *Consider the system (4.4) with a supply rate w. Then, it is dissipative with respect to w if and only if the available storage function*

$$S_a(x) = \sup_{u(\cdot), T \geq 0} \left(-\int_0^T w(u,y)\, dt \right), \quad x(0) = x \tag{4.7}$$

is well defined, i.e., $S_a(x) < \infty$, $\forall x \in \mathcal{X}$. Moreover, if $S_a(x) < \infty$, $\forall x \in \mathcal{X}$, then S_a is itself a storage function, and it provides a lower bound on all other storage functions, i.e., for any other storage function S

$$S_a(x) \leq S(x)$$

Proof: First note that $S_a \geq 0$ (why?). Suppose that S_a is finite. Compare now $S_a(x(t_0))$ with $S_a(x(t_1)) - \int_{t_0}^{t_1} w(u(t), y(t))\,dt$, for a given $u : [t_0, t_1] \to \mathbb{R}^m$ and resulting state $x(t_1)$. Since S_a is given as the supremum over all $u(.)$ it immediately follows that $S_a(x(t_0)) \geq S_a(x(t_1)) - \int_{t_0}^{t_1} w(u(t), y(t))\,dt$ and thus S_a is a storage function, proving that the system (4.4) is dissipative with respect to the supply rate w.

In order to show the converse, assume that (4.4) is dissipative with respect to w. Then there exists a storage function $S \geq 0$ such that for all $u(.)$

$$S(x(0)) + \int_0^T w(u(t), y(t))\,dt \geq S(x(T)) \geq 0$$

which shows that

$$S(x(0)) \geq \sup_{u(\cdot), T \geq 0} \left(- \int_0^T w(u(t), y(t))\,dt \right) = S_a(x(0))$$

proving finiteness of S_a, as well as $S_a(x) \leq S(x)$.

□

Remark 4.4 *Note that in linking dissipativity with the existence of the function S_a, we have removed attention from the satisfaction of the dissipation inequality to the existence of the solution to an optimization problem.*

Remark 4.5 *The quantity S_a can be interpreted as the maximal energy which can be extracted from the system* (4.4) *starting at an initial condition x_0.*

Consider the dissipation inequality in the limit where $t_1 \to t_0$. Then it may be seen that satisfaction of the dissipation inequality is equivalent to fulfilling the partial differential equation (assuming S is differentiable)

$$\dot{S}(x) = \frac{\partial S(x)}{\partial x} f(x, u) \leq w(u, h(x, u)), \forall x \in \mathbb{R}^n, u \in \mathbb{R}^m \tag{4.8}$$

This version of the dissipation property is called the *differential dissipation inequality*. Having this differential version, we can establish a connection to what we have already seen in the lecture on Lyapunov stability theory. But before we do that, let us first see what the rate at which the system (4.4) dissipates energy is.

Definition 4.6 *The function $d : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ is the dissipation rate of the dissipative system (4.4) with supply rate w and storage function S, if $\forall t_0, t_1 \in \mathbb{R}^+$, $x_0 \in \mathcal{X}$, and $u \in \mathcal{U}$, the following equality holds*

$$S(x(t_0)) + \int_{t_0}^{t_1} (w(t) + d(t))\, dt = S(x(t_1))$$

Of course, we would require that d is non-negative in order to obtain dissipation!

Lemma 4.7 *Let S be a continuously differentiable storage function for the system (4.4) and assume that the supply rate w satisfies*

$$w\,(0, y) \leq 0, \quad \forall y \in \mathcal{Y}$$

Let the origin $x = 0$ be a strict local minimum of $S\,(x)$. Then $x = 0$ is a locally stable equilibrium for the unforced system $\dot{x} = f(x, 0)$ and $V\,(x) = S\,(x) - S\,(0) \geq 0$ is a local Lyapunov function.

Proof: Consider the Lyapunov function candidate $V(x) = S(x) - S(0)$, which is positive definite (why?). Under $u = 0$, we have that

$$\dot{V}(x) = \dot{S}(x) = \frac{\partial S(x)}{\partial x} f(x, 0) \leq w\,(0, y) \leq 0$$

$\square$

We can also show that the feedback interconnection of dissipative systems is stable.

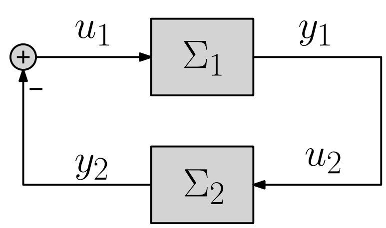


Figure 4.3: Feedback interconnection of dissipative systems

Lemma 4.8 *Consider the two systems*

$$(\Sigma_i) : \begin{cases} \dot{x}_i = f_i(x_i, u_i) \\ y_i = h(x_i, u_i) \end{cases} \tag{4.9}$$

connected in feedback as shown in Figure 4.3. Assume that both systems are dissipative with respect to supply rates w_i and positive definite storage functions S_i. Assume further that

$$w_1(u, y) + w_2(y, -u) = 0, \quad \forall u, y$$

Then, the feedback system is stable.

Proof: Consider the Lyapunov function candidate $V(x) = S_1(x_1) + S_2(x_2)$.

$$\begin{aligned} \dot{V}(x_1, x_2) = \dot{S}(x_1) + \dot{S}(x_2) &\leq w_1(u_1, y_1) + w_2(u_2, y_2) \\ &= w_1(-y_2, y_1) + w_2(y_1, y_2) = 0 \end{aligned}$$

and the result follows.

□

Lemma 4.8 is an extremely powerful one and captures many of the stability results in the frequency domain.

Example 4.9 *Consider the RLC circuit shown in Figure 4.4. Define the states $x_1 = v_c$ and $x_2 = i$, the input $u = v_{in}$ and the output $y = x_2$. The state-space model is given by*

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{L} \end{bmatrix} u, \quad y = Hx = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

For simplicity, we shall take $L = C = R = 1$. The energy storage in the system is captured by the inductor and the capacitor, i.e., the storage function in the system is given by

$$S(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$$

which is positive definite, and the supply rate to the system is given by

$$w(u, y) = uy \tag{4.10}$$

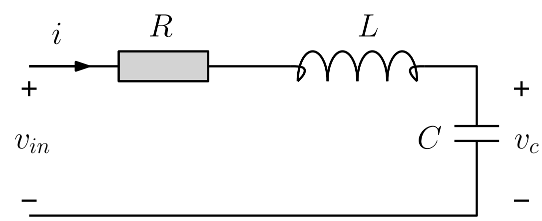


Figure 4.4: *RLC* electric circuit

which is the power injected (extracted) into (from) the system. Now,

$$\dot{S}(x) = x_1\dot{x}_1 + x_2\dot{x}_2 = -x_2^2 + uy \leq w(u, y) \tag{4.11}$$

Hence, the system is dissipative. Now, if we let $u = 0$, i.e., we short-circuit the terminals, we obtain

$$\dot{S}(x) = -x_2^2 \leq 0$$

and the energy that was initially stored in the capacitor and/or inductor is cycled in the system and dissipated in the resistor. We can show in this case, that using the so called strong Lyapunov function *with $u = 0$ we get asymptotic stability. Consider the Lyapunov function candidate*

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \epsilon x_1 x_2 \tag{4.12}$$

where $\epsilon \in (0, 1)$ (show that this choice of ϵ makes $V(x)$ positive definite!). Taking the derivative of V along the trajectories of the closed-loop system, we obtain

$$\begin{aligned}
\dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 + \epsilon x_1\dot{x}_2 + \epsilon x_2\dot{x}_1 \\
&= x_1x_2 + x_2(-x_1 - x_2) + \epsilon x_1(-x_1 - x_2) - \epsilon x_2^2 \\
&= -\epsilon x_1^2 - (1+\epsilon)x_2^2 - \epsilon x_1 x_2 \\
&\leq -\epsilon x_1^2 - (1+\epsilon)x_2^2 + \frac{\epsilon^2}{2}x_1^2 + \frac{1}{2}x_2^2 \\
&= -\epsilon\left(1 - \frac{1}{2}\epsilon\right)x_1^2 - \left(\frac{1}{2} + \epsilon\right)x_2^2 < 0
\end{aligned}$$

for our choice of ϵ. This way we have avoided using LaSalle's invariance principle, which we could have used instead to show asymptotic stability of the system.

4.3 Passive Systems

Passive systems are a special subclass of dissipative systems, and they have a special type of supply rate, given by

$$w(u,y) = u^T y \tag{4.13}$$

with the implicit condition that the number of inputs and outputs is the same, i.e., $u, y \in \mathbb{R}^p$. We can also differentiate among various types of passive dynamical systems according to the following definition:

Definition 4.10 *A state space system (4.4) is called*

1. *passive if it is dissipative with respect to the supply rate* $w\left(u,y\right) = u^T y$
2. *lossless if* $\dot{S}(x) = u^T y$
3. *input-feedforward passive if it is dissipative with respect to the supply rate* $w(u,y) = u^T y - u^T \varphi(u)$ *for some function* φ
4. *input strictly passive if it is dissipative with respect to the supply rate* $w(u,y) = u^T y - u^T \varphi(u)$ *and* $u^T \varphi(u) > 0,\ \forall u \neq 0$
5. *output feedback passive if it is dissipative with respect to the supply rate* $w(u,y) = u^T y - y^T \rho(y)$ *for some function* ρ
6. *output strictly passive if it is dissipative with respect to the supply rate* $w(u,y) = u^T y - y^T \rho(y)$ *for some function* $y^T \rho(y) > 0,\ \forall y \neq 0$
7. *strictly passive if it is dissipative with respect to the supply rate* $w(u,y) = u^T y - \psi(x)$ *for some positive definite function* ψ

Example 4.11 *Consider an integrator model given by*

$$\dot{x} = u, \quad y = x \tag{4.14}$$

with the supply rate $w(u,y) = uy$*. Take the storage function* $S(x) = \frac{1}{2}x^2$*. The derivative* $\dot{S}(x) = x\dot{x} = uy$*, and hence the system is lossless.*

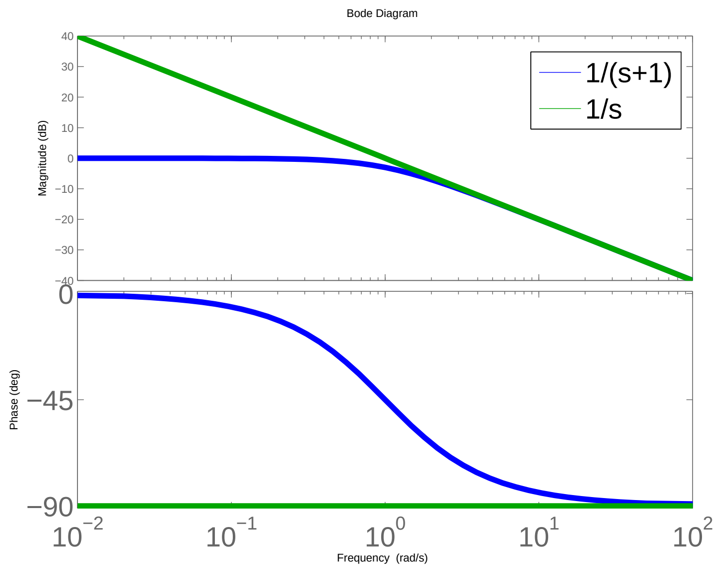


Figure 4.5: Bode plot of the two systems in Examples 4.11 and 4.12

Example 4.12 *Now assume that instead of the pure integrator in Example 4.11 with transfer function* $\frac{1}{s}$*, we consider the low pass filter* $\frac{1}{s+1}$*, which has the state-space representation*

$$\dot{x} = -x + u, \quad y = x \tag{4.15}$$

Consider the storage function $S(x) = \frac{1}{2}x^2$*. The derivative* $\dot{S}(x) = x\dot{x} = -x^2 + uy$*. Hence the system is strictly dissipative, and globally asymptotically stable for* $u = 0$ *(convince yourself of the latter fact). This is one of the main reasons that we would not implement 'pure' integrators in embedded systems, but instead opt for low pass filters due to their inherent stable behavior. Finally, notice from Figure 4.5 that both systems behave similarly for high frequencies.*

4.3.1 Characterizations of Passivity for Linear Systems

Passive systems are particularly interesting in the linear systems case, because we can get characterizations of passivity both in the frequency domain and in the time domain. In the frequency domain, we think of transfer functions and we can relate passivity to certain conditions being statisfied for the transfer function.

Definition 4.13 *A $p \times p$ proper transfer function matrix $G(s)$ is called positive real if all the following conditions are satisfied:*

1. *the poles of all elements of $G(s)$ have non-positive real part*
2. *for all real frequencies ω for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G(-j\omega)^T$ is positive semi-definite*
3. *any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s \to jw}(s - jw)G(s)$ is positive semidefinite Hermitian*

$G(s)$ is called strictly positive real if $G(s - \epsilon)$ is positive real for some $\epsilon > 0$.

Remark 4.14 *For $p = 1$, the second condition of Definition 4.13 reduces to $Re[G(jw)] \geq 0, \forall w \in \mathbb{R}$. Moreover, this condition is satisfied only if the relative degree of the transfer function $G(s)$ is at most one.*

The positive real property of transfer matrices can be equivalently characterized as follows.

Lemma 4.15 *Let $G(s)$ be a proper rational $p \times p$ transfer function matrix. Suppose that $det(G(s) + G(-s)^T)$ is not equivalent to zero for all s. Then $G(s)$ is strictly positive real if and only if the following three conditions are satisfied*

1. *$G(s)$ is Hurwitz,*
2. *$G(j\omega) + G(-j\omega)^T$ is positive definite $\forall \omega \in \mathbb{R}$*
3. *either $G(\infty) + G(\infty)^T$ is positive definite or it is positive semidefinite and $\lim_{\omega \to \infty} \omega^2 M^T (G(j\omega) + G(-j\omega)^T) M$ is positive definite for any full rank $p \times (p - q)$ matrix M such that $M^T(G(\infty) + G(\infty)^T)M = 0$, where $q = rank(G(\infty) + G(\infty)^T)$.*

Example 4.16 *Recall the RLC circuit in Example 4.9. The transfer function (for $R = L = C = 1$) is given by*

$$G(s) = H(sI - A)^{-1}B = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & -1 \\ 1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{s}{s^2 + s + 1} \tag{4.16}$$

Let us check if it is positive real. Note: We can show that condition 2. of Lemma 4.15 does not hold, and hence the transfer function is not strictly positive real.

The poles of $G(s)$ are given by $s_i = \frac{-1 \pm j\sqrt{3}}{2}$, $i = 1, 2$, thus $G(s)$ is Hurwitz. We also have that

$$Re[G(jw)] = \frac{w^2}{(1-w^2)^2 + w^2} \geq 0, \forall w \in \mathbb{R}$$

Finally, we have no pure imaginary poles. Therefore, we can conclude that the transfer function (4.16) is positive real.

One can also look at the state-space system directly and conclude that the transfer function is actually positive real, as shown by the celebrated KYP lemma below.

Lemma 4.17 (Kalman-Ykubovich-Popov) *Consider the $m \times m$ transfer function matrix $G(s) = C(sI - A)^{-1}B + D$, where the pair (A, B) is controllable and the pair (A, C) is observable. $G(s)$ is strictly positive real if and only if there exist matrices $P = P^T > 0$, L, and W, and $\epsilon > 0$ such that the following equalities hold*

$$PA + A^T P = -L^T L - \epsilon P \tag{4.17}$$

$$PB = C^T - L^T W \tag{4.18}$$

$$W^T W = D + D^T \tag{4.19}$$

Remark 4.18 *If $\epsilon = 0$ in Lemma 4.15, then the transfer function $G(s)$ is simply positive real.*

Example 4.19 *Consider again the RLC system in Example 4.9. Since there is no direct feedthrough in the system, we have $D = 0$ and by (4.19), $W = 0$. As such, (4.18) implies $P\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix} \Rightarrow P = \begin{bmatrix}p & 0\\0 & 1\end{bmatrix}$, $p > 0$. Plugging P into (4.17)*

$$PA + A^T P = -\begin{bmatrix}0 & 1-p\\1-p & 2\end{bmatrix} \leq 0 \Leftrightarrow p = 1 \Rightarrow L^T = \begin{bmatrix}0\\\sqrt{2}\end{bmatrix}$$

Therefore, again we reach the same conclusion that the system is passive. However, we cannot satisfy the condition (4.17) with $\epsilon > 0$ and hence the system is not strictly passive.

Finally, we are ready to state the connection/equivalence between positive realness and passivity for Linear Time-Invariant (LTI).

Theorem 4.20 *The LTI system*

$$\begin{aligned}\dot{x} &= Ax + Bu\\ y &= Cx + Du\end{aligned}$$

with the corresponding transfer matrix $G(s) = C(sI - A)^{-1}B + D$ *is*

- *passive if* $G(s)$ *is positive real*
- *strictly passive, if* $G(s)$ *is strictly positive real.*

Proof: Consider the storage function $S(x) = \frac{1}{2}x^T Px$. Then,

$$\begin{aligned}
\dot{S}(x) &= x^T P(Ax + Bu) = \frac{1}{2}x^T(PA + A^T P)x + x^T PBu\\
&= \frac{1}{2}x^T(PA + A^T P)x + x^T C^T u - x^T L^T W u\\
&= \frac{1}{2}x^T(PA + A^T P)x + y^T u - \frac{1}{2}u^T(D^T + D)u - x^T L^T W u\\
&= \frac{1}{2}x^T(PA + A^T P)x + y^T u - \frac{1}{2}u^T W^T W u - x^T L^T W u\\
&\quad - \frac{1}{2}x^T L^T Lx + \frac{1}{2}x^T L^T Lx\\
&= \frac{1}{2}x^T(PA + A^T P)x + y^T u - \frac{1}{2}(Wu + Lx)^T(Wu + Lx) + \frac{1}{2}x^T L^T Lx\\
&\leq \frac{1}{2}x^T(PA + A^T P)x + y^T u + \frac{1}{2}x^T L^T Lx\\
&= \begin{cases} -\frac{\epsilon}{2}x^T Px + y^T u, \ \epsilon > 0, \ \text{strictly passive}\\ y^T u, \ \epsilon = 0, \ \text{passive}\end{cases}
\end{aligned}$$

□

4.3.2 Stability of Passive Systems

Let us consider again the nonlinear system (4.4), where $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is locally Lipschitz, $h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ is continuous, with $f(0,0) = 0$ and $h(0,0) = 0$.

Lemma 4.21 *Assume that the system* (4.4) *is passive with a positive definite storage function* $S(x)$*, then the origin of* $\dot{x} = f(x, 0)$ *is stable.*

Proof: Take $V(x) = S(x)$ as Lyapunov function candidate. Then $\dot{V}(x) \leq u^T y = 0$, and stability follows.

□

We can further strengthen the previous Lemma, as follows.

Lemma 4.22 *Assume that the system* (4.4) *is strictly passive with some storage function* $S(x)$*, then the origin of* $\dot{x} = f(x, 0)$ *is asymptotically stable. Furthermore, if* $S(x)$ *is radially unbounded, then the origin is globally asymptotically stable.*

Proof: Let $V(x) = S(x)$ be a Lyapunov function candidate. Since the system is strictly passive with a storage function $V(x)$, it follows that (for $u = 0$)

$$\dot{V}(x) \leq -\psi(x) + u^T y = -\psi(x)$$

Now consider any $x \in \mathbb{R}^n$ and let $\phi(t, x)$ be the solution to the differential equation $\dot{x} = f(x, 0)$, starting at x and time $t = 0$. As such, we have that

$$V(\phi(t, x)) - V(x) \leq -\int_0^t \psi(\phi(\tau, x))d\tau \quad \forall t \in [0, \delta] \tag{4.20}$$

for some positive constant δ. Since $V(\phi(t, x)) \geq 0$, then

$$V(x) \geq \int_0^t \psi(\phi(\tau, x))d\tau$$

Suppose now that there exists some $\bar{x} \neq 0$ such that $V(\bar{x}) = 0$. This implies that

$$\int_0^t \psi(\phi(\tau, \bar{x}))d\tau = 0, \forall t \in [0, \delta] \Rightarrow \psi(\phi(\tau, \bar{x})) \equiv 0 \Rightarrow \bar{x} = 0$$

which gives a contradiction. Hence, $V(x) > 0$ for all $x \neq 0$, i.e., positive definite. Combining this with the $\dot{V}(x) \leq -\psi(x)$, yields asymptotic stability of the origin. Finally, if $V(x)$ is radially unbounded, we obtain asymptotic stability.

□

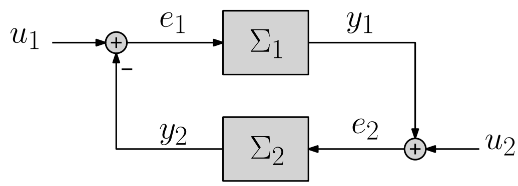


Figure 4.6: Feedback interconnection of passive systems

We shall look at some of the stability properties of passive systems, when connected in a feedback structure as shown in Figure 4.6.

Theorem 4.23 *The feedback connection of two passive systems is passive.*

Proof: Let $S_i(x_i)$ be the storage function of system Σ_i, $i = 1, 2$. Since both systems are passive,we have that

$$\dot{S}_i(x_i) \leq e_i^T y_i$$

Using the feedback structure in Figure 4.6, we have that

$$\begin{aligned} \dot{S}(x) &= \dot{S}_1(x_1) + \dot{S}_2(x_2) \leq e_1^T y_1 + e_2^T y_2 && (4.21) \\ &= (u_1 - y_2)^T y_1 + (u_2 + y_1)^T y_2 = u_1^T y_1 + u_2^T y_2 = u^T y && (4.22) \end{aligned}$$

and the result follows. Note that if any of the two systems are memoryless, i.e., $y_i = h_i(u_i)$, then the corresponding storage function can be taken to be 0.

□

4.3.3 Passivity-Based Control

Having done some analysis on passive systems, we can now show a glimpse of how this theory can be used to design the so-called passivity-based controllers. Consider the dynamical system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \qquad (4.23)$$

with the usual Lipschitz assumption on f and continuity of h. Moreover, assume that $f(0, 0) = 0$ and $h(0) = 0$.

Definition 4.24 *The system (4.23) is called zero-state observable, if no solution of the unforced system $\dot{x} = f(x, 0)$ can stay identically in the set $\{h(x) = 0\}$ other than the trivial solution $x(t) \equiv 0$.*

Theorem 4.25 *Assume that the system (4.23) is*

1. *passive with a radially unbounded positive definite storage function, and*
2. *is zero-state observable,*

then the origin can be globally stabilized with a control law $u = -\phi(y)$, where ϕ is any locally Lipschitz function such that $\phi(0) = 0$, and $y^T\phi(y) > 0$, $\forall y \neq 0$.

Proof: Let $V(x)$ be a storage function of the system and use it as a candidate Lyapunov function for the closed-loop system with $u = -\phi(y)$. We have that

$$\dot{V}(x) \leq u^T y = -\phi(y)^T y \leq 0$$

Hence, $\dot{V}(x) \leq 0$, and $\dot{V}(x) = 0$ if and only if $y = 0$. By zero-state observability, the only solution that can stay in the set $\{y = h(x) = 0\}$ is the trivial solution $x(t) \equiv 0$, and we can conclude using LaSalle's invariance principle that the origin is globally asymptotically stable.

□

This last theorem is very useful when designing control laws for a large number of electrical and mechanical systems. Moreover, instead of starting with systems for which the origin is open-loop stable, we can design control laws that convert a nonpassive system into a passive one, a technique known as feedback passivation.

Example 4.26 *Consider the system*

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^2 + u \\ y &= x_2\end{aligned}$$

The open-loop system ($u = 0$) is unstable, and hence the system is not passive. However, we can design the control law

$$u = -x_2^2 - x_1^3 + v$$

that yields the system passive with respect to the supply rate $w = vy$. Let $S(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$, then

$$\dot{S}(x) = -x_2^2 + vy$$

and the system is passive. Noting that $v = 0$ and $y(t) \equiv 0$ imply $x(t) \equiv 0$. Therefore, all the conditions of Theorem 4.25 are satisfied, and accordingly we can design a globally stabilizing control law, for example $v = -kx_2$ or $v = -tan^{-1}(x_2)$ and any $k > 0$.

For further reading on the subject, please see [4, 6, 7].

4.4 Exercises

1. The motion of a rigid n-link robot manipulator is described by the following equations:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + \frac{\partial P(q)}{\partial q}^{\top} = u$$

where $q \in \mathbb{R}^n$ and $\dot{q} \in \mathbb{R}^n$ are the so-called generalized coordinates of the system, $u \in \mathbb{R}^n$ represents the input (torques or forces), and $P(q)$ is the potential energy that is positive definite. The system above satisfies the following skew-symmetry properties:

- The matrix $L = \dot{M}(q) - 2C(q,\dot{q})$ is skew-symmetric, i.e. $L^{\top} = -L$;
- $M(q)^{\top} = M(q) \geq \lambda I$, $\forall q \in \mathbb{R}^n$ with $\lambda > 0$.

(a) Show that the system is passive from input u to output $\dot{q}$, using the storage function $S(q,\dot{q}) = \frac{1}{2}\dot{q}^{\top}M(q)\dot{q}$. Only for this point, consider $P(q) = 0$.

(b) Now let us use the following control law $u = -B\dot{q}$, where $B = B^{\top} > 0$ is a damping matrix. Under the assumption that $\frac{\partial P(q)}{\partial q}^{\top}$ has an isolated root at the origin, show that this control law is asymptotically stabilizing using $V(q,\dot{q}) = S(q,\dot{q}) + P(q)$ as a Lyapunov fucntion.

(c) Now, consider the following control law $u = \frac{\partial P(q)}{\partial q}^{\top} - B\dot{q} - K(q - q^d)$, where $B = B^T > 0$, $K = K^T > 0$, and q^d is some desired set point. Show that the equilibrium point $(q,\dot{q}) = (q^d, 0)$ is asymptotically stable using the Lyapunov function candidate $V = \frac{1}{2}\dot{q}M\dot{q} + \frac{1}{2}(q - q^d)^{\top}K(q - q^d)$ (Note that $\dot{q}_d = 0$).

2. Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h(x_1) - ax_2 + u \\ y &= \alpha x_1 + x_2 \end{aligned}$$

where $0 < \alpha < a$, $yh(y) > 0$ for all $y \neq 0$.

(a) Show that the system is passive by choosing an opportune storage candidate function. *Hint:* In your candidate function, you need to integrate the nonlinear function $h(\cdot)$.

(b) How would you render the system globally asymptotically stable by using output feedback?

3. Euler equations for a rotating rigid spacecraft are given by

$$\begin{aligned} J_1\dot{\omega}_1 &= (J_2 - J_3)\omega_2\omega_3 + u_1 \\ J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 + u_2 \\ J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_1\omega_2 + u_3 \end{aligned}$$

where ω_1 to ω_3 are the components of the angular velocity vector along the principal axes, u_1 to u_3 are the torque inputs applied about the principal axes, and J_1 to J_3 are the principal moment of inertia.

(a) Show that the map from $u = [u_1, u_2, u_3]^\top$ to $\omega = [\omega_1, \omega_2, \omega_3]^\top$ is lossless.

(b) Show that, for the control law $u = -K\omega$, where K is a positive definite matrix, the origin $\omega = 0$ is globally asymptotically stable.

Part II

Control Design Techniques

Chapter 5

Feedback Control Design

5.1 Introduction

Control design procedures require the use of feedback in order to steer the state of the system towards some desired steady-state value, or a given output towards some reference.

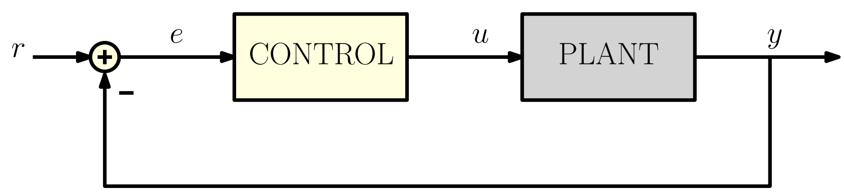


Figure 5.1: Feedback Control

When dealing with nonlinear systems, one has a variety of design techniques that can be used. We summarize in the table below some of the dimensions that we shall consider in the design phase of the course.
In this lecture, we shall look into three control design techniques: Linearization, integral control, and gain scheduling, all of which are based on our knowledge of control design for linear systems.

5.2 Linearization

Based on the ideas we have already explored in Lyapunov's indirect method, we can consider designing a controller by linearizing the system about the de-

Requirements / Goals	Tools	Difficulties
Stabilization Tracking Disturbance rejection Disturbance attenuation Transient response ...	Linearization Integral Control Gain Scheduling Robust control Adaptive control Feedback linearization Sliding mode control ...	Nonlinearities Lack of measurements Noise Uncertainties ...

sired equilibrium point, and then designing a stabilizing linear state feedback by pole placement, or other similar techniques.

Consider the nonlinear system

$$\dot{x} = f(x, u) \tag{5.1}$$

where $f(x,u)$ is continuously differentiable in a domain $\mathcal{X} \times \mathcal{U} \subseteq \mathbb{R}^n \times \mathbb{R}^m$ that contains the origin. Let $(x_{ss}, u_{ss}) \in \mathcal{X} \times \mathcal{U}$ be the steady state point for which

$$0 = f(x_{ss}, u_{ss}) \tag{5.2}$$

We can linearize the system (5.1) around the steady-state operation point (x_{ss}, u_{ss}) using the Taylor series expansion to obtain,

$$\begin{aligned}
\frac{d}{dt}(x - x_{ss}) &= f(x,u) \\
&\approx f(x_{ss}, u_{ss}) + \left.\frac{\partial f}{\partial x}\right|_{x=x_{ss},u=u_{ss}}(x - x_{ss}) + \left.\frac{\partial f}{\partial u}\right|_{x=x_{ss},u=u_{ss}}(u - u_{ss}) \\
&= A(x - x_{ss}) + B(u - u_{ss})
\end{aligned} \tag{5.3}$$

where $A = \left.\frac{\partial f}{\partial x}\right|_{x=x_{ss},u=u_{ss}}$ and $B = \left.\frac{\partial f}{\partial u}\right|_{x=x_{ss},u=u_{ss}}$. Assume that (A, B) is stabilizable, that is the eigenvalues that aren't controllable have all negative real parts. Then $\exists K \in \mathbb{R}^{m\times n}$ such that $(A - BK)$ has all eigenvalues in the left hand side of the complex plane. If *(A,B)* is controllable, we can place all the all the poles at the desired locations in the left-half complex plane. The resulting feedback control law for the nonlinear system (5.1) is given by

$$u = u_{ss} - K(x - x_{ss}) \tag{5.4}$$

and the corresponding closed-loop system is given by

$$\dot{x} = f(x, u_{ss} - K(x - x_{ss}))$$

which is asymptotically stable in some small neighborhood around (x_{ss}, u_{ss}).

Example 5.1 *Consider the pendulum shown in Figure 5.2 equation*

$$\ddot{\theta} = -a\sin\theta - b\dot{\theta} + cT \tag{5.5}$$

where $a = g/l$, $b = k/m \geq 0$, $c = 1/ml^2 > 0$, and T is the torque applied to the pendulum. We would like to stabilize the pendulum at some angle θ_{ss}.

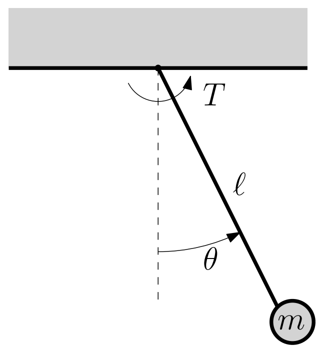


Figure 5.2: Pendulum

For the pendulum to have an equilibrium at $(\theta_{ss}, 0)$, the input torque must have a steady state component T_{ss} that satisfies

$$T_{ss} = \frac{a}{c}\sin(\theta_{ss}) \tag{5.6}$$

Define the state variables as $x_1 = \theta - \theta_{ss}$, $x_2 = \dot{\theta}$ and the control variable as $u = T - T_{ss}$. Accordingly, the state equations are given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a[\sin(x_1 + \theta_{ss}) - \sin(\theta_{ss})] - bx_2 + cu \end{aligned} \tag{5.7}$$

We linearize this system at the origin to obtain the system matrices $A = \begin{bmatrix} 0 & 1 \\ -a\cos(\theta_{ss}) & -b \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ c \end{bmatrix}$. It is not difficult to show that the pair (A, B) is controllable (show this). Take the feedback matrix $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$, then $(A - BK)$ is Hurwitz for any choice of the gains that satisfies [1]

$$k_1 > -\frac{a\cos(\theta_{ss})}{c}, k_2 > -\frac{b}{c}$$

[1] you can drive this by looking at $det(\lambda I - (A - BK)))$ and enforcing that the sign of all the coefficients of the characteristic polynomial is the same

Finally, the input torque that we would apply to the pendulum is given by

$$T = T_{ss} - Kx = \frac{a}{c}\sin(\theta_{ss}) - k_1(\theta - \theta_{ss}) - k_2\dot{\theta}$$

Assume now that we do not have access to the full state, but rather a nonlinear function of the state, i.e., we have the nonlinear system

$$\begin{aligned} \dot{x} &= f(x, u) \\ y &= h(x) \end{aligned} \tag{5.8}$$

We shall linearize the system (5.8) around the steady-state point (x_{ss}, u_{ss}), for which the following conditions are satisfied

$$f(x_{ss}, u_{ss}) = 0, \quad h(x_{ss}) = 0 \tag{5.9}$$

to obtain

$$\frac{d}{dt}(x - x_{ss}) = A(x - x_{ss}) + B(u - u_{ss}) \tag{5.10}$$

$$y = C(x - x_{ss}) \tag{5.11}$$

where $A = \frac{\partial f}{\partial x}\big|_{x=x_{ss},u=u_{ss}}$, $B = \frac{\partial f}{\partial u}\big|_{x=x_{ss},u=u_{ss}}$, and $C = \frac{\partial h}{\partial x}\big|_{x=x_{ss}}$. If we assume that (A, B) is stabilizable and (A, C) is detectable, then we can design an observer-based linear dynamic output feedback controller

$$\begin{aligned} \frac{d}{dt}(\hat{x} - x_{ss}) &= A(\hat{x} - x_{ss}) + B(u - u_{ss}) + L(y - C(\hat{x} - x_{ss})) \\ u - u_{ss} &= -K(\hat{x} - x_{ss}) \end{aligned} \tag{5.12}$$

such that the closed-loop system

$$\frac{d}{dt}\begin{bmatrix} x - x_{ss} \\ x - \hat{x} \end{bmatrix} = \mathcal{A}_{cl}\begin{bmatrix} x - x_{ss} \\ x - \hat{x} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix}\begin{bmatrix} x - x_{ss} \\ x - \hat{x} \end{bmatrix} \tag{5.13}$$

is asymptotically (exponentially) stable, i.e., the matrix $\mathcal{A}_{cl}$ is Hurwitz. Finally, the corresponding closed-loop nonlinear system is given by

$$\begin{aligned} \dot{x} &= f(x, u_{ss} - K(\hat{x} - x_{ss})) \\ \dot{\hat{x}} &= (A - BK - LC)(\hat{x} - x_{ss}) + Lh(x) \end{aligned}$$

5.3 Integral Control

The approach before is useful if there are no modeling errors. Integral control is a method to handle some common uncertainties. We will present an integral control approach that ensures asymptotic regulation under all parameter perturbations that do not destroy the stability of the closed-loop system.

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= f(x, u, w) \\ y &= h(x, w) \end{aligned} \tag{5.14}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the control input, $y \in \mathbb{R}^p$ is the output, and $w \in \mathbb{R}^l$ is a vector of unknown *constant* parameters and disturbances. We assume that f and h are continuously differentiable in x and u, and continuous in w within some domain $\mathcal{X} \times \mathcal{U} \times \mathcal{W} \subset \mathbb{R}^n \times \mathbb{R}^p \times \mathbb{R}^l$. We would like to design a control law that guarantees that $y(t) \to r$ as $t \to \infty$, for some constant reference r. We shall assume that there is a unique pair (x_{ss}, u_{ss}) which depends on both r and w, such that

$$\begin{aligned} 0 &= f(x_{ss}(w, r), u_{ss}(w, r), w) \\ r &= h(x_{ss}(w, r), w) \end{aligned}$$

where x_{ss} is the desired equilibrium point and u_{ss} is the corresponding steady-state control. As the name of the method suggests, we would like to introduce an integral action with repsect to the error signal $e = y - r$. This goal is achieved by introducing the extra state (multi-dimensional)

$$\dot{\sigma} = y - r = h(x, w) - r \tag{5.15}$$

Accordingly, the augmented system is given by

$$\begin{aligned} \dot{x} &= f(x, u, w) \\ \dot{\sigma} &= h(x, w) - r \end{aligned} \tag{5.16}$$

In general, it is quite difficult to find a globally stabilizing control law for the system (5.16). As such, we shall proceed by seeking a locally stabilizing control via the linearization technique we saw in the previous section.

We propose the linear control structure

$$u = -K_1 x - K_2 \sigma \tag{5.17}$$

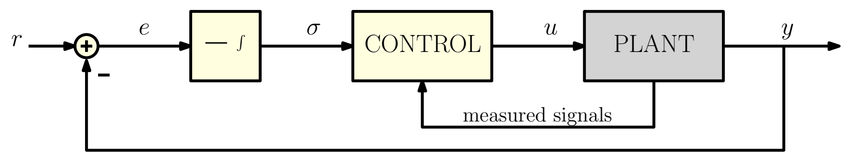


Figure 5.3: Integral Control

and require that the matrix gain K_2 be nonsingular in order guarantee the existence of a unique equilibrium $(x_{ss}, u_{ss}, \sigma_{ss})$ to the steady-state equations

$$\begin{aligned} 0 &= f(x_{ss}, u_{ss}, w) \\ 0 &= h(x_{ss}, w) - r \\ u_{ss} &= -K_1 x_{ss} - K_2 \sigma_{ss} \end{aligned}$$

with the understanding that the equilibrium point depends on both the known reference r and the unknown but constant parameters w.

We shall proceed by using the linearization design technique that we saw in the previous section to obtain the system

$$\begin{aligned} \dot{x} &= A(x - x_{ss}) + B(u - u_{ss}) \\ \dot{\sigma} &= C(x - x_{ss}) \end{aligned} \tag{5.18}$$

where $A = A(w,r) = \left.\frac{\partial f(x,u,w)}{\partial x}\right|_{x=x_{ss},u=u_{ss}}$, $B = B(w,r) = \left.\frac{\partial f(x,u,w)}{\partial u}\right|_{x=x_{ss},u=u_{ss}}$, and $C = C(w,r) = \left.\frac{\partial h(x,w)}{\partial x}\right|_{x=x_{ss}}$ are dependent on the constant parameters w and the constant reference vector r. Now if we substitute the value steady-state input u_{ss} into the linearized system (5.18), we obtain the following augmented closed-loop system

$$\dot{\xi} = (\mathcal{A} - \mathcal{B}\mathcal{K})\xi = \begin{bmatrix} A - BK_1 & -BK_2 \\ C & 0 \end{bmatrix} \xi \tag{5.19}$$

where $\xi = \begin{bmatrix} x - x_{ss} \\ \sigma - \sigma_{ss} \end{bmatrix}$. We have the following Lemma that guarantees the possibility of designing a stabilizing control matrix $\mathcal{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$.

Lemma 5.2 *Assume that the pair (A, B) is controllable and that*

$$rank \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = n + p \tag{5.20}$$

Then, the pair $(\mathcal{A}, \mathcal{B})$ is controllable.

Proof: See the exercises.

Assuming that the conditions of Lemma 5.2 are satisfied, we can design the state feedback integral controller as

$$u = -K_1 x - K_2 \sigma \tag{5.21}$$
$$\dot{\sigma} = h(x, w) - r \tag{5.22}$$

where $\mathcal{K} = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$ is designed so that it renders $(\mathcal{A} - \mathcal{B}\mathcal{K})$ Hurwitz for all possible values of the uncertain vector w in some known set $\mathcal{W}$ (i.e., we have robust stability).

Example 5.3 *Consider again the Pendulum system in Example 5.1. This time we don't know the parameters in the system, but we have that $a = g/l > 0$, $b = k/m \geq 0$, and $c = 1/(ml^2) > 0$. We have the nonlinear system (using the same definition for the state and input as before)*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin(x_1 + \theta_{ss}) - bx_2 + cu \\ y &= x_1 \end{aligned} \tag{5.23}$$

with the equilibrium point

$$x_{ss} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad u_{ss} = \frac{a}{c}\sin(\theta_{ss}) \tag{5.24}$$

The linearization matrices are computed as

$$A = \begin{bmatrix} 0 & 1 \\ -a\cos(\theta_{ss}) & -b \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ c \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{5.25}$$

Since the parameter $c > 0$ we can deduce that the pair (A, B) is controllable (verify this). Moreover, the rank condition (5.20) is satisfied (verify this as well). Hence the conditions of Lemma 5.2 are satisfied and we can design as stabilizing controller $u = -\begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \sigma \end{bmatrix}$, where $\dot{\sigma} = x_1$. The resulting closed-loop matrix is given by

$$(\mathcal{A} - \mathcal{B}\mathcal{K}) = \begin{bmatrix} 0 & 1 & 0 \\ -(a\cos(\theta_{ss}) + ck_1) & -b - ck_1 & -ck_3 \\ 1 & 0 & 0 \end{bmatrix} \tag{5.26}$$

which is Hurwitz if the control gains satisfy

$$b + ck_2 > 0, \quad ck_3 > 0, \quad (b + ck_2)(a\cos(\theta_{ss}) + ck_1) > ck_3$$

These three inequalities have to be satisfied for all possible realization of the system parameters a, b, and c. For example, a choice of $k_2 > 0$, $k_3 > 0$, and $k_1 > \max\left(\frac{a}{c}\right) + \frac{k_3}{k_2}\max\left(\frac{1}{c}\right)$ would render the closed-loop matrix Hurwitz (assuming that the maximum values are attained).

5.4 Gain Scheduling

So far we have looked at design methods that guarantee stability in some neighborhood of the equilibrium point. However, this is very limiting as we would like to stabilize the system over a large region of the state space about several equilibrium points. This can be achieved by linearizing the system about each equilibrium point and design a local controller for stability. Then, we can online schedule the gains to go from one equilibrium point to another. This design methodology which scales the previous design methods can be described by the following steps:

1. Linearize the given nonlinear system about several equilibrium points, which are parametrized by scheduling variables
2. Design a parametrized family of linear controllers to locally stabilize the system around each of the equilibrium points
3. Construct a gain-scheduled controller
4. Check the performance of the gain-scheduled controller by simulating the nonlinear closed-loop model

Remark 5.4 *The region of attraction of one equilibrium point should contain the neighboring equilibrium point, as shown in Figure 5.4. Moreover, we should guarantee that the overshoot that happens whenever switching from one steady-state point ($x_{ss,i}$) to another ($x_{ss,i+1}$) does not drive the trajectory of the system beyond the region of attraction of the latter steady state point ($x_{ss,i+1}$).*

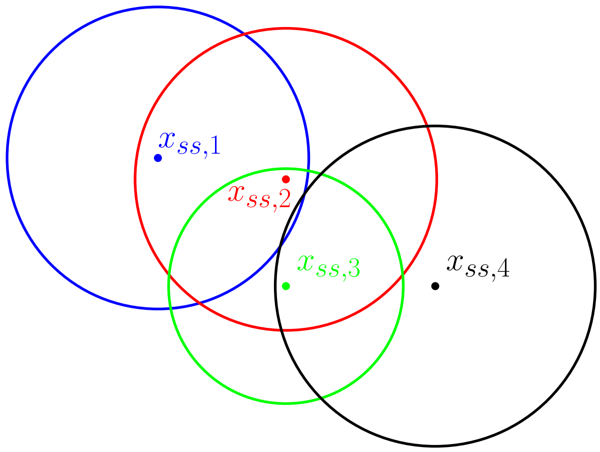


Figure 5.4: Gain scheduling example

Consider the nonlinear system

$$\dot{x} = f(x, u, v, w) \tag{5.27}$$
$$y = h(x, w) \tag{5.28}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input, $v \in \mathbb{R}^q$ is a known exogenous input, $w \in \mathbb{R}^l$ is a vector of constant but unknown parameters. We would like to design a state feedback controller that achieves the regulation of the output y to a given reference vector $r \in \mathbb{R}^p$. Let

$$\rho = \begin{bmatrix} v \\ r \end{bmatrix} \tag{5.29}$$

be the vector of known exogenous quantities in the system. We shall use ρ as a scheduling variable. For each fixed value of $\rho = \alpha = \begin{bmatrix} \alpha_v \\ \alpha_r \end{bmatrix}$, we use integral control to regulate the error $e = y - r$ to zero in the presence of the unknown vector w. Moreover, we rely on gain scheduling to achieve small error whenever ρ is varying slowly over time.

In order to design the integral control, we assume (as we have seen before) that there exists a unique pair (x_{ss}, u_{ss}) that is continuously differentiable in α and continuous in w, such that

$$0 = f(x_{ss}(\alpha, w), u_{ss}(\alpha, w), \alpha_v, w) \tag{5.30}$$
$$\alpha_r = h(x_{ss}(\alpha, w), w) \tag{5.31}$$

for all values of (α, w). Now, as in the previous section, we can design the following integral controller

$$\dot{\sigma} = y - r \tag{5.32}$$
$$u = -K_1 x - K_2 \sigma \tag{5.33}$$

where $K_1 = K_1(\alpha)$ and $K_2 = K_2(\alpha)$, such that the matrix

$$\mathcal{A} = \begin{bmatrix} A - BK_1 & -BK_2 \\ C & 0 \end{bmatrix} \tag{5.34}$$

is Hurwitz for every (α, w), where

$$A = A(\alpha, w) = \left. \frac{\partial f(x, u, v, w)}{\partial x} \right|_{(x,u,v)=(x_{ss}, u_{ss}, \alpha_v)} \tag{5.35}$$
$$B = B(\alpha, w) = \left. \frac{\partial f(x, u, v, w)}{\partial u} \right|_{(x,u,v)=(x_{ss}, u_{ss}, \alpha_v)} \tag{5.36}$$
$$C = C(\alpha, w) = \left. \frac{\partial h(x, w)}{\partial x} \right|_{(x,u,v)=(x_{ss}, u_{ss}, \alpha_v)} \tag{5.37}$$

With such a design choice, we know that the closed-loop nonlinear system

$$\begin{aligned} \dot{x} &= f(x, u, v, w) \\ \dot{\sigma} &= y - r = h(x, w) - r \\ u &= -K_1(\alpha) x - K_2(\alpha) \sigma \end{aligned}$$

has a locally asymptotically stable equilibrium point for a fixed $\rho = \alpha$.

Remark 5.5 *The fact that $\mathcal{A}$ is stable for every "frozen" $\rho = \alpha$ does not guarantee stability of the closed loop system when $\rho = \alpha(t)$. However, there are theoretical developments that relate the slow rate of change of $\rho(t)$ to the boundedness of the system response. Moreover, if $\rho(t)$ converges asymptotically to some constant value, then the error in output tracking would asymptotically converge to zero.*

Remark 5.6 *We can extent the method to the case when we don't have full state feedback, but rather we need to design an estimator of the state.*

Example 5.7 *Consider the second order system*

$$\begin{aligned} \dot{x} &= \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \tan(x_1) + x_2 \\ x_1 + u \end{bmatrix} = f(x,u) \\ y &= h(x) = x_2 \end{aligned} \tag{5.38}$$

Assume that we measure the full state, and would like the output to track a reference signal r. Now, for any fixed value of the reference $r = \alpha$, we obtain the following (unique) steady-state solution

$$x_{ss}(\alpha) = \begin{bmatrix} -\tan^{-1}(\alpha) \\ \alpha \end{bmatrix}, \quad u_{ss}(\alpha) = \tan^{-1}(\alpha) \tag{5.39}$$

We use the following integral controller

$$\dot{\sigma} = y - r = y - \alpha \tag{5.40}$$

$$u = -K_1(\alpha)x - K_2(\alpha)\sigma \tag{5.41}$$

The linearized system matrices of (5.38) are given by

$$A = A(\alpha) = \left.\frac{\partial f(x,u)}{\partial x}\right|_{x_{ss},u_{ss}} = \begin{bmatrix} 1+\alpha^2 & 1 \\ 1 & 0 \end{bmatrix}, \; B = \left.\frac{\partial f(x,u)}{\partial u}\right|_{x_{ss},u_{ss}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Therefore, we would like to design the gains $K_1(\alpha)$ and $K_2(\alpha)$ such that the closed-loop matrix of the linearized system

$$\mathcal{A}(\alpha) = \begin{bmatrix} A(\alpha) - BK_1(\alpha) & -BK_2(\alpha) \\ C & 0 \end{bmatrix} \tag{5.42}$$

is Hurwitz. The gains are designed as

$$\begin{aligned} K_1(\alpha) &= \begin{bmatrix} (1+\alpha^2)(3+\alpha^2) + 3 + \frac{1}{1+\alpha^2} & 3+\alpha^2 \end{bmatrix} \\ K_2(\alpha) &= -\frac{1}{1+\alpha^2} \end{aligned}$$

which would assign the closed-loop eigenvalues at $-1, -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$. The response of the closed-loop system is shown in Figure 5.5.

Additional performance can be achieved if the derivative of the output y can be reliably estimated. In those cases, one can extend the original system with the equation

$$\dot{y} = \frac{\partial h}{\partial x}\frac{dx}{dt} = \frac{\partial h}{\partial x} f$$

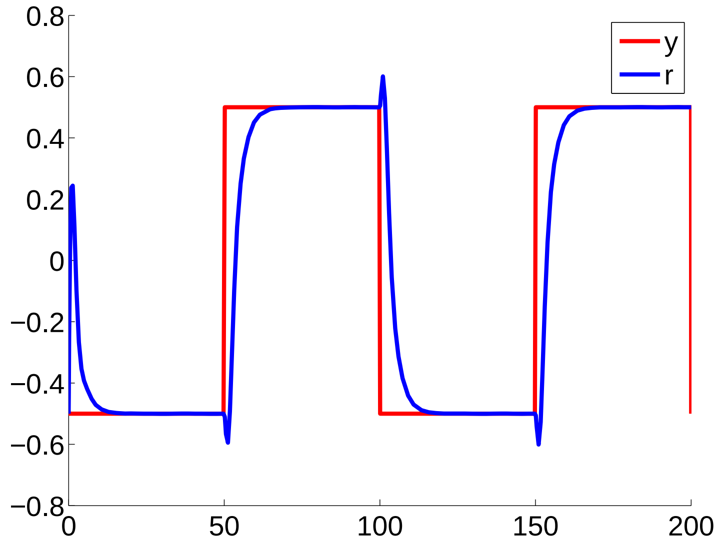


Figure 5.5: Response of integral control design

and apply the same procedures and ideas that have been discussed in this chapter to the new vector of measurements given by $[y; \dot{y}]$.

The result of applying this method to the previous example can be seen in Figure 5.6. In the first plot, we see the linearization failing once we move away from the linearisation point. In the second plot, we see the good effect of adapting the linearization point using gain scheduling, but we see large transients for some setpoints. The last plot shows that the large transients disappear when the derivative of the observation is added to the original system as new measurement.

For further reading on these design methods, see [4, Ch. 12].

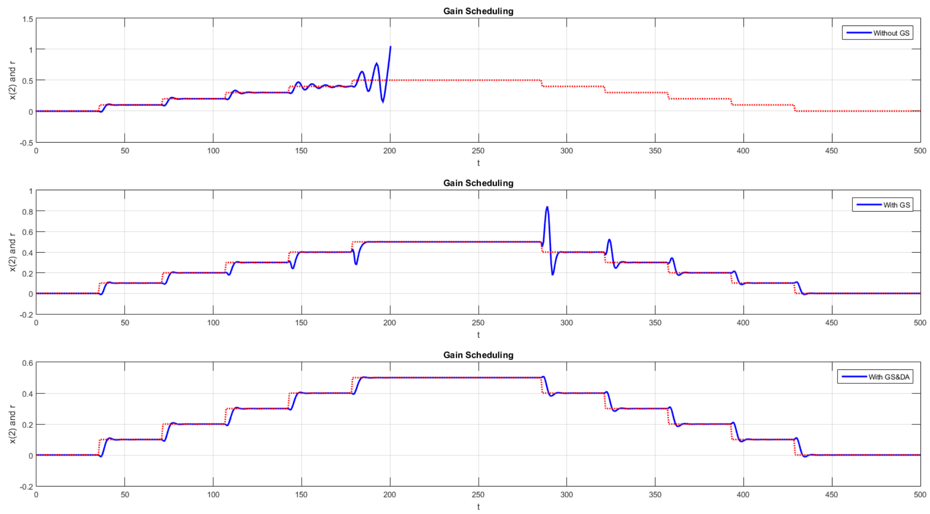


Figure 5.6: Response of integral control design with added derivative action

5.5 Exercises

1. A magnetic suspension system is modelled as

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= g - \frac{k}{m}x_2 - \frac{L_0 a x_3^2}{2m(a+x_1)^2} \\
\dot{x}_3 &= \frac{1}{L(x_1)}\left(-Rx_3 + \frac{L_0 a x_2 x_3}{(a+x_1)^2} + u\right)
\end{aligned}$$

where $x_1 = y$ is the elevation, $x_2 = \dot{y}$, $x_3 = I$ the current, and $u = V$ is the input voltage. Use the following numerical data $m = 0.1$ kg, $k = 0.001$ N/(m s), $g = 9.81$ m/s^2, $a = 0.05$ m, $L_0 = 0.01$ H, $L_1 = 0.02$ H, $R = 1\ \Omega$, and $L(x_1) = L_1 + \frac{L_0}{1+\frac{x_1}{a}}$.

   (a) Find the steady-state values I_{ss} and V_{ss} of I and V, respectively, which are needed to keep the ball at a given position $y = r$.

   (b) Is the equilibrium point obtained above stable ?

   (c) Linearize the system and design a state feedback control law to stabilize the ball at $y = 0.05$ m.

2. A simplified model of the low-frequency motion of a ship is given by

$$\tau\ddot{\psi} + \dot{\psi} = k\sigma$$

where ψ is the heading angle of the ship and σ is the rudder angle, viewed as the control input. The time constant τ and the gain k depends on the forward speed of the ship v, according to the expressions $\tau = \tau_0 \frac{v_0}{v}$ and $k = k_0 \frac{v}{v_0}$, where τ_0, k_0, and v_0 are constants. Assuming a constant forward speed, design a state feedback integral controller so that ψ tracks a desired angle ψ_r.

3. Suppose we have a controllable system $\dot{x} = Ax + Bu$. We now augment the system with an additional integral state $\dot{\sigma} = Cx \in \mathbb{R}^p$, p the number of inputs, and obtain the augmented system

$$\dot{x}_a = \mathcal{A}_a x_a + \mathcal{B}_a u$$

where $x_a = [x^T, \sigma^T]^T$ and

$$\mathcal{A}_a = \begin{pmatrix} A & 0 \\ C & 0 \end{pmatrix}, \quad \mathcal{B}_a = \begin{pmatrix} B \\ 0 \end{pmatrix}$$

Show that the augmented system is also controllable if and only if

$$\operatorname{rank} \begin{pmatrix} A & B \\ C & 0 \end{pmatrix} = n + p.$$

4. Consider the following non linear system for (x_1, x_2) and measurements y

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 \\ \dot{x}_2 &= -\beta \frac{x_2^3}{3} - \gamma x_2 + u \\ y &= x_1 \end{aligned}$$

with $\beta > 0$, $\gamma > 0$ parameters. We want to output y to track a reference signal r. Design, if possible, an integral controller that performes the task and is robust with respect to parameter variation.

Chapter 6

Feedback Linearization

6.1 Introduction

Consider a class of single-input-single-output (SISO) nonlinear systems of the form

$$\dot{x} = f(x) + g(x)u \tag{6.1}$$

$$y = h(x) \tag{6.2}$$

where $x \in \mathcal{D} \subset \mathbb{R}^n$, $u, y \in \mathbb{R}^1$, $f : \mathcal{D} \to \mathbb{R}^n$, $g : \mathcal{D} \to \mathbb{R}^n$, and the domain $\mathcal{D}$ contains the origin. [1]

In this lecture, we shall answer the following two (tightly related) questions:

1. Does there exist a nonlinear change of variables $z = \begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x)$, and a control input $u = \alpha(x) + \beta(x)v$ that would transform the system (6.1)-(6.2) into the following partially linear form?

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi} &= A\xi + Bv \\ y &= C\xi \end{aligned}$$

2. Does there exist a nonlinear change of variables $z = T(x)$ and a control input $u = \alpha(x) + \beta(x)v$ that would transform the system (6.1) into the

[1] This condition is not a limitation to the theory of feedback linearization and the results can be extended to the multiple-input-multiple-output (MIMO) case

following fully linear form?

$$\dot{z} = \tilde{A}z + \tilde{B}v$$

If the answer to question 2. is positive, then we say that the system (6.1) is feedback linearizable. If the answer to question 1. is positive, then we say that the system (6.1)-(6.2) is input-output linearizable. Both scenarios are very attractive from the control design point of view, since in either case we can rely on linear design techniques to render the closed-loop system stable.

Since we cannot expect every nonlinear system to possess such properties of feedback linearization, it is interesting to understand the structural properties that the nonlinear system should possess that render it feedback linearizable.

Before we dive into the theoretical developments of feedback linearization, let us first look into two simple examples.

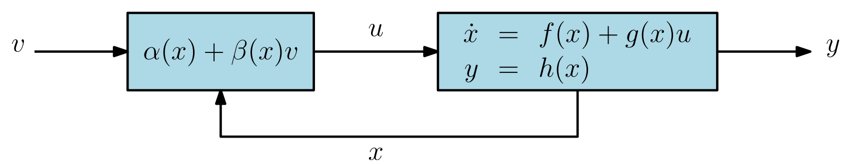


Figure 6.1: General idea of feedback linearization

Example 6.1 *Consider the example of the pendulum that we have seen in previous lectures. The dynamics are given by*

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -a\left[\sin\left(x_1 + \delta\right) - \sin\delta\right] - bx_2 + cu
\end{aligned}$$

If we choose the control

$$u = \frac{a}{c}\left[\sin\left(x_1 + \delta\right) - \sin\delta\right] + \frac{v}{c}$$

we can cancel the nonlinear term $a\left[\sin\left(x_1 + \delta\right) - \sin\delta\right]$*. The resulting linear system is given by*

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -bx_2 + v
\end{aligned}$$

As such, the stabilization problem for the nonlinear system has been reduced to a stabilization problem for a controllable linear system. We can proceed to design a stabilizing linear state feedback control

$$v = -k_1 x_1 - k_2 x_2$$

that renders the closed-loop system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_1 x_1 - (k_2 + b)\, x_2 \end{aligned}$$

asymptotically stable. The overall state feedback control law comprises linear and nonlinear parts

$$u = \left(\frac{a}{c}\right) [\sin(x_1 + \delta) - \sin\delta] - \frac{1}{c}(k_1 x_1 + k_2 x_2)$$

Example 6.2 *Consider the system*

$$\begin{aligned} \dot{x}_1 &= a \sin x_2 \\ \dot{x}_2 &= -x_1^2 + u \end{aligned}$$

We cannot simply choose u to cancel the nonlinear term $a \sin x_2$. However, let us first apply the following change of variables

$$\begin{aligned} z_1 &= x_1 \\ z_2 &= a \sin x_2 = \dot{x}_1 \end{aligned}$$

Then, the new variables z_1 and z_2 satisfy

$$\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= a \cos x_2 . \dot{x}_2 = a \cos\left(\sin^{-1} \frac{z_2}{a}\right)\left(u - z_1^2\right) \end{aligned}$$

and the nonlinearities can be canceled by the control

$$u = x_1^2 + \frac{1}{a \cos x_2} v$$

which is well defined for $-\frac{\pi}{2} < x_2 < \frac{\pi}{2}$. The state equation in the new coordinates can be found by inverting the transformation to express (x_1, x_2) in the terms of (z_1, z_2), that is,

$$\begin{aligned} x_1 &= z_1 \\ x_2 &= \sin^{-1}\left(\frac{z_2}{a}\right) \end{aligned}$$

which is well defined for $-a < z_2 < a$. *The transformed state equation is given by*

$$\dot{z}_1 = z_2$$
$$\dot{z}_2 = a\cos\left(\sin^{-1}\left(\frac{z_2}{a}\right)\right)\left(-z_1^2 + u\right)$$

which is in the required form to use state feedback. Finally, the control input that we would use is of the following form

$$u = x_1^2 + \frac{1}{a\cos x_2}(-k_1 z_1 - k_2 z_2) = x_1^2 + \frac{1}{a\cos x_2}(-k_1 x_1 - k_2 a\sin(x_2))$$

6.2 Input-Output Linearization

Consider the single-input-single-output system (6.1)-(6.2), where the vector fields $f : \mathcal{D} \to \mathbb{R}^n$ and $g : \mathcal{D} \to \mathbb{R}^n$ and the function $h : \mathcal{D} \to \mathbb{R}$ are sufficiently smooth.

Definition 6.3 *f is called smooth if $f \in C^\infty$, that is, f is continuous and all its derivatives of all orders are continuous.*

The goal is to derive conditions under which the input-output map can be rendered linear. *The idea is to take a sufficient number of derivatives of the output, until the input appears.*

We proceed by taking the derivative of y, which is given by

$$\dot{y} = \frac{\partial h}{\partial x}\left[f\left(x\right) + g\left(x\right)u\right] = L_f h\left(x\right) + L_g h\left(x\right)u$$

where $L_f h\left(x\right) \triangleq \frac{\partial h}{\partial x} f\left(x\right)$ is called the Lie Derivative of h with respect to (along) f. If $L_g h\left(x\right) = 0$, then $\dot{y} = L_f h\left(x\right)$, independent of u and we repeat the differentiation process again. Calculating the second derivative of y, denoted by $y^{(2)}$, we obtain

$$y^{(2)} = \frac{\partial\left(L_f h\right)}{\partial x}\left[f\left(x\right) + g\left(x\right)u\right] = L_f^2 h\left(x\right) + L_g L_f h\left(x\right)u$$

Once again, if $L_g L_f h\left(x\right) = 0$, then $y^{(2)} = L_f^2 h\left(x\right)$ is independent of u and we repeat the process. Actually, we repeat the processes of taking derivatives of the output until we see that $h\left(x\right)$ satisfies

$$L_g L_f^{i-1} h\left(x\right) = 0, i = 1, 2, ..., \rho - 1; \quad L_g L_f^{\rho-1} h\left(x\right) \neq 0$$

Therefore, u does not appear in the expressions of y, $\dot{y}$,...,$y^{(\rho-1)}$ and appears in the expression of $y^{(\rho)}$ with a nonzero coefficient, i.e.,

$$y^{(\rho)} = L_f^{\rho} h(x) + L_g L_f^{\rho-1} h(x)\, u \tag{6.3}$$

We can clearly see from (6.3), that the system is input-output linearizable, since the state feedback control

$$u = \frac{1}{L_g L_f^{\rho-1} h(x)} \left(-L_f^{\rho} h(x) + v \right) \tag{6.4}$$

reduces the input-output map (in some domain $\mathcal{D}_0 \subset \mathcal{D}$) to

$$y^{(\rho)} = v$$

which is a chain of ρ integrators. In this case, the integer ρ is called the ***relative degree*** of the system.

Definition 6.4 *The nonlinear system (6.1)-(6.2) is said to have a relative degree ρ, $1 \leq \rho \leq n$, in the region $\mathcal{D}_0 \subset \mathcal{D}$ if*

$$\begin{cases} L_g L_f^i h(x) = 0, \ i = 0, \cdots, \rho - 2 \\ L_g L_f^{\rho-1} h(x) \neq 0 \end{cases} \tag{6.5}$$

for all $x \in \mathcal{D}_0$.

Example 6.5 *Consider the controlled van der Pol equation*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \varepsilon \left(1 - x_1^2\right) x_2 + u \end{aligned}$$

with output $y = x_1$. Calculating the derivatives of the output, we obtain

$$\begin{aligned} \dot{y} &= \dot{x}_1 = x_2 \\ \ddot{y} &= \dot{x}_2 = -x_1 + \varepsilon (1 - x_1^2) x_2 + u \end{aligned}$$

Hence, the system has relative degree two in $\mathbb{R}^2$. For the output $y = x_1 + x_2^2$, we have that

$$\dot{y} = x_2 + 2x_2 \left[-x_1 + \varepsilon \left(1 - x_1^2\right) x_2 + u \right]$$

and the system has relative degree one in $\mathcal{D}_0 = \{x \in \mathbb{R}^2 | x_2 \neq 0\}$. As such, we can see that the procedure of input-output linearization is dependent on the choice of the output map $h(x)$.

Continuing with our derivations, we now let

$$z = T(x) = \begin{bmatrix} T_1(x) \\ --- \\ T_2(x) \end{bmatrix} \triangleq \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1}h(x) \end{bmatrix} \triangleq \begin{bmatrix} \phi(x) \\ --- \\ \psi(x) \end{bmatrix} \triangleq \begin{bmatrix} \eta \\ --- \\ \xi \end{bmatrix} \tag{6.6}$$

where $\phi_1(x)$ to $\phi_{n-\rho}(x)$ are chosen such that

$$\frac{\partial \phi_i}{\partial x} g(x) = 0, \quad 1 \leq i \leq n-\rho \tag{6.7}$$

which ensures that the η-dynamics

$$\dot{\eta} = \frac{\partial \phi}{\partial x}\left[f(x) + g(x)u\right] = \left.\frac{\partial \phi}{\partial x} f(x)\right|_{x=T^{-1}(z)}$$

are independent of u. Note also that our definition of $T_2(x)$ in (6.6) results in $\xi = \begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(\rho-1)} \end{bmatrix}$.

Of course, it is crucial to know at this point if such functions ϕ exist in order to define the transformation T; the next result shows that.

Definition 6.6 *A continuously differentiable transformation T with a continuously differential inverse is a called a diffeomorphism.*

Theorem 6.7 *Consider the system (6.1)-(6.2), and suppose that it has a relative degree $\rho \leq n$ in $\mathcal{D}$. If $\rho = n$, then for every $x_0 \in \mathcal{D}$, a neighborhood N of x_0 exists such that the map*

$$T(x) = \begin{bmatrix} h(x) \\ L_f h(x) \\ \vdots \\ L_f^{n-1}h(x) \end{bmatrix}$$

restricted to N is a diffeomorphism on N. If $\rho < n$, then for every $x_0 \in \mathcal{D}$, a neighborhood N of x_0 and smooth maps $\phi_1(x), \cdots, \phi_{n-\rho}(x)$ exist such that the map such that the map

$$T(x) = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_{n-\rho}(x) \\ --- \\ h(x) \\ \vdots \\ L_f^{\rho-1} h(x) \end{bmatrix}$$

restricted to N, is a diffeomorphism on N.

We can now apply the change of variables $z = T(x)$ to transform the system (6.1)-(6.2). [2] into

$$\dot{\eta} = f_0(\eta, \xi) \tag{6.8}$$

$$\dot{\xi} = A_c\xi + B_c\gamma(x)[u - \alpha(x)] \tag{6.9}$$

$$y = C_c\xi \tag{6.10}$$

where $\xi \in \mathbb{R}^\rho$, $\eta \in \mathbb{R}^{n-\rho}$, (A_C, B_C, C_C) is a canonical form representation of a chain of ρ integrators, i.e.,

$$A_c = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & & & 0 & 1 \\ 0 & \cdots & & 0 & 0 \end{bmatrix}, \ B_c = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, C_c = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$$

[2]The η-dynamics can be derived as follows

$$\begin{aligned} \dot{\eta} = \dot{\phi}(x) &= \frac{\partial \phi}{\partial x}\dot{x} = \frac{\partial \phi}{\partial x}(f(x) + g(x)u) \\ &= \frac{\partial \phi}{\partial x}f(x) = \frac{\partial \phi}{\partial x}f(x)\bigg|_{x=T^{-1}(x)} \triangleq f_0(\eta, \xi) \end{aligned}$$

The ξ-dynamics can be derived as follows

$$\begin{aligned} \dot{\xi} = \begin{bmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \vdots \\ \dot{\xi}_\rho \end{bmatrix} &= \frac{d}{dt}\begin{bmatrix} y \\ y^{(1)} \\ \vdots \\ y^{(\rho-1)} \end{bmatrix} = \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ \vdots \\ y^{(\rho)} \end{bmatrix} = \begin{bmatrix} \xi_2 \\ \vdots \\ \xi_\rho \\ y^{(\rho)} \end{bmatrix} \\ &= A_c\xi + B_c y^{(\rho)} = A_c\xi + B_c(L_gL_f^{\rho-1}h(x))\left[(u - \left(-\frac{L_f^\rho h(x)}{L_gL_f^{\rho-1}h(x)}\right)\right] \end{aligned}$$

we can now define

$$\gamma(x) \triangleq L_gL_f^{\rho-1}h(x), \quad \alpha(x) \triangleq -\frac{L_f^\rho h(x)}{L_gL_f^{\rho-1}h(x)}$$

Finally, our choice of the transformation $T_2(x)$ yields

$$y = h(x) = \xi_1 = C_c\xi$$

and

$$f_0(\eta, \xi) = \frac{\partial \phi}{\partial x} f(x) \bigg|_{x=T^{-1}(z)} \tag{6.11}$$

$$\gamma(x) = L_g L_f^{\rho-1} h(x) \tag{6.12}$$

$$\alpha(x) = -\frac{L_f^\rho h(x)}{L_g L_f^{\rho-1} h(x)} \tag{6.13}$$

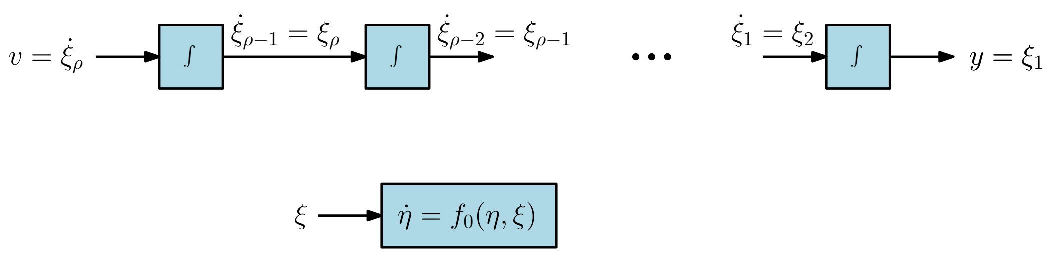


Figure 6.2: Transformed system (6.8)-(6.10) and the input defined as in (6.4)

We have kept α and γ expressed in the original coordinates. These two functions are uniquely determined in terms of f, g, and h and are independent of the choice of ϕ. They can be expressed in the new coordinates by setting

$$\alpha_0(\eta, \xi) = \alpha\left(T^{-1}(z)\right), \quad \gamma_0(\eta, \xi) = \gamma\left(T^{-1}(z)\right) \tag{6.14}$$

but now they are dependent on the choice of the functions ϕ. Regarding the definition of the equilibrium point for the transformed system, assume that $\bar{x}$ is the open-loop equilibrium of the system (6.1), then

$$\bar{\eta} = \phi(\bar{x}), \quad \bar{\xi} = \begin{bmatrix} h(\bar{x}) & 0 & \cdots & 0 \end{bmatrix}^T \tag{6.15}$$

The transformed system (6.8)-(6.10) is said to be in the normal form. This form decomposes the system into an external part ξ and an internal part η. The external part is linearized by the state feedback control

$$u = \alpha(x) + \beta(x) v \tag{6.16}$$

where $\beta(x) = \gamma^{-1}(x)$, while the internal part is made unobservable by the same control (see Figure 6.2). Setting $\xi = 0$ in the internal dynamics (6.8), results in

$$\dot{\eta} = f_0(\eta, 0) \tag{6.17}$$

which is called the ***zero dynamics*** of the system. If the zero dynamics of the system are (globally) asymptotically stable, the system is called ***minimum phase***.

Finally, the linearized system may then be stabilized by the choice of an appropriate state feedback:

$$v = -K\xi.$$

6.2.1 Relationship to Linear Systems

The notions of relative degree and minimum phase can also be found in linear systems. Consider a linear system represented by the transfer function

$$G(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + ... + b_0}{s^n + a_{n-1}s^{n-1} + ... + a_0} \tag{6.18}$$

where $m < n$ and $b_m \neq 0$. We can realize the transfer function in (6.18) in state-space form as

$$\dot{x} = Ax + Bu, \quad y = Cx \tag{6.19}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & & 0 \\ 0 & 0 & 1 & \cdots & & 0 \\ \vdots & \vdots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \\ 0 & & & & 0 & 1 \\ -a_0 & -a_1 & \cdots & & & -a_{n-1} \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$
$$C = \begin{bmatrix} b_0 & b_1 & \cdots & b_m & 0 & \cdots & 0 \end{bmatrix}$$

This linear space model is a special case of

$$\dot{x} = f(x) + g(x)u, \quad y = h(x)$$

where $f(x) = Ax$, $g = B$, and $h(x) = Cx$. We know that the relative degree of the transfer function is given by $n - m$. However, let us verify that this is the case. We take the derivative of the output, i.e.,

$$\dot{y} = C\dot{x} = CAx + CBu \tag{6.20}$$

Now, if $m = n-1$, then $CB = b_{n-1} \neq 0$ and the system has relative degree one. In general we have that

$$y^{(n-m)} = CA^{(n-m)}x + CA^{(n-m-1)}Bu \tag{6.21}$$

and we have the conditions that

$$CA^{(i-1)}B = 0, \forall i = 1, 2, \cdots, n-m-1, \text{ and } CA^{(n-m-1)}B = b_m \neq 0 \tag{6.22}$$

and the relative degree of the system is $n-m$, i.e., the difference between the order of the denominator and the numerator of the transfer function $G(s)$.

Moreover, we know that a minimum phase system means that the zeros of the transfer function $G(s)$ lie in the open left-half complex plane. We can show as well that there is a correspondence of the zeros of $H(s)$ and the normal form zero dynamics.

6.3 Full State Feedback Linearization

We know from the previous section, that the system (6.1) is feedback linearizable if we can find a sufficiently smooth function $h(x)$ in a domain $\mathcal{D}$ such that the system (6.1)-(6.2) has relative degree n within some region $\mathcal{D}_0 \subset \mathcal{D}$. This is because the normal form would have no zero dynamics. In fact, we can show that this result holds as an if and only if statement. As such, it remains to show that such a function $h(x)$ exists.

We begin with some definitions. For any two vector fields f and g on $\mathcal{D} \subset \mathbb{R}^n$, the **Lie bracket** $[f, g]$ is a third vector field that is defined as

$$[f, g](x) \triangleq \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) = L_f g(x) - L_g f(x) \tag{6.23}$$

Taking Lie brackets can be repeated, as such we define the following notation:

$$\begin{aligned} ad_f^0 g(x) &= g(x) \\ ad_f g(x) &= [f, g](x) \\ ad_f^k g(x) &= [f, ad_f^{k-1} g](x), \quad k \geq 1 \end{aligned}$$

Example 6.8 *Consider the two vector fields $f(x) = Ax$ and g is constant. Then, $ad_f g(x) = [f, g](x) = -Ag$, $ad_f^2 g(x) = [f, ad_f g](x) = -A(-Ag) = A^2 g$, and $ad_f^k g = (-1)^k A^k g$.*

Definition 6.9 (Distribution) *Consider the vector fields $f_1, \cdots, f_k$ on a domain $\mathcal{D} \subset \mathbb{R}^n$. The distribution, denoted by*

$$\Delta = \{f_1, f_2, \cdots, f_k\} \tag{6.24}$$

is the collection of all vectors spaces $\Delta(x)$ for $x \in \mathcal{D}$, where

$$\Delta(x) = span\{f_1(x), f_2(x), \cdots, f_k(x)\}$$

is the subspace of $\mathbb{R}^n$ spanned by the vectors $f_1(x), \cdots\ f_k(x)$.

Definition 6.10 *A distribution Δ is called involutive, if*

$$g_1 \in \Delta,\ \text{and}\ g_2 \in \Delta \Rightarrow [g_1, g_2] \in \Delta$$

And now we are ready to state the main result of this section.

Theorem 6.11 (Feedback Linearization) *The SISO system (6.1) is feedback linearizable if and only if*

1. *the matrix $\mathcal{M}(x) = [g(x), ad_f g(x), \cdots, ad_f^{n-1} g(x)]$ has full rank ($= n$) for all $x \in \mathcal{D}_0$, and*
2. *the distribution $\Delta = span\{g, ad_f g, \cdots, ad_f^{n-2} g\}$ is involutive on $\mathcal{D}_0$.*

Example 6.12 *Consider the system*

$$\dot{x} = f(x) + gu = \begin{bmatrix} a\sin(x_2) \\ -x_1^2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

We compute the Lie bracket

$$ad_f g = [f, g](x) = -\frac{\partial f}{\partial x} g = \begin{bmatrix} -a\cos(x_2) \\ 0 \end{bmatrix}$$

Accordingly, the matrix

$$\mathcal{M}(x) = [g, ad_f g] = \begin{bmatrix} 0 & -a\cos(x_2) \\ 1 & 0 \end{bmatrix}$$

has full rank for any x such that $\cos(x_2) \neq 0$. Moreover, the distribution $\Delta = span\{g\}$ is involutive. Therefore, the conditions of Theorem (6.11) are *satisfied and the system is feedback linearizable in the domain* $\mathcal{D}_0 = \{x \in \mathbb{R}^2 \mid \cos(x_2) \neq 0\}$. *Let us now find the function* $h(x)$ *for which the system is feedback linearizable.* $h(x)$ *should satisfy the following conditions*

$$\frac{\partial h}{\partial x} g = 0, \quad \frac{\partial (L_f h)}{\partial x} g \neq 0, \quad h(0) = 0$$

Now, $\frac{\partial h}{\partial x} g = \frac{\partial h}{\partial x_2} = 0$ *which implies that* $h(x)$ *is independent of* x_2. *Therefore,* $L_f h(x) = \frac{\partial h}{\partial x_1} a \sin(x_2)$. *The condition* $\frac{\partial (L_f h)}{\partial x} g = \frac{\partial (L_f h)}{\partial x_2} = \frac{\partial h}{\partial x_1} a \cos(x_2) \neq 0$ *is satisfied in* $\mathcal{D}_0$ *for any choice of* h *that satisfies* $\frac{\partial h}{\partial x_1} \neq 0$. *There are many such choices for* h, *for example,* $h(x) = x_1$ *or* $h(x) = x_1 + x_1^3$.

6.4 Stability

Consider again the input-output linearized system in normal form (6.8)-(6.10), and assume that we have designed the feedback control law

$$u = \alpha(x) + \beta(x) v$$

where $\beta(x) = \gamma^{-1}(x)$, and $v = -K\xi$, such that $(A_c - B_c K)$ is Hurwitz. The resulting system reduces to the following dynamics

$$\dot{\eta} = f_0(\eta, \xi) \tag{6.25}$$

$$\dot{\xi} = (A_c - B_c K)\xi \tag{6.26}$$

Theorem 6.13 *The origin of the system (6.25)-(6.25) is asymptotically stable, if the origin of the zero dynamics* $\dot{\eta} = f_0(\eta, 0)$ *is asymptotically stable (minimum phase).*

Proof: The idea is to construct a special Lyapunov function. Since the zero dynamics are asymptotically stable, there exists (by converse Lyapunov theorem) a continuously differentiable function $V_\eta(\eta)$ such that

$$\frac{\partial V_\eta}{\partial \eta} f(\eta, 0) < -W(\|\eta\|)$$

in some neighborhood of $\eta = 0$, where W is a strictly increasing continuous function with $W(0) = 0$. Let $P = P^T > 0$ be the solution of the Lyapunov equation.

$$P(A_c - B_c K) + (A_c - B_c K)^T P = -I$$

Consider the function

$$V(\eta,\xi) = V_\eta(\eta) + k\sqrt{\xi^T P\xi}, \quad k>0 \tag{6.27}$$

The derivative of V is given by

$$\begin{aligned}
\dot{V} &= \frac{\partial V_\eta}{\partial \eta} f_0(\eta,\xi) + \frac{k}{2\sqrt{\xi^T P\xi}}\xi^T \underbrace{\left[P(A_C - B_C K) + (A_C - B_C K)P\right]}_{-I}\xi \\
&= \frac{\partial V_\eta}{\partial \eta} f(\eta,0) + \frac{\partial V_\eta}{\partial \eta}\left(f_0(\eta,\xi) - f_0(\eta,0)\right) - \frac{k}{2\sqrt{\xi^T P\xi}}\xi^T\xi \\
&\leq -W(\|\xi\|) + k_1(\|\xi\|) - kk_2(\|\xi\|)
\end{aligned}$$

for $k_1, k_2 > 0$[3]. By choosing k large enough, we guarantee $\dot{V} < 0$, and the result follows.

□

Remark 6.14 *The result in the last theorem is local and does not extend to the global setup, even if the zero dynamics are globally asymptotically stable. In order to make the result global, we have to impose more restrictive requirements regarding the zero dynamics, namely the notion of input-to-state stability of the zero dynamics (which is beyond the current scope of the course).*

6.5 Robustness

Feedback linearization is based on exact cancellation of the nonlinearities in the system dynamics, which is practically very difficult to achieve. In a realistic setup, we would have only approximations $\hat{\alpha}$, $\hat{\beta}$ and $\hat{T}(x)$ of the true α, β, and $T(x)$. The feedback control law has then the form

$$u = \hat{\alpha}(x) + \hat{\beta}(x)v = \hat{\alpha}(x) - \hat{\beta}(x)K\xi = \hat{\alpha}(x) - \hat{\beta}(x)K\hat{T}_2(x)$$

Accordingly, the closed-loop system of the normal form is given by

$$\dot{\eta} = f_0(\eta,\xi) \tag{6.28}$$

$$\dot{\xi} = A\xi + B\gamma(x)\left[(\hat{\alpha}(x) - \alpha(x)) - \hat{\beta}(x)K\hat{T}_2(x)\right] \tag{6.29}$$

[3] $k_1 > 0$ follows from local continuity and differentiability of the function f_0, and $k_2 > 0$ follows from P

Adding and subtracting $BK\xi$ to the $\dot{\xi}$ equation we obtain

$$\dot{\eta} = f(\eta, \xi) \tag{6.30}$$

$$\dot{\xi} = (A - BK)\,\xi + B\delta(z) \tag{6.31}$$

where

$$\begin{aligned}\delta(z) = \gamma(x)[(\hat{\alpha}(x) - \alpha(x)) - (\hat{\beta}(x) - \beta(x))KT \\ - \hat{\beta}(x)K(\hat{T}_2(x) - T(x))]|_{x=T^{-1}(z)}\end{aligned}$$

Hence, the local closed loop system differs from the nominal one by an additive perturbation. Thus, we can show that locally the closed-loop system remains stable, despite small uncertainties in the feedback linearization maps.

For more details on feedback linearization concepts and their extensions, see [2].

6.6 Exercises

1. Consider the fourth order single input system

$$\dot{x} = f(x) + gu$$

with

$$f(x) = \begin{pmatrix} x_2 \\ -a \sin x_1 - b(x_1 - x_3) \\ x_4 \\ c(x_1 - x_3) \end{pmatrix}, \quad g = \begin{pmatrix} 0 \\ 0 \\ 0 \\ d \end{pmatrix}$$

where a, b, c and d are strictly positive constants. Give the transformation $T = [T_1, \ldots, T_4]$ that transforms the system into the standard form. Design a controller using feedback linearization to stabilize the origin.

2. Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1 + x_2 - x_3 \\ \dot{x}_2 &= -x_1 x_3 - x_2 + u \\ \dot{x}_3 &= -x_1 + u \\ y &= x_3 \end{aligned}$$

Is the system input-output linearizable? Is the system minimum phase?

3. Consider the following SISO nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 + \sin x_1 \\ \dot{x}_2 &= x_1^2 + \gamma u \\ y &= x_1 \end{aligned}$$

where γ is a given scalar. Design a continuous feedback controller via feedback linearization so that the output y tracks a given signal $v(t) = \sin t$.

Chapter 7

Sliding Mode Control

7.1 Introduction

Consider the second-order nonlinear system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= f(x) + g(x)u\end{aligned} \tag{7.1}$$

where $f(x)$ and $g(x) \geq g_0 > 0$ are unknown nonlinear functions and g_0 is a lower bound on the function g. The main goal is to design a feedback control law $u(x)$ that would guarantee stability in the presence of the uncertainty, i.e., a robustly stabilizing control law.

Had we known the nonlinearities exactly, the control design problem would be simple, i.e.,

$$u = \frac{1}{g(x)}(-f(x) - k_1 x_1 - k_2 x_2) \tag{7.2}$$

where any choice of the gains $k_1, k_2 > 0$ guarantees stability of the closed-loop system. However, since we do not know the nonlinear functions, our goal is to design a potentially switched controller that guarantees robust stability of the system.

For this reason, assume for a moment that we can directly manipulate x_2, i.e., use x_2 as an input to the first equation in (7.1), then we can pick $x_2 = -ax_1$ with $a > 0$, and make the first equation exponentially stable. This idea motivates us to define the so-called *sliding surface* or *sliding manifold*

$$\sigma = ax_1 + x_2 = 0 \tag{7.3}$$

As such, if we can control the system to the manifold σ and keep it there, the state trajectory would just asymptotically slide towards the origin. Differentiating the function σ along the trajectories of the system yields

$$\dot{\sigma} = a\dot{x}_1 + \dot{x}_2 = ax_2 + f(x) + g(x)u$$

Assume further that the unknown functions satisfy the following bound

$$\left|\frac{ax_2 + f(x)}{g(x)}\right| \leq \rho(x), \quad \forall x \in \mathbb{R}^2 \tag{7.4}$$

for some known function $\rho(x)$. Taking $V = \frac{1}{2}\sigma^2$ as a Lyapunov function candidate, we have that

$$\dot{V} = \sigma\dot{\sigma} = \sigma(ax_2 + f(x) + g(x)u) \leq g(x)\left[|\sigma|\rho(x) + \sigma u\right]$$

We can now design

$$u = -(\beta_0 + \rho(x))sgn(\sigma), \quad \beta_0 > 0, \quad sgn(\sigma) = \begin{cases} 1, & \sigma > 0 \\ -1, & \sigma < 0 \end{cases} \tag{7.5}$$

which yields

$$\dot{V} \leq -g_0\beta_0|\sigma| \tag{7.6}$$

We can now define the function $W(\sigma) \triangleq \sqrt{2V}$, which satisfies[1]

$$W(\sigma(t)) \leq W(\sigma(0)) - \beta_0 g_0 t \tag{7.7}$$

This shows that the trajectory of the closed-loop system (7.1) and (7.5) reaches the surface σ in finite time, and once on this surface, it cannot leave it, as seen by the inequality (7.6). The fact that the first phase is finished in finite time is very important in order to claim asymptotic stability (why?).

We can now categorize the behavior of the closed-loop system under sliding mode control into two phases:

Reaching Phase in which the designed input drives all trajectories that start off the surface σ towards σ in finite time and keeps them there

Sliding Phase in which the motion of the system is confined to the surface σ and can be described by some reduced order (stable) dynamics

Such a choice of surface σ is called a sliding surface and the corresponding input is called sliding mode control.

[1] The inequality (7.7) can be formally proven using the so-called comparison lemma and the notion of Dini derivatives, which are beyond the scope of the course.

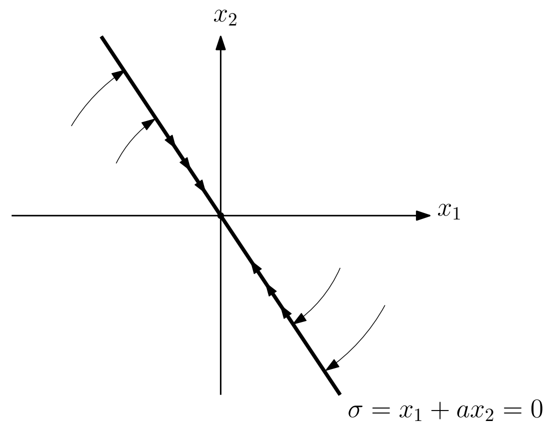


Figure 7.1: Illustration of the sliding mode control in the plane

7.2 Sliding Mode Control

Let us now consider the more general scenario of a nonlinear system given by

$$\dot{x} = f(x) + B(x)[G(x)E(x)u + \delta(x,u)] \tag{7.8}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input, f, B, G, and E are sufficiently smooth functions within some domain $\mathcal{D} \in \mathbb{R}^n$ that contains the origin. We assume that f, B, and E are known, while G and δ are uncertain, and that $\delta(x,u)$ is sufficiently smooth in both arguments for $(x,u) \in \mathcal{D} \times \mathbb{R}^m$. Assume further that $E(x)$ is invertible and that $G(x)$ is a diagonal matrix with all elements being bounded away from zero, i.e., $g_i(x) \geq g_0 > 0$, for all $x \in \mathcal{D}$. Let $f(0) = 0$, so that if $\delta = 0$ then the origin is an open-loop equilibrium point.

Remark 7.1 *The equation (7.8) shows that we have the scenario of matched uncertainty, i.e., the input u and the uncertainty δ affect the dynamics via the same $B(x)$ matrix.*

Let $T : \mathcal{D} \to \mathbb{R}^n$ be a diffeomorphism such that

$$\frac{\partial T}{\partial x} B(x) = \begin{bmatrix} 0 \\ I_{m \times m} \end{bmatrix} \tag{7.9}$$

The dynamics of the new variables defined by the transformation T, i.e.,

$z = \begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x) = \begin{bmatrix} T_1(x) \\ T_2(x) \end{bmatrix}$ are given by

$$\dot{\eta} = \frac{\partial T_1}{\partial x} f(x)\bigg|_{x=T^{-1}(z)} = f_a(\eta, \xi) \tag{7.10}$$

$$\dot{\xi} = \frac{\partial T_2}{\partial x} f(x)\bigg|_{x=T^{-1}(z)} + G(x)E(x)u + \delta(x,u) = f_b(\eta,\xi) + G(x)E(x)u + \delta(x,u) \tag{7.11}$$

Consider the sliding manifold

$$\sigma = \xi - \phi(\eta) = 0 \tag{7.12}$$

such that when we restrict the motion of the system to the manifold, the reduced order model

$$\dot{\eta} = f_a(\eta, \phi(\eta)) \tag{7.13}$$

is asymptotically stable to the origin. In order to solve for the function $\phi(\eta)$ we just need to look for a stabilizing feedback in which ξ is taken as the control input for the reduced dynamics (just as we have seen in the Introduction section). The choice of the design method is open, and we can use any of the design techniques we have learned in this course.

Assume that we have solved the stabilization problem for the system (7.13) and obtained a stabilizing continuously differentiable function $\phi(\eta)$, such that $\phi(0) = 0$. We now need to design the input u that achieves the reaching phase to the sliding manifold and maintains the trajectory on the sliding manifold. The derivative of σ is given by

$$\dot{\sigma} = \dot{\xi} - \frac{\partial \phi}{\partial \eta}\dot{\eta} = f_b(\eta,\xi) + G(x)E(x)u + \delta(x,u) - \frac{\partial \phi}{\partial \eta} f_a(\eta,\xi) \tag{7.14}$$

Let $\hat{G}(x)$ be the nominal model of $G(x)$ and consider the control input

$$u = E^{-1}(x)\left[-\hat{G}^{-1}(x)\left(f_b(\eta,\xi) - \frac{\partial \phi}{\partial \eta} f_a(\eta,\xi)\right) + v\right] \tag{7.15}$$

where v is a free input that we would like to design. Substituting (7.15) into (7.14), we obtain

$$\dot{\sigma} = G(x)v + \Delta(x,v) \tag{7.16}$$

where

$$\Delta(x,v) = (I - G(x)\hat{G}^{-1}(x))(f_b(\eta,\xi) - \frac{\partial\phi}{\partial\eta} f_a(\eta,\xi))$$
$$+ \delta\left(x, E^{-1}(x)\left[-\hat{G}^{-1}(x)\left(f_b(\eta,\xi) - \frac{\partial\phi}{\partial\eta} f_a(\eta,\xi)\right) + v\right]\right)$$

Assume that the ratio of the diagonal elements Δ_i/g_i satisfies

$$\left|\frac{\Delta_i(x,v)}{g_i(x)}\right| \le \rho(x) + \kappa_0 \|v\|_\infty, \quad \forall (x,v) \in \mathcal{D} \times \mathbb{R}^m, \quad \forall i \in \{1,2,\cdots,m\} \tag{7.17}$$

where $\rho(x) \ge 0$ is a known continuous function and $\kappa_0 \in [0,1)$ is a known constant.

Consider now the candidate Lyapunov function $V = \frac{1}{2}\sigma^T\sigma$, whose derivative can be computed as

$$\begin{aligned}
\dot{V} &= \sigma^T\dot{\sigma} = \sigma^T(G(x)v + \Delta(x,v)) \\
&= \sum_{i=1}^m (\sigma_i g_i(x) v_i + \sigma_i \Delta_i(x,u)) \le \sum_{i=1}^m g_i(x)\left(\sigma_i v_i + |\sigma_i|\left|\frac{\Delta_i(x,v)}{g_i(x)}\right|\right) \\
&\le \sum_{i=1}^m g_i(x)\left(\sigma_i v_i + |\sigma_i|\left(\rho(x) + \kappa_0\|v\|_\infty\right)\right)
\end{aligned}$$

Choose the extra inputs v_i as

$$v_i = -\beta(x) sgn(\sigma_i), \quad i \in \{1,2,\cdots,m\} \tag{7.18}$$
$$\beta(x) \ge \frac{\rho(x)}{1-\kappa_0} + \beta_0, \quad \forall x \in \mathcal{D} \tag{7.19}$$

where $\beta_0 > 0$. As such, we have that

$$\dot{V} \le \sum_{i=1}^m g_i(x)(-\beta(x) + \rho(x) + \kappa_0\beta(x))|\sigma_i| \le -\sum_{i=1}^m g_0\beta_0(1-\kappa_0)|\sigma_i|$$

which ensures that all the trajectories that do not start on σ converge to the manifold σ in finite time, and that those that start on σ stay on it.

Remark 7.2 *We have used the sgn function for the design of the input v, which poses theoretical questions in terms of the existence and uniqueness of* *solutions of the closed-loop system of ODEs. However, this difficulty can be alleviated via the using continuous approximations of the sgn function, for example,*

$$v_i = -\beta(x) sat\left(\sigma_i/\epsilon\right), \quad \forall i \in \{1, 2, \cdots, m\}, \text{ and } \epsilon > 0$$

The difference of these two choices is illustrated in Figure 7.2

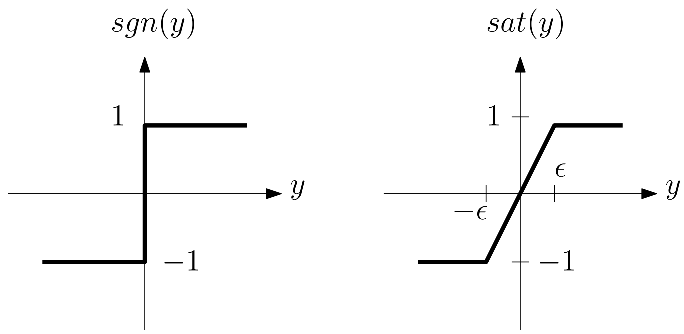


Figure 7.2: Illustrations of the *sgn* and *sat* functions

Remark 7.3 *In practice, we would use a digital implementation of the any control method. As such, it we may not be able to instantaneously switch the control input v. This results in the so-called chattering phenomenon. The effect of chattering may result in high heat losses and wearing out mechanical systems. This effect can be alleviated by introducing some time and/or space hysteresis bands that govern the switching commands whenever the trajectory is close to the sliding surfaces.*

Example 7.4 *Consider the second-order system*

$$\begin{aligned}\dot{x}_1 &= x_2 + p_1 x_1 \sin(x_2) \\ \dot{x}_2 &= p_2 x_2^2 + x_1 + u\end{aligned}$$

where the parameters p_1 and p_2 are unknown, but satisfy $|p_1| \leq a$ and $|p_2| \leq b$, where a and b are known. We take $\eta = x_1$ and $\xi = x_2$, and we would like to robustly stabilize the $\eta = x_1$ dynamics to the origin by using x_2 as an input. To this end, we design $x_2 = -kx_1$, where $k > a$, which yields

$$\dot{x}_1 = -(k - p_1 \sin(-kx_1))x_1$$

which is a robustly stable system for our choice of k. Accordingly, define the sliding manifold as

$$\sigma = x_2 + kx_1 = 0$$

The derivative of σ satisfies

$$\dot{\sigma} = \dot{x}_2 + k\dot{x}_1 = (x_1 + kx_2) + \big(p_2 x_2^2 + kp_1 x_1 \sin(x_2)\big) + u$$

We design the input u so as to cancel the known term, i.e.,

$$u = -(x_1 + kx_2) + v$$

For this choice of u, we have that

$$\dot{\sigma} = v + \Delta(x)$$

where the uncertainty $\Delta(x)$ satisfies the bound

$$|\Delta(x)| \leq bx_2^2 + ka|x_1| = \rho(x)$$

Consider the Lyapunov function candidate $V = \frac{1}{2}\sigma^2$, then

$$\begin{aligned} \dot{V} &= \sigma\dot{\sigma} = \sigma(v + \Delta(x)) \\ &\leq \sigma v + |\sigma|\rho(x) \end{aligned}$$

which proves finite time stability to the manifold σ, for any choice of $v = -\beta(x)sgn(s)$ (or some continuous approximation) with $\beta(x) = \rho(x) + \beta_0$, $\beta_0 > 0$. Finally, the sliding mode controller is given by

$$u = -(x_1 + kx_2) - (bx_2^2 + ka|x_1| + \beta_0)sgn(x_2 + kx_1)$$

7.3 Tracking

Consider the system, comprised of a chain of integrators (SISO)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_n &= f(x) + g(x)u + \delta(x, u) \\ y &= x_1 \end{aligned} \tag{7.20}$$

where $f(x)$ is a known function, while $g(x)$ and $\delta(x)$ are uncertain functions that satisfy, the first of which satisfies

$$g(x) \geq g_0 > 0$$

Remark 7.5 *The system* (7.20) *may be the result of an input output feedback linearization approach, i.e., the linearized dynamics in* ξ *that we have seen in the lecture on feedback linearization.*

We would like the output y to asymptotically track a given signal $r(t)$, where $r(t)$ and its derivatives $r^{(1)}(t), \cdots, r^{(n)}(t)$ are bounded for all t, $r^{(n)}(t)$ is piecewise continuous, and all these signals are available online.

Let $e = \begin{bmatrix} x_1 - r \\ x_2 - r^{(1)} \\ \vdots \\ x_n - r^{(n-1)} \end{bmatrix}$ be the error vector, then we have the following error dynamics

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ &\vdots \\ \dot{e}_{n-1} &= e_n \\ \dot{e}_n &= f(x) + g(x)u + \delta(x) - r^{(n)}(t) \end{aligned} \tag{7.21}$$

Let us now assume that we can directly design e_n to stabilize the subsystem

$$\begin{aligned} \dot{e}_1 &= e_2 \\ \dot{e}_2 &= e_3 \\ &\vdots \\ \dot{e}_{n-1} &= e_n \end{aligned} \tag{7.22}$$

In this case, we can take any feedback of the form

$$e_n = -(k_1 e_1 + \cdots + k_{n-1} e_{n-1})$$

where the coefficients are chosen so that the polynomial

$$s^{n-1} + k_{n-1} s^{n-2} + \cdots + k_1$$

is Hurwitz. Accordingly, we can now define the sliding manifold as

$$\sigma = k_1 e_1 + \cdots + k_{n-1} e_{n-1} + e_n = 0 \tag{7.23}$$

The dynamics of σ are given by

$$\dot{\sigma} = k_1 e_2 + \cdots + k_{n-1} e_n + f(x) + g(x)u + \delta(x) - r^{(n)}(t) \tag{7.24}$$

We can now design the input

$$u = -\frac{1}{\hat{g}(x)}\left(k_1 e_2 + \cdots + k_{n-1} e_n + f(x) - r^{(n)}(t)\right) + v$$

to yield

$$\dot{\sigma} = g(x)v + \Delta(t, x, v)$$

where

$$\begin{aligned}\Delta(t, x, v) = &\left(I - \frac{g(x)}{\hat{g}(x)}\right)\left(k_1 e_2 + \cdots + k_{n-1} e_n + f(x) - r^{(n)}(t)\right) \\ &+ \delta\left(x, -\frac{1}{\hat{g}(x)}\left(k_1 e_2 + \cdots + k_{n-1} e_n + f(x) - r^{(n)}(t)\right) + v\right)\end{aligned}$$

Assume that we have the following bound

$$\left|\frac{\Delta(t, x, v)}{g(x)}\right| \leq \rho(x) + \kappa_0 |v|, \quad \kappa_0 \in [0, 1)$$

then designing $v = -\beta(x) sgn(\sigma)$ with $\beta(x) \geq \frac{\rho(x)}{1-\kappa_0} + \beta_0$ and $\beta_0 > 0$ achieves asymptotic tracking. You can show this using exactly the same argument as in the previous section with a Lyapunov function candidate $V = \frac{1}{2}\sigma^2$.

7.4 Exercises

1. Given the system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^2 + \gamma u \\ y &= x_1 \end{aligned} \tag{7.25}$$

   with $\gamma \in [1, 3]$ uncertain parameter, design a Sliding Mode Control that tracks asymptotically a signal $r(t) = \sin t$.

2. The motion of a rigid n-link robot manipulator is described by the following equations:

$$H(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau \tag{7.26}$$

   where $q \in \mathbb{R}^n$ and $\dot{q} \in \mathbb{R}^n$ are the so-called generalized coordinates of the system, $\tau \in \mathbb{R}^n$ represents the input (torques or forces). The system above satisfies the following skew-symmetry properties:

   - The matrix $L = \dot{H}(q) - 2C(q, \dot{q})$ is skew-symmetric, i.e. $L^\top = -L$;
   - $H(q)^\top = H(q) \geq \lambda I$, $\forall q \in \mathbb{R}^n$ with $\lambda > 0$.

   Design a control τ that tracks the coordinates q_d and $\dot{q}_d$, with the following approaches:

   (a) Design a feedback linearization control, assuming that the matrices H, C and G are known exactly.

   (b) Assume now that H, G and C are uncertain, and we only know some nominal $\hat{H}$, $\hat{C}$ and $\hat{G}$. Design a SMC which accounts for this mismatch and asymptotically stabilizes the system.

   Hint: Take as sliding manifold $\sigma = \dot{q} - \dot{q}_d + K(q - q_d)$ with an accordingly defined K and leverage the positive definiteness of $H(q)$ by defining the Lyapunov function $V = \frac{1}{2}\sigma^\top H(q)\sigma$.

Chapter 8

Optimal Control

We consider first simple unconstrained optimization problems, and then demonstrate how this may be generalized to handle constrained optimization problems. With the observation that an optimal control problem is a form of constrained optimization problem, variational methods are used to derive an optimal controller, which embodies Pontryagin's Minimum Principle. We provide necessary and sufficient conditions for usage in the context of this important result.

8.1 Unconstrained Finite-Dimensional Optimization

Consider a function

$$L : \mathbb{R} \to \mathbb{R}$$

We want to find

$$\min_x L(x)$$

Let us assume that L is sufficiently smooth, and consider the Taylor expansion at the point x^*:

$$L(x) = L(x^*) + \frac{dL}{dx}\Big|_{x=x^*}(x - x^*) + \frac{1}{2}\frac{dL^2}{dx^2}\Big|_{x=x^*}(x - x^*)^2 + ..$$

In order for the point x^* to be a minima, we should have the following condition holding (at least locally)

$$L(x^*) \leq L(x), \quad \forall x : |x - x^*| < \epsilon$$

Then we have the following first-order necessary condition

$$\frac{dL}{dx}\bigg|_{x=x^*} = 0$$

and the second-order sufficient condition

$$\frac{dL^2}{dx^2}\bigg|_{x=x^*} > 0$$

for optimality of x^*. Note that these are only conditions for a local minimum. Additional conditions are required to find the global minimum if the function is non-convex. If we have a function with more than one variable, that is $L : \mathbb{R}^n \to \mathbb{R}$ we have the following conditions

$$\left(\tfrac{\partial L}{\partial x_1} \quad \tfrac{\partial L}{\partial x_2} \quad \cdots \quad \tfrac{\partial L}{\partial x_n}\right)\Big|_{x=x^*} = 0$$

and

$$\frac{\partial^2 L}{\partial x^2}\bigg|_{x=x^*} = \left(\begin{array}{ccc} \frac{\partial^2 L}{\partial x_1^2} & \cdots & \frac{\partial^2 L}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 L}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 L}{\partial x_n^2} \end{array}\right)\Bigg|_{x=x^*} > 0$$

i.e. is positive definite.

Example 8.1 *Find the minimum of the function*

$$L(x_1, x_2) = 6x_1^2 + 2x_2^2 + 3x_1x_2 - 8x_1 + 3x_2 - 4.$$

Necessary condition

$$\frac{\partial L}{\partial x}\bigg|_{x=x^*} = \left(\begin{array}{c} 12x_1 + 3x_2 - 8 \\ 4x_2 + 3x_1 + 3 \end{array}\right)\Bigg|_{x=x^*} = 0$$

Solving these equations we find $x^* = (\frac{41}{39}, -\frac{20}{13})$ and when we insert x^* into the Hessian Matrix we see that

$$\frac{\partial^2 L}{\partial x^2}\bigg|_{x=x^*} = \left(\begin{array}{cc} 12 & 3 \\ 3 & 4 \end{array}\right).$$

the resulting matrix is positive definite. We conclude that the point x^* is a minimum.

The plot in Figure 8.1 shows the function for which we are finding the minimum.

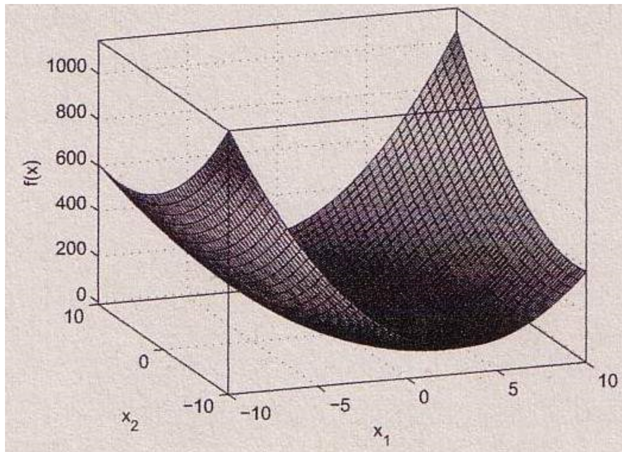


Figure 8.1: Convex Function

8.2 Constrained Finite-Dimensional Optimization

Theorem 8.2 *Consider the problem*

$$\begin{aligned} \min_x \quad & L(x) \\ s.t. \quad & f(x) = 0 \end{aligned} \tag{8.1}$$

with $L : \mathbb{R}^n \to \mathbb{R}$ and $f : \mathbb{R}^n \to \mathbb{R}^m$. Then this problem is equivalent[1] *to solving*

$$\min_{x,\lambda} \quad H(x,\lambda) := L(x) + \lambda^T f(x). \tag{8.2}$$

The function $H(x,\lambda)$ is called the **Lagrangian** of the problem and the coefficients $\lambda \in \mathbb{R}^m$ are called **Lagrangian multpliers**.

Sketch of Proof: We shall split the proof into several steps:

1. The equality constraints $f_1(x) = f_2(x) = \cdots = f_m(x) = 0$ define a feasible surface D in $\mathbb{R}^n$, i.e., if x^* is an optimum, it should belong to D.

[1] in the sense that first order necessary conditions are identical for both formulations

2. Assume that x^* is a regular point of D, i.e., the partial derivatives
$$\left\{ \frac{\partial f_1}{\partial x}\Big|_{x^*}, \cdots, \frac{\partial f_m}{\partial x}\Big|_{x^*} \right\}$$
are linearly independent.

3. Feasible directions/perturbations of the the optimal point x^* belong to the tangent space of the surface D at the point x^*, denoted by $T_{x^*}D$, i.e., we can have the first-order approximation of the cost
$$L(x) \approx L(x^*) + \frac{\partial L}{\partial x}\Big|_{x^*}(x - x^*)$$
perturbed using vectors $d = x - x^* \in T_{x^*}D$.

4. Necessary condition for optimality of the original problem can be stated as
$$\frac{\partial L}{\partial x}\Big|_{x^*} = 0, \quad \forall d \in T_{x^*}D$$
or equivalently
$$\frac{\partial L}{\partial x}\Big|_{x^*} d = 0, \quad \forall d : \frac{\partial f_1}{\partial x}\Big|_{x^*} d = 0, \forall i = 1, \cdots m$$
or equivalently
$$\frac{\partial L}{\partial x}\Big|_{x^*} \in span \left\{ \frac{\partial f_1}{\partial x}\Big|_{x^*}, \forall i = 1, \cdots m \right\}$$
or equivalently, $\exists \lambda_1, \cdots, \lambda_m \in \mathbb{R}$ such that
$$\boxed{\frac{\partial L}{\partial x}\Big|_{x^*} + \lambda_1 \frac{\partial f_1}{\partial x}\Big|_{x^*} + \cdots + \lambda_m \frac{\partial f_m}{\partial x}\Big|_{x^*} = 0}$$

Now, let's look at the necessary conditions for (8.2). Taking partial derivaties and setting them equal to zero, we get
$$\frac{\partial H}{\partial x}\Big|_{x^*} = 0 \Rightarrow \frac{\partial L}{\partial x}\Big|_{x^*} + \lambda^T \frac{\partial f}{\partial x}\Big|_{x^*}$$
which is the same necessary condition as the original problem. Moreover,
$$\frac{\partial H}{\partial \lambda}\Big|_{x^*} = 0 \Rightarrow f(x^*) = 0$$
which is the equality constraint of the original problem.

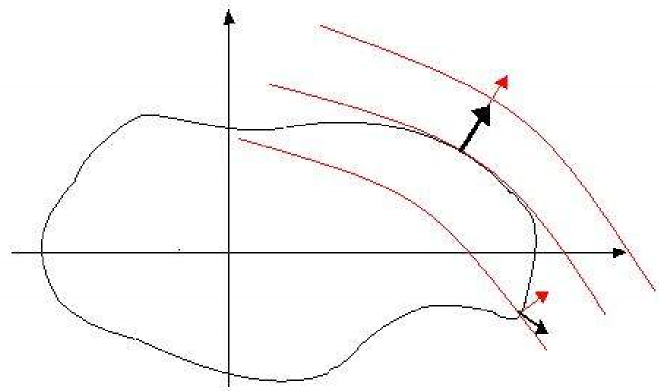


Figure 8.2: Gradients of L and f must be colinear at the extrema

$\square$

Example 8.3 *Find the minimum of the function*

$$L(x_1, x_2) = x_1^2 - 4x_1 + x_2^2$$

with the constraints

$$x_1 + x_2 = 0.$$

Indeed, the Hamiltonian is given by

$$H(x, \lambda) = x_1^2 - 4x_1 + x_2^2 + \lambda(x_1 + x_2).$$

Thus, the expressions for

$$\left.\frac{\partial H}{\partial x}\right|_{(x^*,\lambda^*)} = 0, \quad \left.\frac{\partial H}{\partial \lambda}\right|_{(x^*,\lambda^*)} = 0$$

are

$$\begin{aligned} 0 = \left.\frac{\partial H}{\partial x_1}\right|_{(x^*,\lambda^*)} &= 2x_1^* - 4 + \lambda^* \\ 0 = \left.\frac{\partial H}{\partial x_2}\right|_{(x^*,\lambda^*)} &= 2x_2^* + \lambda^* \\ 0 = \left.\frac{\partial H}{\partial \lambda}\right|_{(x^*,\lambda^*)} &= x_1^* + x_2^* \end{aligned}$$

Solving this system of linear equations we find the solution $(x_1^*, x_2^*, \lambda^*) = (1, -1, 2)$*. This solution is only a candidate for solution. One must now test whether these coordinates represented the solution we seek.*

Example 8.4 *Consider the following optimization problem*

$$\begin{aligned} \min_x \quad & L(x) = x_1^2 + x_2^2 + x_3^2 \\ s.t. \quad & f(x) = x_1x_2 + x_2x_3 + x_3x_1 - 1 = 0 \end{aligned}$$

The Lagrangian is given by

$$H(x, \lambda) = x_1^2 + x_2^2 + x_3^2 + \lambda(x_1x_2 + x_2x_3 + x_3x_1 - 1)$$

The necessary conditions are given by

$$\frac{\partial H}{\partial \lambda}\Big|_{x^*}^T = \begin{bmatrix} 2x_1^* + \lambda x_2^* + \lambda x_3^* \\ 2x_2^* + \lambda x_1^* + \lambda x_3^* \\ 2x_3^* + \lambda x_1^* + \lambda x_2^* \end{bmatrix} = \begin{bmatrix} 2 & \lambda & \lambda \\ \lambda & 2 & \lambda \\ \lambda & \lambda & 2 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The equalities above can be satisfied, if $x_1^* = x_2^* = x_3^3 = 0$ *OR if the matrix* $\begin{bmatrix} 2 & \lambda & \lambda \\ \lambda & 2 & \lambda \\ \lambda & \lambda & 2 \end{bmatrix}$ *has no trivial null space, i.e., if*

$$det\left(\begin{bmatrix} 2 & \lambda & \lambda \\ \lambda & 2 & \lambda \\ \lambda & \lambda & 2 \end{bmatrix}\right) = 0 \Rightarrow 2\lambda^3 - 6\lambda^2 + 8 = 0$$

which implies that the Lagrange multiplier can take values $\lambda^* = -1$ *or* $\lambda^* = 2$*.*

Let's analyse all these cases using the final conditon of the necessary conditions $\frac{\partial H}{\partial \lambda}\Big|_{x^*} = f(x^*) = 0$*.*

If $x_1^* = x_2^* = x_3^3 = 0$*, then the condition* $f(x^*) = 0$ *cannot be satisifed, therefore this is not an optimum point.*

If $\lambda = 2$*, then the null space is spanned by the following two vectors:* $\begin{bmatrix} 2 & -1 & -1 \end{bmatrix}$ *and* $\begin{bmatrix} 0 & 1 & -1 \end{bmatrix}$*, which means that we have the following two potential solutions: (*$2x_1^* = -x_2^*$*,* $x_2^* = x_3^*$*) OR (*$x_1^* = 0$*,* $x_2^* = -x_3^*$*), both of which do not yield a real-valued solution when substituted into the constraint* $f(x^*) = 0$*.*

If $\lambda = -1$*, then the null space is spanned by* $\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$*, which means that* $x_1^* = x_2^* = x_3^*$*, which when coupled with the constraint* $f(x^*) = 0$*, yields* $x_1^* = x_2^* = x_3^* = \sqrt{1/3}$ *which is a candidate for an optimum. We still need to check the sufficient conditions for this candidate solution to the problem.*

Finally, note that The second-order sufficient condtion for optimality of the problem (8.1) is given by

$$d^T H_{xx}|_{(x^*,\lambda^*)} d > 0, \quad \forall d \in T_{x^*} D$$

where $H_{xx}|_{(x^*,\lambda^*)} = \frac{\partial^2 L}{\partial x^2}\Big|_{x^*} + \sum_{i=1}^{m} \lambda_i \frac{\partial^2 f_1}{\partial x^2}\Big|_{x^*}$ or equivalently

$$d^T H_{xx}|_{(x^*,\lambda^*)} d > 0, \quad \forall d : \frac{\partial f_i}{\partial x}\Big|_{x^*} d = 0, \quad \forall i = 1, \cdots, m$$

8.3 Pontryagin's Minimum Principle

Consider the system

$$\dot{x} = f(x,u), \quad x(0) = x_0 \tag{8.3}$$

with associated performance index

$$J[x_0, u(\cdot)] = \phi(x(T)) + \int_0^T L(x(t), u(t))dt \tag{8.4}$$

and final state constraint

$$\psi(x(T)) = 0 \tag{8.5}$$

The following terminology is customary:

1. $J[x_0, u(\cdot)]$ is called cost function.
2. $\phi(x(T))$ is called end constraint penalty
3. $L(x,u)$ is called running cost
4. $H(x,u,\lambda) = L(x,u) + \lambda f(x,u)$ is called the **Hamiltonian**.

The ***Optimal Control Problem*** is: Find the control function

$$u : [0,T] \mapsto \mathbb{R}^m$$

whereas u is to sought in an appropriate function class, such that the performance index is minimized and the final state constraint and the system equations are satisfied.

Theorem 8.5 *Solutions of the Optimal Control Problem also solve the following set of differential equations:*

$$\begin{aligned}
\textit{State Equation:} \quad \dot{x} \quad &= \quad H_\lambda = f(x,u), && (8.6)\\
\textit{Co-State Equation:} \quad -\dot{\lambda} \quad &= \quad H_x = \frac{\partial L}{\partial x} + \lambda \frac{\partial f}{\partial x} && (8.7)\\
\textit{Optimality Condition:} \quad 0 \quad &= \quad H_u = \frac{\partial L}{\partial u} + \lambda \frac{\partial f}{\partial u} && (8.8)\\
\textit{State initial condition:} \quad x(0) \quad &= \quad x_0 && (8.9)\\
\textit{Co-state final condition:} \quad \lambda(T) \quad &= \quad (\phi_x + \psi_x \nu)|_{x(T)} && (8.10)
\end{aligned}$$

where ν is the Lagrange multiplier corresponding to end condition given by Equation 8.5.

Remark 8.6 *The expression regarding $u(.)$ in (8.8) is a special case of a more general minimum condition. In general we must seek*

$$u^* = \arg\min H(x,u,\lambda)$$

for the optimal values of $x(t)$ and $\lambda(t)$. In other words, the Pontryagins Minimum Principle states that the Hamiltonian is minimized over all admissible u for optimal values of the state and co-state.

Proof We use the Lagrange multipliers to eliminate the constraints. Since the main constraints are now given by a dynamical system $\dot{x} = f(x,u)$ it is intuitive clear that they must hold for all t and thus the vector function $\lambda : [0,T] \to \mathbb{R}^n$ is a function of time.

Using the notation $H(x,u,\lambda) := L(x,u) + \lambda f(x,u)$, the unconstrained minimization problem can be written as

$$\begin{aligned}
J(x,u,\lambda) \quad &= \quad \phi(x(T)) + \psi(x(T))\nu + \int_0^T [L(x,u) + \lambda(f(x,u) - \dot{x})]dt\\
&= \quad \phi(x(T)) + \psi(x(T))\nu + \int_0^T [H(x,u,\lambda) - \lambda\dot{x})]dt.
\end{aligned}$$

The differentials are written as

$$\begin{aligned}
\delta J = [\phi_x(x) + \psi_x(x)\nu]\delta x)|_{x=x(T)} + \\
\int_0^T [H_x \delta x + H_u \delta u + H_\lambda \delta\lambda - \lambda\delta\dot{x} - \delta\lambda\dot{x}]dt + \psi(x)|_{x(T)}\delta\nu
\end{aligned}$$

Note that by integrating by parts:

$$-\int_0^T \lambda\delta\dot{x} = -\lambda\delta x|_{t=T} + \lambda\delta x|_{t=0} + \int_0^T \dot{\lambda}\delta x dt$$

Furthermore, since $x(0) = x(0)$ is constant, it holds that $\lambda\delta x|_{t=0} = 0$. So we can rewrite the previous expression as

$$\delta J = [\phi_x(x) + \psi_x(x)\nu - \lambda]\delta x)|_{x=x(T)} + \int_0^T [(H_x + \dot{\lambda})\delta x + (H_\lambda - \dot{x})\delta\lambda + H_u\delta u]dt + \psi(x)|_{x(T)}\delta\nu$$

Now for the function $u : [0, T] \to \mathbb{R}^m$ to minimize the cost function, δJ must be zero for any value of the differentials. Thus, all the expressions before the differentials have to be zero for every $t \in [0, T]$. This observation gives the equations as required.

□

Remark 8.7 *Just as in the static optimization case, where the zeros of the derivatives represent candidates to be tested for extremum, the solutions of the system described in Theorem 8.5 are to be seen as candidates to be the optimal solution and their optimality must be tested for each particular case. In other words, the Pontriaguin Maximum Principle delivers necessary, but not sufficient, conditions for optimality.*

Remark 8.8 *The system of equations in Theorem 8.5 is a so called "Two Point Boundary Value" problem. generally speaking, these problems are only solvable by dedicated software implementing suitable numerical methods.*

Remark 8.9 *Special attention is needed in the case where $H_u = const$ for all u, in which case the optimal solution is found where the constraints are active. This sort of solution is often called "Bang-Bang solution" to acknowledge for the fact that they are often discontinuous and take values at the bounds.*

Example 8.10 *Consider the minimization problem*

$$J[x_0, u(\cdot)] = 0.5\int_0^T u^2(t)dt$$

subject to

$$\begin{aligned}\dot{x_1} &= x_2, \quad x_1(0) = x_{10} \\ \dot{x_2} &= u, \quad x_2(0) = x_{20}\end{aligned}$$

and with final constraint

$$\psi(x(T)) = x(T) = 0.$$

Applying Theorem 8.5, we transform the system into

$$\begin{aligned}\dot{x_1} &= H_{\lambda_1} = x_2; \quad x_1(0) = x_{10} \\ \dot{x_2} &= H_{\lambda_2} = u; \quad x_2(0) = x_{20} \\ \dot{\lambda}_1 &= -H_{x_1} = 0; \quad \lambda_1(T) = \nu_1 \\ \dot{\lambda}_2 &= -H_{x_2} = -\lambda_1; \quad \lambda_2(T) = \nu_2\end{aligned}$$

where

$$H(x, u, \lambda) = 0.5u^2 + \lambda_1 x_2 + \lambda_2 u$$

Now we see that

$$H_u = u + \lambda_2 = 0$$

Thus, we can solve the differential equations and see that

$$\begin{aligned}\lambda_1(t) &= \nu_1 \\ \lambda_2(t) &= -\nu_1(t - T) + \nu_2 \\ u(t) &= \nu_1(t - T) - \nu_2\end{aligned}$$

Placing these linear expressions in the dynamic equations for x and using the initial conditions $x(0) = x_0$, we obtain a linear system of equations with respect to (ν_1, ν_2), which gives us the final parametrization of the control law u. Figure 8.3 shows the result of applying this control law with $T = 15$.

Example 8.11 *Consider the same problem as in Example 8.10, but with a performance reflecting minimal time, i.e.*

$$J[x_0, u(\cdot)] = \int_0^T 1 \quad dt$$

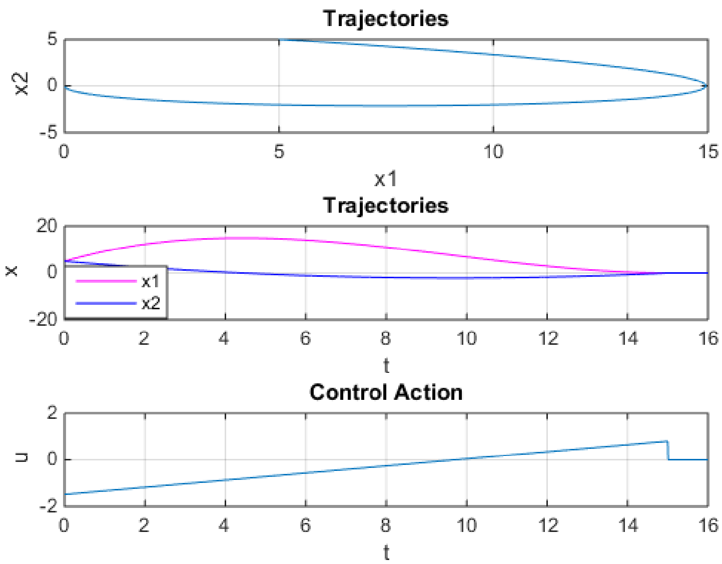


Figure 8.3: Trajectories for Minimal Energy Case. Arrival time $T = 15$.

and constrained input

$$-1 \leq u(t) \leq 1 \quad \forall t \geq 0.$$

Now

$$H(x, u, \lambda) = 1 + \lambda_1 x_2 + \lambda_2 u.$$

We notice that $H_u = \lambda_2$, i.e. H_u does not depend on u and thus the extremum is to be reached at the boundaries, i.e.

$$u^*(t) = \pm 1, \quad \forall t \geq 0.$$

Using

$$u^* = \arg\min_u H(u) = \arg\min_u \lambda_2(t) u(t)$$

we see that

$$u^*(t) = -sign \lambda_2(t).$$

Figure 8.4 depicts the result of deploying this control. Note that the control law is discontinuous. Further, we observe that the system now needs less time to reach the origin than in the previous "Minimal Energy" example.

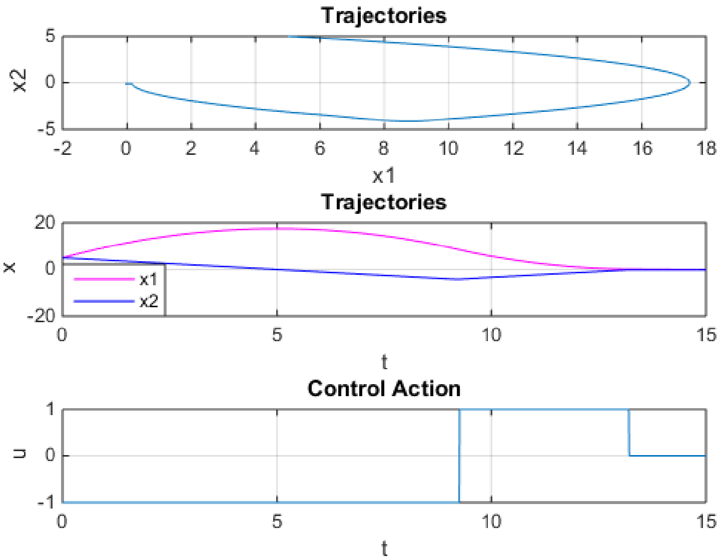


Figure 8.4: Trajectories for Minimal Time Case. Arrival time $T \approx 13$.

Of course, this "success" is obtained at the cost of deploying more actuator energy.

8.4 Sufficient Conditions for Optimality

The simplest sufficient condition for optimality is o course to establish uniqueness of the solution that appears solving the first order necessary condition. In applications it is often the case that this is possible by using application specific knowledge.

A more generic and well understood sufficient condition follows directly from the standard constrained optimization result and is can be stated as follows.

Theorem 8.12 (Sufficiency Conditions I) *Let us assume that* (x^*, λ^*, u^*) *fulfill the conditions in Theorem 8.5. Then, the condition*

$$\frac{\partial^2 H}{\partial u^2}(x^*, u^*, \lambda^*) > 0$$

is sufficient for the optimality of u^*.

Proof The results is proved by contradiction (i.e assuming that u^* might not be a local optimizer) and using the expression

$$\begin{aligned} H(x^*, u, \lambda^*) &\approx H(x^*, u^*, \lambda^*) + (u - u^*)\frac{\partial^2 H}{\partial u^2}(x^*, u^*, \lambda^*)(u - u^*) \\ &\geq H(x^*, u^*, \lambda^*) \end{aligned}$$

and this is a well known results from constrained optimization.

□

Further, useful results can be obtained using the concept of convexity.

Definition 8.13 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called convex when it fulfills the following property: for any given x and x^* we have*

$$f((1-t)x + tx^*) \leq f(x)(1-t) + tf(x^*), t \in [0, 1].$$

A smooth convex function f fulfills the following condition

$$f(x) \geq f(x^*) + \left.\frac{\partial f}{\partial x}\right|_{x=x^*}(x - x^*), \quad \forall x, x^* \in \mathbb{R}^n.$$

Before stating the next result we also note the following fact for the Hamiltonian $H = H(x, u, \lambda)$.

$$\frac{dH}{dx} = \frac{\partial H}{\partial x} + \frac{\partial H}{\partial u}\frac{\partial u}{\partial x}.$$

if u^* is an optimal control policy candidate, then $\frac{\partial H}{\partial u} = 0$ and we establish that

$$\frac{dH}{dx}(x, u^*, \lambda) = \frac{\partial H}{\partial x}(x, u^*, \lambda).$$

Theorem 8.14 (Sufficiency Conditions II) *Let $u^*(t)$, and the corresponding $x^*(t)$ and $\lambda^*(t)$ satisfy the minimum principle necessary condition for all $t \in [0, T]$. Then, u^* is an optimal control if $H(x, u^*, \lambda)$ is convex in x and $\phi(x)$ is convex in x.*

Proof We want to prove that

$$J(x_0, u) - J(x_0, u^*) \geq 0,$$

where x satisfies Equation 8.3. Now $u^* = \arg\min H$ and the convexity of H give us

$$\begin{aligned} H(x, u, \lambda) &\geq H(x, u^*, \lambda) \\ &\geq H(x^*, u^*, \lambda) + \left.\frac{\partial H}{\partial x}\right|_{x^*, u^*, \lambda} (x - x^*) \end{aligned}$$

Or equivalently

$$L(x, u) + \lambda f(x, u) \geq L(x^*, u^*) + \lambda f(x^*, u^*) - \dot{\lambda}(x - x^*)$$

or

$$\begin{aligned} L(x, u) - L(x^*, u^*) &\geq -\lambda f(x, u) + \lambda f(x^*, u^*) - \dot{\lambda}(x - x^*) \\ &= -\lambda[(\dot{x} - \dot{x}^*) - \dot{\lambda}(x - x^*) \\ &= \frac{d}{dt}[-\lambda(x - x^*)] \end{aligned}$$

Integrating, and using both the transversality condition and the fact that all trajectories have the same initial condition $x = x_0$, we observe that

$$\begin{aligned} \int_0^T [L(x, u) - L(x^*, u^*, \lambda)]dt &= -\lambda(T)(x(T) - x^*(T)) + \lambda(0))(x^*(0) - x(0)) \\ &= -\left.\frac{\partial \phi}{\partial x}\right|_{x^*} (x - x^*) \end{aligned}$$

Then, since ϕ is convex, we know that

$$\phi(x) - \phi(x^*) \geq \left.\frac{\partial \phi}{\partial x}\right|_{x = x^*} (x - x^*)$$

This implies

$$-\left.\frac{\partial \phi}{\partial x}\right|_{x = x^*} (x - x^*) = -\phi(x) + \phi(x^*) + \alpha, \quad \alpha \geq 0.$$

We now recall that

$$J[x_0, u(\cdot)] = \phi(x(T)) + \int_0^T L(x(t), u(t))dt.$$

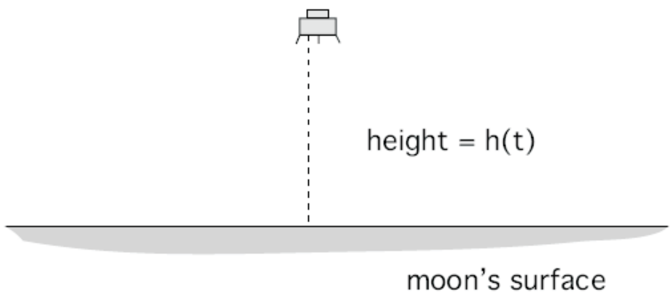
Thus, after some manipulations, we obtain

$$J(x_0, u) - J(x, u^*) = \alpha \geq 0,$$

which proves the optimality of u^*.

8.5 Exercises

1. Let us consider a spacecraft willing to land on the lunar surface. The quantities $h(t)$, $v(t)$, $m(t)$, and $u(t)$ represent the height over the surface, the velocity, the mass of the spacecraft and the vertical thrust (control action), respectively. A sketch is provided in the figure below.



The dynamics is described by

$$\dot{h} = v$$
$$\dot{v} = -g + \frac{u}{m}$$
$$\dot{m} = -ku$$

with k and g positive constants, and the control constrained to $0 \leq u(t) \leq 1$. The landing has to be soft, i.e. we want $h(t^*) = 0$, $v(t^*) = 0$ at the landing time t^*. The initial conditions $h(0)$, $v(0)$, $m(0)$ are fixed. Use Pontryagin's minimum principle to find a candidate control $u^\circ(t)$ to minimize the fuel consumption

$$J(u) = \int_0^{t^*} u(t)dt \,.$$

2. Let us consider a unidimensional optimal control problem with linear dynamics and quadratics costs

$$\min_{|u(t)|\leq 10} \int_0^1 \frac{x^2}{2} + \frac{u^2}{2}\, dt + \frac{x^2(1)}{2}$$
$$\text{s.t.} \quad \dot{x} = x + u, \quad x(0) = 1 \,.$$

Use the Pontryagin's minimum principle to find a candidate control $u^\circ(t)$. Is such candidate optimal? [*Hint: is the Hamiltonian convex?*]

3. This exercise is taylored to show that the Pontryagin's minimum principle only provides a necessary condition, in general cases when the Hamiltonian is non convex.
   Consider the following

$$\min_{u(t)\in[-1,1]} \int_0^1 -\frac{x^2}{2} dt$$
$$s.t. \quad \dot{x} = u, \quad x(0) = 0\,.$$

   Do $p^\circ(t) = 0$, $x^\circ(t) = 0$ and $u^\circ(t) = 0$ satisfy Pontryagin's minimum principle equations? Is $u^\circ(t) = 0$ optimal? If not, by just looking and the problem, can you come up with an optimal control $u^*(t)$?

Chapter 9

State-Dependent Riccati Equation

9.1 State-Independent Riccati Equation Method

9.1.1 Results on Control Design for LTI Systems

The Linear Quadratic Regulator (LQR)

$$\dot{x} = Ax + Bu \tag{9.1}$$

and the performance criteria

$$J[x_0, u(\cdot)] = \int_0^\infty [x^T Q x + u^T R u] dt, \dots Q \geq 0, R > 0 \tag{9.2}$$

Problem: Calculate function $u : [0, \infty] \mapsto \mathbb{R}^p$ such that $J[u]$ is minimized.

The LQR controller has the following form

$$u(t) = -R^{-1} B^T P x(t) \tag{9.3}$$

where $P \in \mathbb{R}^{n \times n}$ is given by the positive (symmetric) semi definite solution of

$$0 = PA + A^T P + Q - PBR^{-1} B^T P. \tag{9.4}$$

This equation is called Riccati equation. It is solvable if and only if the pair (A, B) is controllable and (Q, A) is detectable.

Note that

1. LQR assumes full knowledge of the state x.
2. The pair (A, B) is given by "the process dynamics" and can not be modified at this stage
3. The pair (Q, R) make the controller design parameters. Large Q penalizes transients of x, large R penalizes usage of control action u.

The Linear Quadratic Gaussian Regulator (LQG) In LQR we assumed that the whole state is available for control at all times (see formula for control action above). This is unrealistic as the very least there is always measurement noise.

One possible generalization is to look at

$$\begin{aligned} \dot{x} &= Ax + Bu + w \qquad (9.5) \\ y(t) &= Cx + v \qquad (9.6) \end{aligned}$$

where v, w are stochastic processes called measurement and process noise respectively. For simplicity in the explanation, we assume these processes to be white noise (ie zero mean, uncorrelated, Gaussian distribution).

Crucially, now only $y(t)$ is available for control. It turns out that for linear systems a separation principle holds

1. First, calculate $x_e(t)$ estimate the full state $x(t)$ using the available information
2. Secondly, apply the LQR controller, using the estimation $x_e(t)$ replacing the true (but unknown!) state $x(t)$.

Observer Design (Kalman Filter) The estimation $x_e(t)$ is calculated by integrating in real time the following ODE

$$\dot{x}_e = Ax_e + Bu + L(y - Cx_e) \qquad (9.7)$$

With the following matrices calculated offline:

$$\begin{aligned} L &= PC^T R_e^{-1} \qquad (9.8) \\ 0 &= AP + PA^T - PC^T R_e^{-1} CP + Q_e, P \geq 0 \qquad (9.9) \\ Q_e &= E(ww^T) \qquad (9.10) \\ R_e &= E(vv^T) \qquad (9.11) \end{aligned}$$

The Riccati equation above has its origin in the minimization of the cost functional

$$J[x_e(\cdot)] = \int_{-\infty}^{0} [(x_e - x)(x_e - x)^T]dt$$

9.2 State-Dependent Riccati Equation Approach

The LQR/LQG method is extremely powerful and widely used in applications, where linearisations of the nonlinear process representations are valid over large operating areas. How can this framework be extended beyond that obvious case?

The State-Dependent Riccati Equation (SDRE) strategy provides an effective algorithm for synthesizing nonlinear feedback controls by allowing for nonlinearities in the system states, while offering design flexibility through state-dependent weighting matrices.

The method entails factorization of the nonlinear dynamics into the state vector and its product with a matrix-valued function that depends on the state itself. In doing so, the SDRE algorithm brings the nonlinear system to a non-unique linear structure having matrices with state-dependent coefficients. The method includes minimizing a nonlinear performance index having a quadratic-like structure. An algebraic Riccati equation (ARE), as given by the SDC matrices, is then solved on-line to give the suboptimum control law.

The coefficients of this Riccati equation vary with the given point in state space. The algorithm thus involves solving, at a given point in the state space, an algebraic state-dependent Riccati equation. The non-uniqueness of the factorization creates extra degrees of freedom, which can be used to enhance controller performance.

Extended Linearization of a Nonlinear System Consider the deterministic, infinite-horizon nonlinear optimal regulation (stabilization) problem, where the system is full- state observable, autonomous, nonlinear in the state, and affine in the input, represented in the form

$$\dot{x} = f(x) + B(x)u(t) \tag{9.12}$$
$$x(0) = x_0 \tag{9.13}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input vector, with smooth functions f and B of appropriate domains such that

- $B(x) \neq 0$ for all x
- $f(0) = 0$.

Extended linearisation is the process of factorizing a nonlinear system into a linear-like structure which contains SDC matrices. Under the fairly generic assumptions

$$f(0) = 0, \quad f \in \mathbb{C}^1(\mathbb{R}^n)$$

a continuous nonlinear matrix-valued function

$$A : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$$

always exists such that

$$f(x) = A(x)x \tag{9.14}$$

where $A : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ is found by algebraic factorization and is clearly non-unique when $n > 1$.

After extended linearization of the input-affine nonlinear system (9.12) becomes

$$\begin{aligned} \dot{x} &= A(x)x + B(x)u(t) \\ x(0) &= x_0. \end{aligned}$$

which has a linear structure with *state dependent* matrices $A(x)$ and $B(x)$.

- Note that these parameterizations are not unique for $n > 1$. For instance, if $A(x)x = f(x)$, then $(A(x)+E(x))x = f(x)$ for any $E(x)$ such that $E(x)x = 0$.
- We also note that given $A_1(x)x = f(x)$ and $A_2(x)x = f(x)$, then for any $\alpha \in \mathbb{R}$

  $$A(x, \alpha)x = [\alpha A_1(x) + (1-\alpha)A_2(x)]x = \alpha f(x) + (1-\alpha)f(x) = f(x)$$

  is also a valid parametrization.

Remark 9.1 *In general, one needs to answer the question about the optimal choice of α for the given application at hand. In principle, α can also be made a function of x, delivering additional degree of freedom for design.*

Pointwise Hurwitz A Matrix does not imply stability We could naively think that if we find a controller $u = K(x)$ such that the closed loop matrix

$$A_{cl}(x) = A(x) - B(x)K(x)$$

is point-wise Hurtwitz then we would have a good design for stabilization of the nonlinear system given by (9.15). However, this condition is not sufficient for closed loop stability as the following example shows.

Example 9.2 *Let us consider the system in form of* (9.15)

$$\begin{pmatrix} \dot{x_1} \\ \dot{x_2} \end{pmatrix} = \begin{pmatrix} -\frac{1}{1+\epsilon\sqrt{x_1^2+x_2^2}} & 1+\sqrt{x_1^2+x_2^2}) \\ 0 & -\frac{1}{1+\epsilon\sqrt{x_1^2+x_2^2})} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \epsilon \geq 0.$$

We note that the eigenvalues of the matrix, given simply by the matrix components a_{11} and a_{22} are negative for any value of x. Moreover, we also notice that for $\epsilon = 0$ the eigenvalues are both equal to -1, with the effect that the state x_2 will converge to zero rapidly. This will make the term a_{12} also small, and eventually lead to x_1 to also converge to zero, see Figure 9.1.

However, if $\epsilon = 1$ then at least for some initial conditions, x_1 may grow fast and make the term a_{22} converge to zero, which will lead to x_2 becoming a constant. Since also $a_{11} = a_{22}$ will be small, the term a_{12} will be dominant in the dynamics of x_1 and will lead x_1 to grow to infinite, see Figure 9.2.

SDRE Method Formulation For the system

$$\dot{x} = A(x)x + B(x)u(t) \tag{9.15}$$

$$x(0) = x_0. \tag{9.16}$$

We shall look at minimization of the infinite-time performance criterion

$$J[x_0, u(\cdot)] = \int_0^\infty [x^T Q(x)x + u^T R(x)u]dt, \ldots Q(x) \geq 0, R(x) > 0. \tag{9.17}$$

Note, that the state and input weighting matrices are also assumed *state dependent.*

Under the specified conditions, we apply point-wise the LQR method for $(A(x), B(x))$,$(Q(x), R(x))$ generating a control law

$$u(t) = -K(x)x = R(x)^{-1}B(x^T)P(x)x(t), \quad K : \mathbb{R}^n \mapsto \mathbb{R}^{p\times n}$$

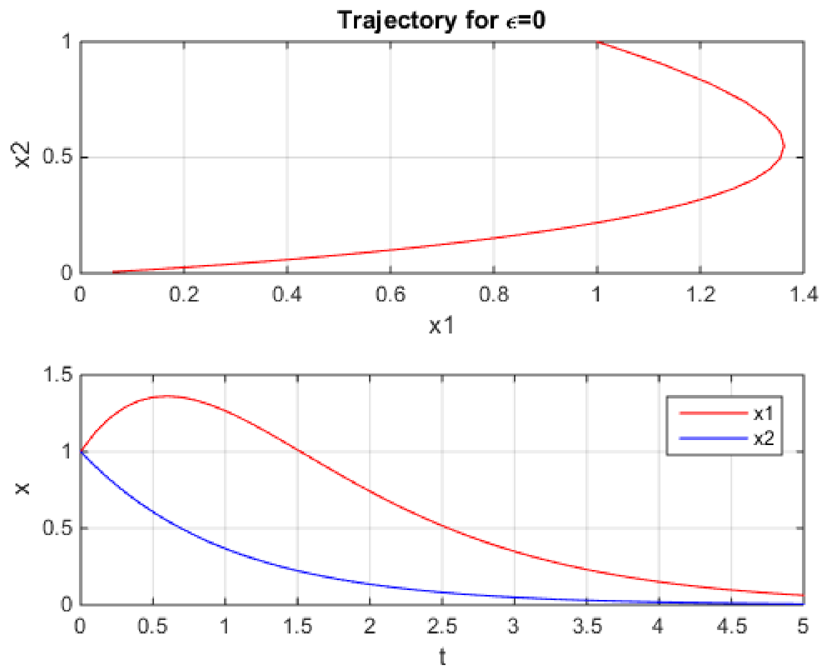


Figure 9.1: System behavior with $\epsilon = 0$

where $P : \mathbb{R}^n \mapsto \mathbb{R}^{n\times n}$ satisfies

$$P(x)A(x) + A(x)^T P(x) - P(x)B(x)R(x)^{-1}B(x)^T P(x) + Q(x) = 0.$$

By applying this method one expects to obtain the good properties of the LQR control design, namely, that

- the control law minimizes the cost given by (9.17)
- regulates the system to the origin $\lim_{t\to\infty} x(t) = 0$.

Main Stability Results Below we establish how close we are to that "wish". Indeed, we have the following theorem.

Theorem 9.3 *(Mracek & Cloutier, 1998) Let us assume the following conditions hold.*

1. *The matrix valued functions* $A(x), B(x), Q(x), R(x) \in C^1(\mathbb{R}^n)$.

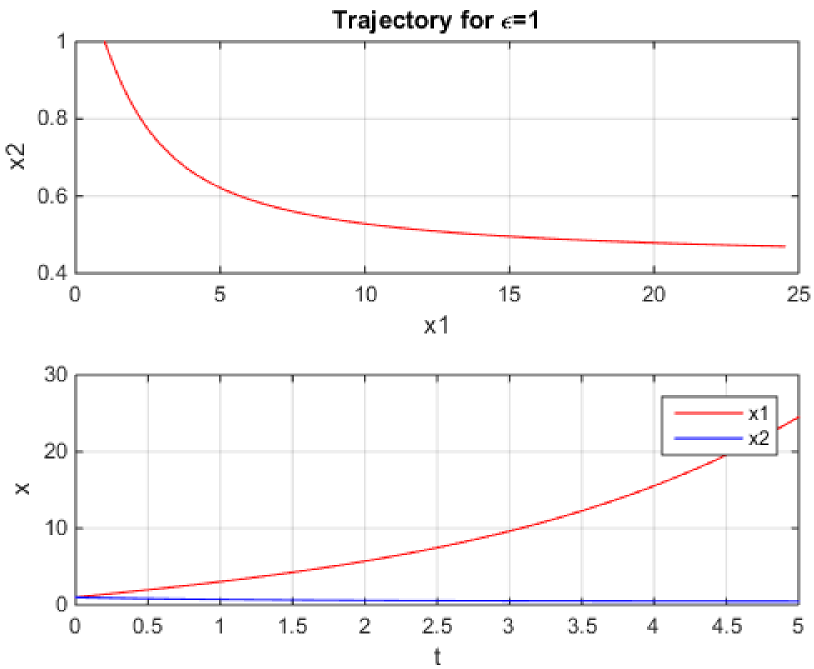


Figure 9.2: System behavior with $\epsilon = 1$

2. *The pairs $(A(x), B(x))$ and $(A(x), Q^{\frac{1}{2}}(x))$ are pointwise stabilizable, respectively, detectable, state dependent parameterizations of the nonlinear system for all $x \in \mathbb{R}^n$.*

Then, the nonlinear multivariable system given by (9.15) is rendered locally asymptotically stable by the control law

$$u(x) = -K(x)x = R^{-1}(x)B^T(x)P(x)x, \quad K : \mathbb{R}^n \mapsto \mathbb{R}^{p\times n} \tag{9.18}$$

where $P(x) \in \mathbb{R}^{n\times n}$ is the unique, symmetric, positive-definite solution of the algebraic State-Dependent Riccati Equation

$$P(x)A(x) + A^T(x)P(x) - P(x)B(x)R^{-1}(x)B^T(x)P(x) + Q(x) = 0. \tag{9.19}$$

Proof: The dynamics is given by the pointwise Hurwitz matrix

$$\dot{x} = [A(x) - B(x)R^{-1}(x)B^T(x)P(x)]x = A_{cl}(x)x.$$

Under the assumptions of the theorem one can show that

$$A_{cl}(x) = A_{cl}(0) + \phi(x), \quad \lim_{x \to 0} \phi(x) = 0.$$

which means that the linear term is dominant near the origin. □

Remark 9.4 *Note that global stability has not been established, this is a local result. In general, as we saw in Example 9.2, even when*

$$A_{cl}(x) = A(x) - B(x)K(x)$$

is Hurtwitz for all x, global stability can not be guaranteed.

Remark 9.5 *One can prove though that if $A_{cl}(x)$ Hurtwitz and* symmetric *for all x, then global stability holds. The proof is simply obtained by showing that under these conditions $V(x) = x^T x$ is a Lyapunov function for system (9.15).*

Theorem 9.6 *Under conditions of Theorem 9.3, the SDRE nonlinear feedback solution and its associated state and costate trajectories satisfy the first necessary condition for optimality of the nonlinear optimal regulator problem*

$$u(x) = \arg\min H(x, \lambda, u), \quad \lambda = P(x)x$$

where

$$H(x, \lambda, u) = 0.5x^T Q(x)x + 0.5u^T R(x)u + \lambda^T [A(x)x + B(x)u]$$

Proof: The Hamiltonian of this problem is given by

$$H(x, \lambda, u) = 0.5x^T Q(x)x + 0.5u^T R(x)u + \lambda^T [A(x)x + B(x)u].$$

Thus, it is clear that

$$H_u = R(x)u + B^T(x)\lambda$$

This implies that for any choice of λ

$$u(x) = -R^{-1}(x)B^T(x)\lambda \implies H_u = 0.$$

In particular, the choice $\lambda = P(x)x$ renders a controller given by (9.18).

□

When using the SDRE method one observes that as the state converges to zero, the solution also converges to the optimal solution given by the Pontriaguin Maximum Principle. This observation is supported by the following result.

Theorem 9.7 *Assume that all state dependent matrices are bounded functions along the trajectories. Then, under the conditions of Theorem 9.3 the Pontriaguin second necessary condition for optimality.*

$$\dot{\lambda} = -H_x(x, \lambda, u)$$

is satisfied approximately by

$$\lambda = P(x)x$$

at a quadratic rate along the trajectory. Here $P(x)$ denotes the solution of Equation 9.19.

Proof: The proof of this statement regarding $\lambda(\cdot)$ is quite technical and is obtained as follows. First, we observe

$$\lambda = P(x)x \implies \dot{\lambda} = \dot{P}(x)x + P(x)\dot{x}.$$

Thus, we need it to hold:

$$-H_x(x, \lambda, u) \approx \dot{P}(x)x + P(x)\dot{x}.$$

Equating these expressions, grouping linear and quadratic terms and using (9.19), one obtains that the residuals are all quadratic functions of x. Theorem 9.3 proves that these quadratic terms in x decay quadratically as x converges to the origin.

$\square$

Example 9.8 . *Steer to $x = (d, 0)$ the following system.*

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a\sin(x_1) - bx_2 + cu(t). \end{aligned}$$

Indeed,

$$A(x) = \begin{pmatrix} 0 & 1 \\ -a\sin(x_1 - d)/(x_1 - d) & -b \end{pmatrix} \quad B(x) = \begin{pmatrix} 0 \\ c \end{pmatrix}$$

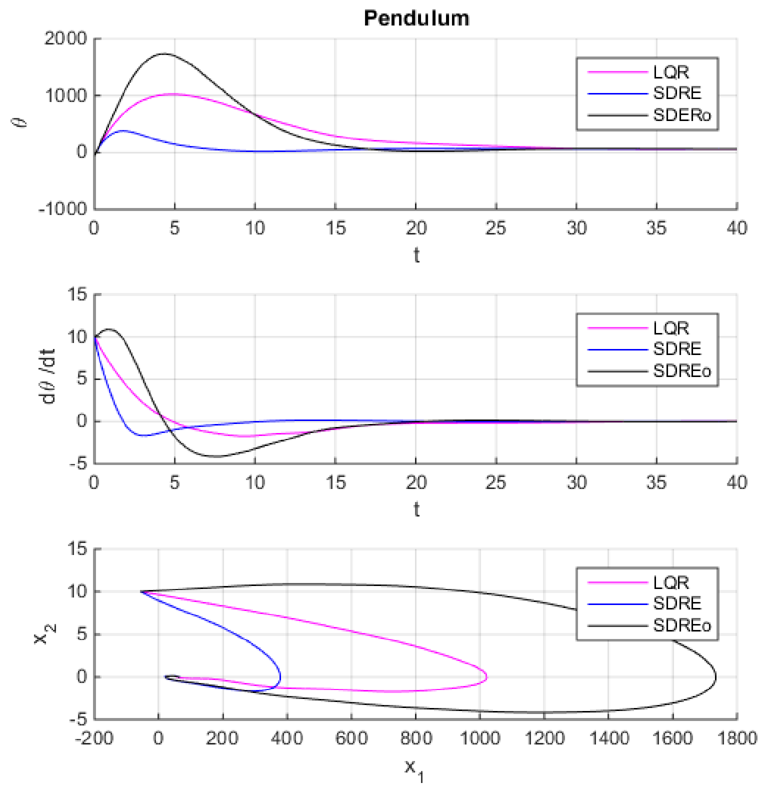


Figure 9.3: SDRE versus LQR

We choose

$$Q(x) = \begin{pmatrix} 1 + (x_1 - d)^2 & 0 \\ 0 & 1 + x_2^2 \end{pmatrix}, \quad R = 1.$$

The choice of $Q(x)$ ensures larger control actions for large deviations from the equilibrium.

The magenta trajectory in Figure 9.3 is obtained using LQR on the standard linearization of the original system with

$$Q(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R = 1.$$

Note how the state is brought faster to the equilibrium in the SDRE case.

Output Feedback and SDRE Approach One can extend these same ideas to the case where the state is not available for control. As expected the method reduces to use the matrix functions $A(\cdot), B(\cdot), C(\cdot)$ to calculate pointwise the corresponding Kalman gain L described in (9.8). Clearly, since x is not available, one must evaluate the matrix functions at the current estimate x_e, which is calculated following (9.7).

In Figure 9.3 the line in black depicts the result of this strategy for the previous example, but using only the position x_1 for calculation the control. One observes that the controller is able to bring the system to the desired position.

We also observe though that the transient is now much longer. The length of this transient being dependent, among other issues, on the quality of the initial estimate and on the measurement noise. Figure 9.4 shows the transients of x and x_e. Note that the controller has only x_e to plan its actions.

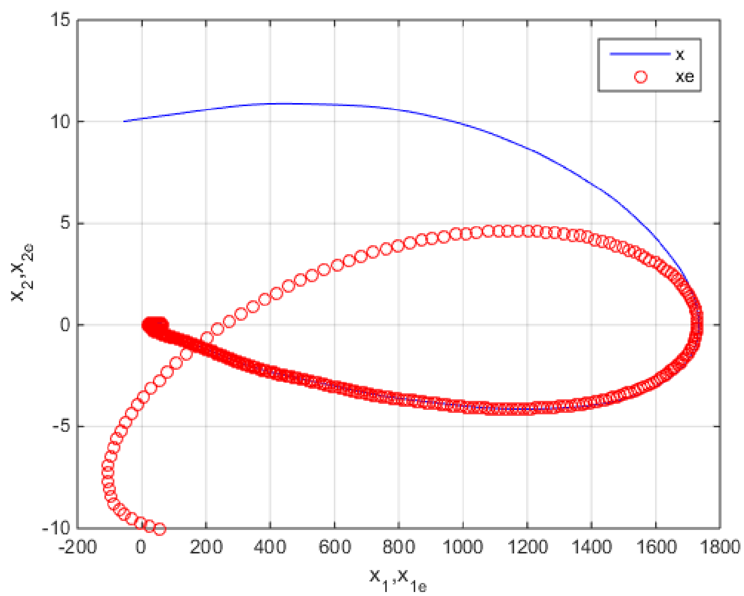


Figure 9.4: Estimate and true state using the SDRE approach

9.2.1 H_∞ control for nonlinear systems

H_∞ control is a systematic method for generating robust controllers for linear systems that implies designing a controller that minimizes the effect on a performance related variable $z = C_z x + D_u u$ of the worst possible disturbance w, see Figure 9.5. In complete analogy to LQR and LQG controllers, the SDRE method is similarly deployed to implement H_∞ controllers for nonlinear systems.

H_∞ control creates a beautiful mathematical apparat related to game theory that unfortunately we will not touch further due to time and space constraints. The interested reader can refer to [1, 3, 5] for further information.

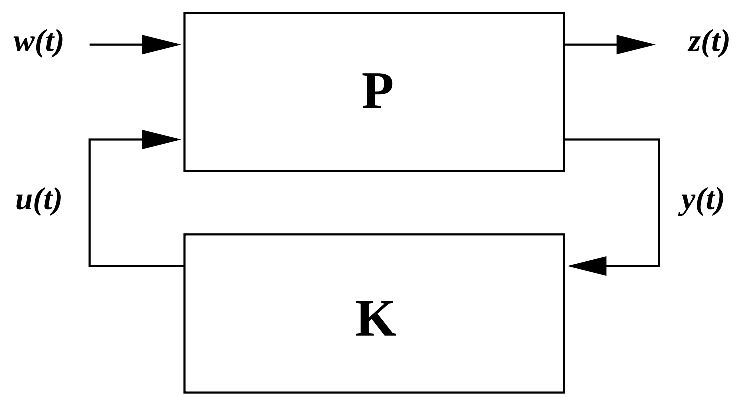


Figure 9.5: H_∞ control components

9.3 Exercises

1. Verify that the Hamiltonian-Jacobi equation for a linear system with quadratic costs reduces to the Riccati equation.

2. Consider the controlled dynamical system

$$\dot{x} = f(x) + b(x)u\,,$$

with $x,\ u \in \mathbb{R}$, $f(x) = x - x^3$ and $b(x) = 1\,.$

   (a) Design, if possible, a SDRE controller $u_s(x)$ to minimize

$$J = \int_0^\infty x^2(s) + u^2(s)ds$$

   (b) Study the stability of the closed loop system. Is there any a priori guarantee you can exploit?

   (c) Recall that the previous optimal control problem can be solved looking at the infinite horizon Hamilton-Jacobi equation. Write the equation and find an explicit solution $u_o(x)$ for the controller.

   (d) Are $u_s(x)$ and $u_o(x)$ the same or different? Did you expect this? Motivate your answer.

3. Given the following system

$$\dot{x} = x^2 + u$$

   (a) Determine its unique factorization.

   (b) Design, if possible, a SDRE controller $u_s(x)$ to minimize $J = \int_0^\infty x^2(s) + u^2(s)ds$, with $Q(x) = 1$ and $R(x) = 1$.

   (c) Check that

$$\dot{\lambda} = H_u(x, \lambda, u)$$

   is satisfied approximately by $\lambda = P(x)x$.

Chapter 10

Nonlinear Model Predictive Control

10.1 Introduction

In the first section we looked at the following simple system.

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u\\ y &= x_1\end{aligned}$$

The goal is keep the output at a given setpoint y_{sp} using the control action $u(\cdot)$, which is bounded $-M \leq u \leq M$. Figure 10.1 shows the result of designing a control using LQR plus saturation at M, i.e.

$$u = \mathrm{sat}(-Kx, M)$$

We observe that as the the value of M decreases this simple strategy fails to keep the system stable. The reason is that although the controller does know about the system dynamics, the controller design ignores the constraint, being over optimistic about its ability to slow down the system once the speed is high.

Intuitively, it is clear that better results would be achieved if we made the controller see not only the dynamics but also the barrier. Model Predictive Control (MPC) offers that framework. Indeed, Figure 10.2 shows the result of applying MPC to this problem. We observe that stability is not lost as M decreases. Model Predictive Control has largely conquered industrial applications by means of being both systematic and intuitive.

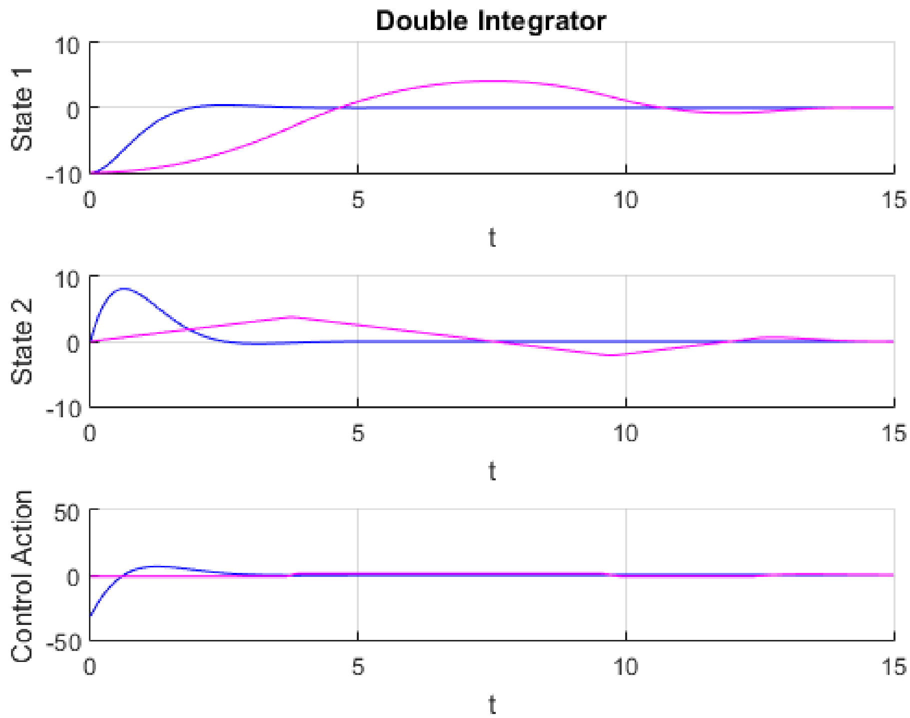


Figure 10.1: LQR plus saturation for double integrator

10.2 Main Theoretical Elements

Model predictive control is nowadays probably the most popular method for handling disturbances and forecast changes. The main ingredients of the method are

1. Plant model
$$\dot{x} = f(t, x, u), \quad x(0) = x_0$$

2. Constraints
$$\begin{aligned} g(t, x, u) &= 0 \\ g_i(t, x, u) &\leq 0 \end{aligned}$$

3. Objective function
$$J[x_0, u(\cdot)] = \int_t^{t+T} f_0(\tau, x(\tau), u(\tau))d\tau + F(x(t+T), u(t+T))$$

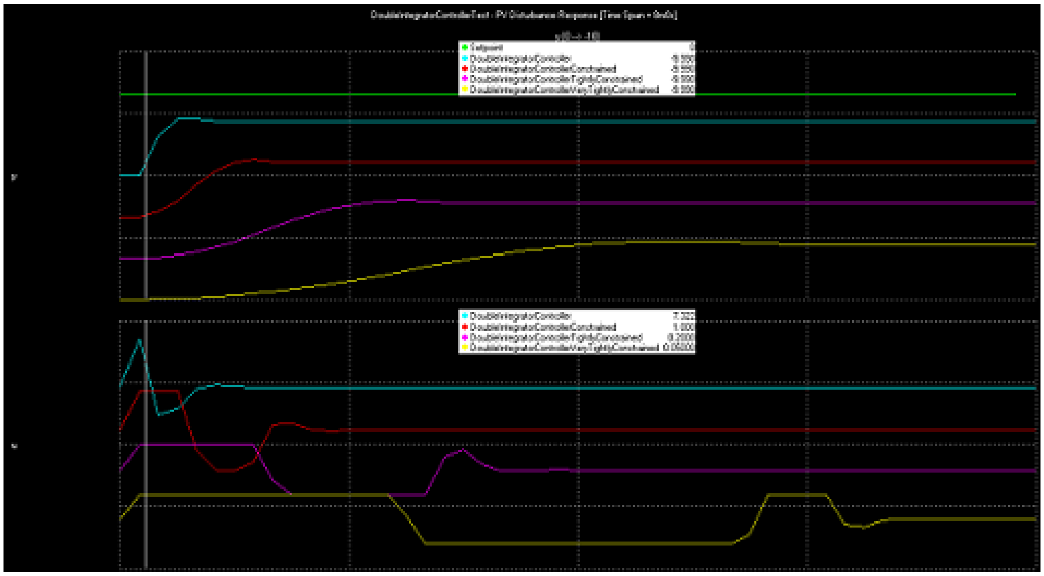


Figure 10.2: MPC for double integrator with constraints

Resembling a chess game as played by a computer algorithm, the method works in iterations comprising the following steps.

1. Evaluate position (=measurement) and estimate system state
2. Calculate the effect on the plant of (possible) sequences of actuator moves
3. Select the best control sequence (mathematical algorithm, optimization)
4. Implement the first move (new actuator set-point)
5. Restart after the opponents move (plant/process reaction to actuator move)

Remark 10.1

1. *The model is used to predict the system behavior into the future.*

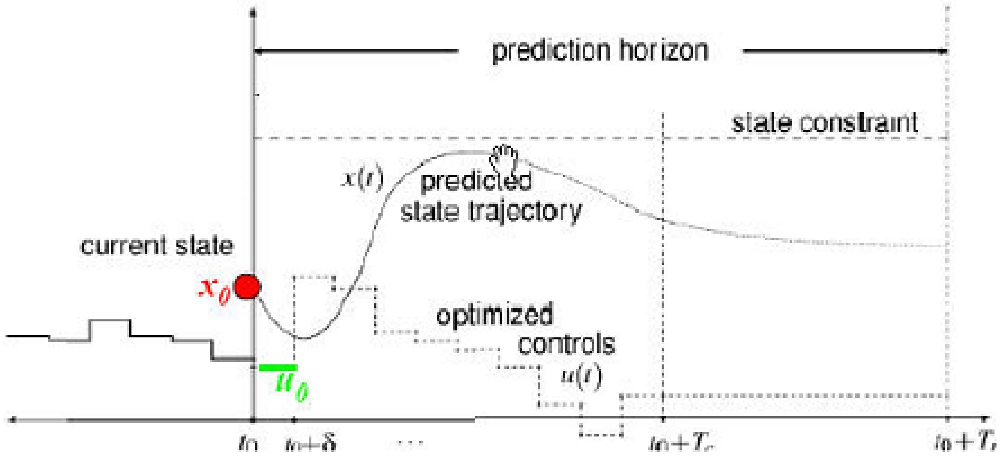


Figure 10.3: Model Predictive Control Trajectories

2. *The method requires solution of optimization problem at every sampling time.*

3. *Additional constraints on the actuators can be added. For instance, that the actuators are to remain constant during the last few steps.*

4. *Normally linear or quadratic cost functions are used. These functions represent trade off among deviations from setpoints, actuator action costs, economical considerations, etc.*

5. *For nonlinear systems, one has to deal with risks like loss of convexity, local minima, increased computational time, etc. Still, the optimization problems to be solved online are highly sparse. This allows for efficiency gains of several orders of magnitudes.*

6. *Useful functionality is added when mathematical models in the "Mixed Logical Dynamical Systems" framework are used. In this case, logical constraints and states can be included in the mathematical model, fact that is extremely useful when dealing with plant wide planning and scheduling problems.*

10.3 Case Study: Tank Level Reference Tracking

Problem Statement
The process consists of a tank with a measurable inflow disturbance and a controllable outflow in form of a pump. Constraints are set on the tank level and the pump capacity. A level reference target is given.

Model Variables

1. Inputs of the model are the known but not controllable tank inflow and the controlled valve opening variation.
2. States are the tank level and the current valve opening:
3. Outputs are the tank level, and the outflow

Model Equations
State dynamics

$$\begin{aligned} volume(t+1) &= volume(t) + f_{in}(t) - f_{out}(t) \\ \\ f_{out}(t) &= \alpha\, level(t)\, u(t) \\ level(t) &= volume(t)/area \\ \\ u(t+1) &= u(t) + du(t) \end{aligned}$$

Outputs

$$\begin{aligned} y_{level}(t) &= level(t) \\ y_{out}(t) &= f_{out}(t) \end{aligned}$$

Inputs

- f_{in}: inflow, measured but not controlled
- u: valve opening, controlled via its increment $du(t) = u(t+1) - u(t)$.

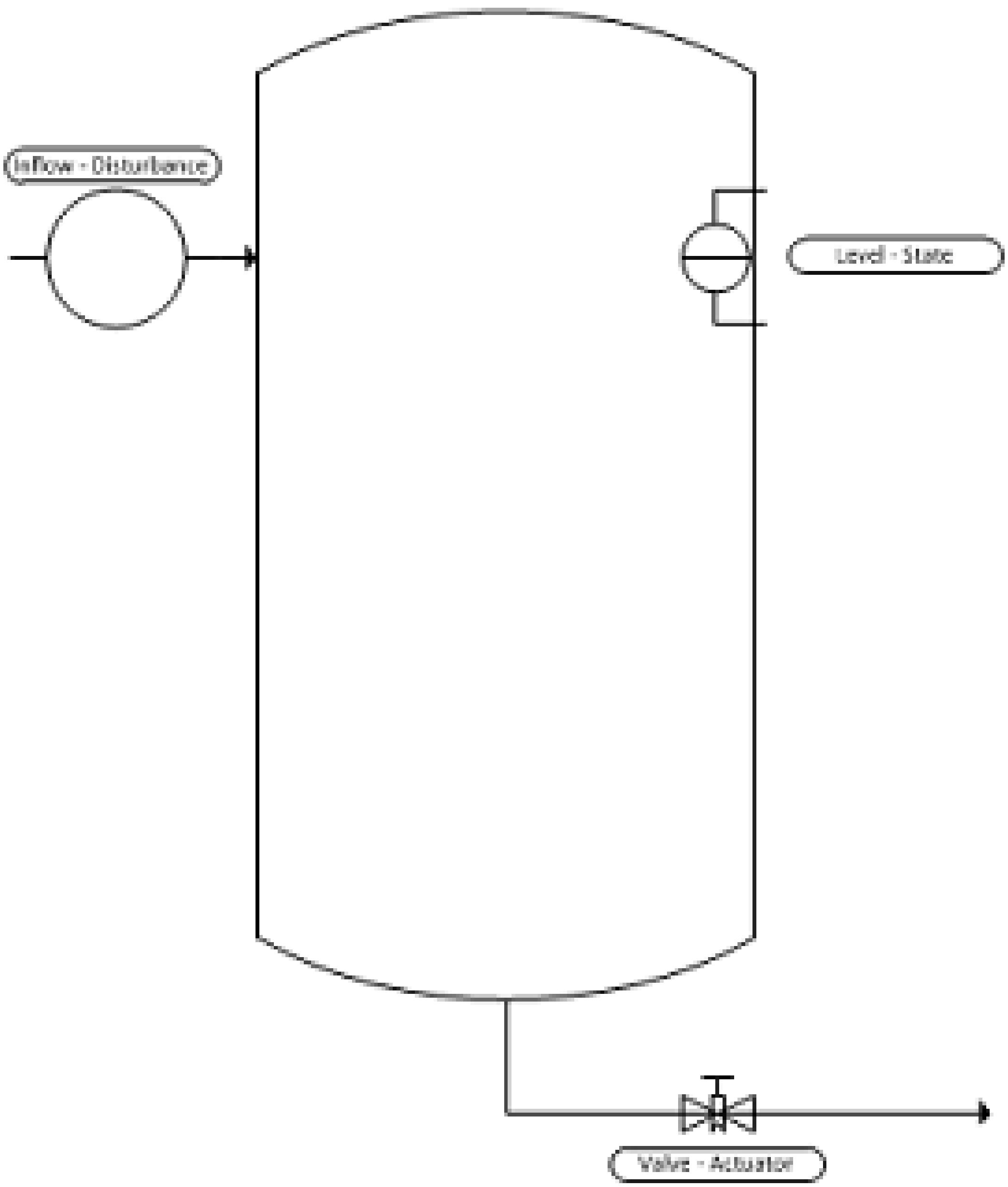


Figure 10.4: Tank Schema

Cost Function

$$J[x_0, u(\cdot)] = \sum_{t=0}^{T} q\,[y_{level}(t) - y_{ref}(t)]^2 + r\,du(t)^2$$

Problem Constraints

1. Level within min/max
2. Outflow within min/max
3. Max valve variation per step

Model Representation Industry has created dedicated software tools aimed to facilitate the tasks of modelling and control design. As a rule, the engineer is provided tools for graphical representation of the model. Figure 10.5 shows the tank model equations depicted in a typical industrial software.

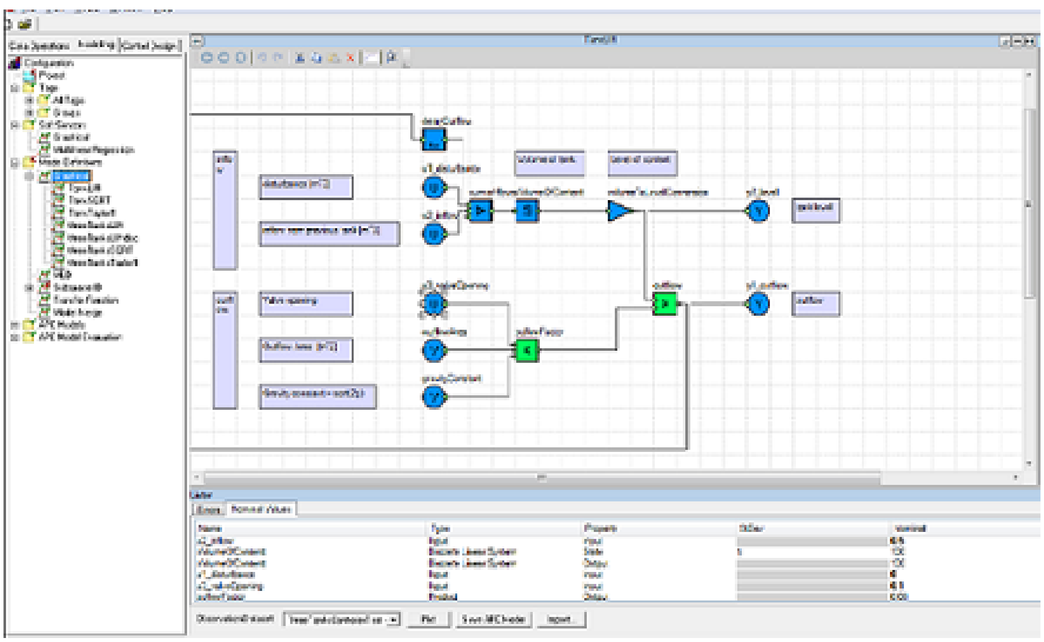


Figure 10.5: Tank Model in Graphical Package

There are also standard displays for fast visualization of the plant dynamics. In Figure 10.6 we see such standard representation. The graphic is a grid of graphs, with as many rows as outputs and as many columns as inputs. Each column represents the response of each output to a step in a given input.

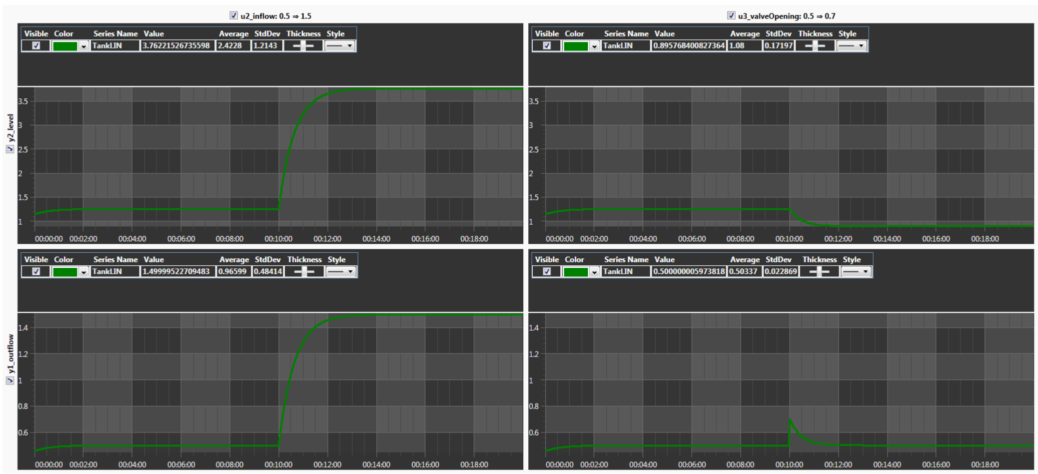


Figure 10.6: Tank Model Step Response

Results Discussion

Figure 10.7 shows optimization results obtained with a receding horizon of T=20 steps. Note how the Model Predictive Controller is able to nicely bring the level to the desired setpoint.

On the other hand, Figure 10.8 shows the same optimization obtained with a receding horizon of T = 2 steps. In this case, the performance considerably deteriorates: stability is lost!

In the last example we look at the problem is controlling 3 interconnected tanks. The problem to solve is the same as before, namely, to keep the tank levels at given setpoints. The problem is now not just larger, but also more complicated due to the interaction among the tanks. We design 2 controllers, one that knows about the true plant dynamics, and one that treats the tanks as if they were independent.

Figure 10.10 shows the step response of this model. Note how the effect of each actuators propagates in the system from one tank to the next.

In Figure 10.11 we represent the closed loop responses of 2 different controllers:

- one that knows about the interconnection among the tanks and can better predict the system behavior and
- one that treats each tanks as independent entities, where the valve is used to control the level.

As expected we observe that the multivariable controller is able to solve the problem in a much more efficient fashion than the ”single input single output” one, showing the benefits of the model predictive control approach over the idea of cascaded SISO controllers.

10.4 Chronology of Model Predictive Control

Below we find some major milestones in the journey leading to the current MPC approach:

1. 1970's: Step response models, quadratic cost function, ad hoc treatment of constraints
2. 1980's: linear state space models, quadratic cost function, linear constraints on inputs and output
3. 1990's: constraint handling: hard, soft, ranked
4. 2000's: full blown nonlinear MPC

10.5 Stability of Model Predictive Controllers

When obtaining stability results for MPC based controllers, one or several of the following assumptions are made

1. Terminal equality constraints
2. Terminal cost function
3. Terminal constraint set

4. Dual mode control (infinite horizon): begin with NMPC with a terminal constraint set, switch then to a stabilizing linear controller when the region of attraction of the linear controller is reached.

In all these cases, the idea of the proofs is to convert the problem cost function into a Lyapunov function for the closed loop system.

Let us consider at least one case in details. For that, we introduce the MPC problem for discrete time systems. Note that in practice, this is the form that is actually used.

Consider the time invariant system

1. Plant model

$$x(k+1) = f(x(k), u(k)), \quad x(0) = x_0$$

2. Constraints

$$\begin{array}{l} g(k, x, u) = 0, \quad k = 0 : \infty \\ g_i(k, x, u) \leq 0, \quad k = 0 : \infty \end{array}$$

3. Objective function

$$J[x_0, u(\cdot)][= \sum_{l=k}^{k+N} L(x(l), u(l))$$

The optimal control is a function

$$u^\star(\cdot) = \arg\min J[x_0, u(\cdot)], \quad u^\star(l), \quad l = k : k+N$$

Theorem 10.2 *Consider an MPC algorithm for the discrete time plant, where $x = 0$ is an equilibrium point for $u = 0$, i.e. $f(0,0) = 0$.*
Let us assume that

- *The problem contains a terminal constraint $x(k+N) = 0$*
- *The function L in the cost function is positive definite in both arguments.*

Then, if the optimization problem is feasible at time k, then the coordinate origin is a stable equilibrium point.

Proof. We use the Lyapunov result on stability of discrete time systems introduced in the Lyapunov stability lecture. Indeed, consider the function

$$V(x) = J^\star(x),$$

where $J^\star$ denotes the performance index evaluated at the optimal trajectory. We note that:

- $V(0) = 0$
- $V(x)$ is positive definite.
- $V(x(k+1)) - V(x(k)) < 0$. The later is seen by noting the following argument. Let

  $$u_k^\star(l), \quad l = k : k+N$$

  be the optimal control sequence at time k. Then, at time $k+1$, it is clear that the control sequence $u(l)$, $l = k+1 : N+1$, given by

  $$\begin{aligned} u(l) &= u_k^\star(l), \quad l = k+1 : N \\ u(N+1) &= 0 \end{aligned}$$

  generates a feasible albeit suboptimal trajectory for the plant. Then, we observe

  $$V(x(k+1)) - V(x(k)) < J(x(k+1), u(\cdot)) - V(x(k)) = -L(x(k), u^\star(k)) < 0$$

  which proves the theorem.

Remark 10.3 *Stability can be lost when receding horizon is too short, see Figure 10.12.*

Remark 10.4 *Stability can also be lost when the full state is not available for control and an observer must be used. More on that topic in the Observers Lecture.*

10.6 Exercises

1. Let us consider the following standard MPC problem :

$$\begin{array}{ll} \min & J = \sum_{i=0}^{N-1} \left(x_i^T Q x_i + u_i^T R u_i \right) + x_N^T P x_N \\ \text{s.t.} & x_{k+1} = A x_k + B u_k, \quad x_0 \text{ is given} \\ & \|u_k\|_\infty \leq \gamma, \quad \forall k \in [0, \ldots, N-1]. \end{array}$$

   (a) Rewrite the cost, dynamics and input constraints as a convex quadratic program in the variable $U = [u_0^T; \ldots; u_{N-1}^T]$.

   (b) Consider $A = -1$, $B = 1$, $x_0 = 10$, $Q = 1$, $R = 1$, $P = 2$, $N = 2$ and $\gamma = 6$. Compute analytically the solution of the quadratic program.

2. Consider the nonlinear system

$$x_{k+1} = -x_k^3 + u.$$

   Design a nonlinear MPC for such a system in Matlab. Close the loop and, starting from an initial point $x_0 = 2$, plot the state evolution in time.

   Hint: Use the Matlab function `fmincon` to solve the nonlinear optimization problem

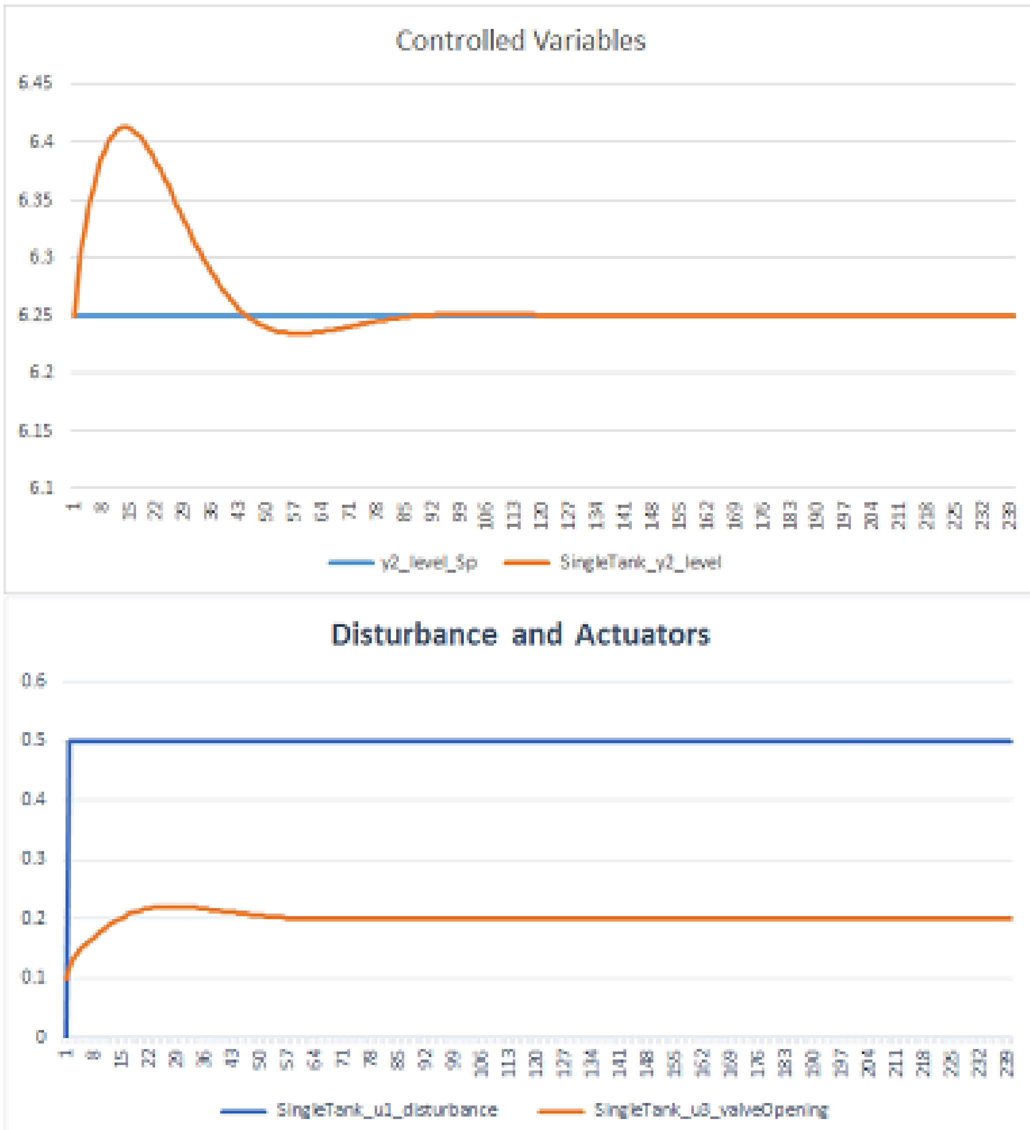


Figure 10.7: Tank Trajectories for T=20

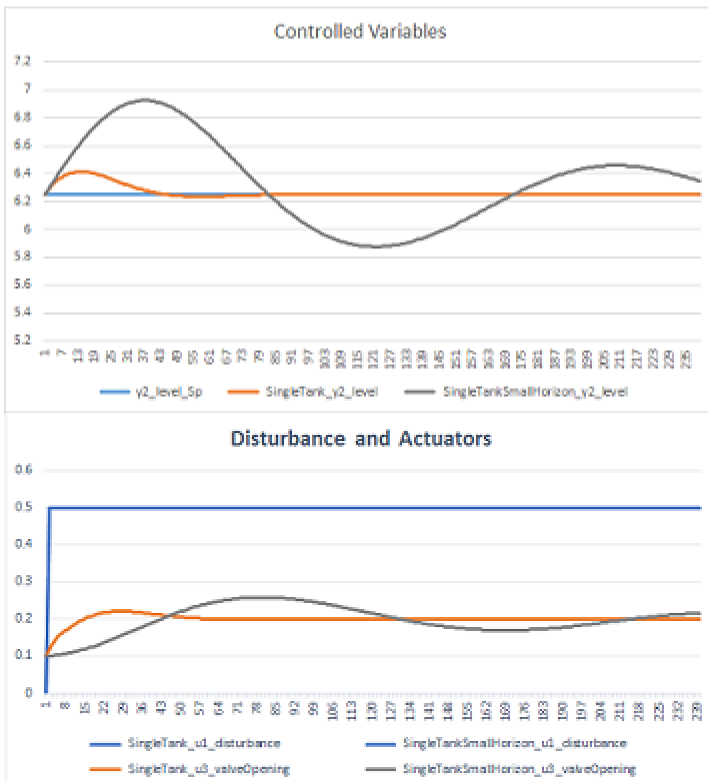


Figure 10.8: Tank Example Trajectories for T=2

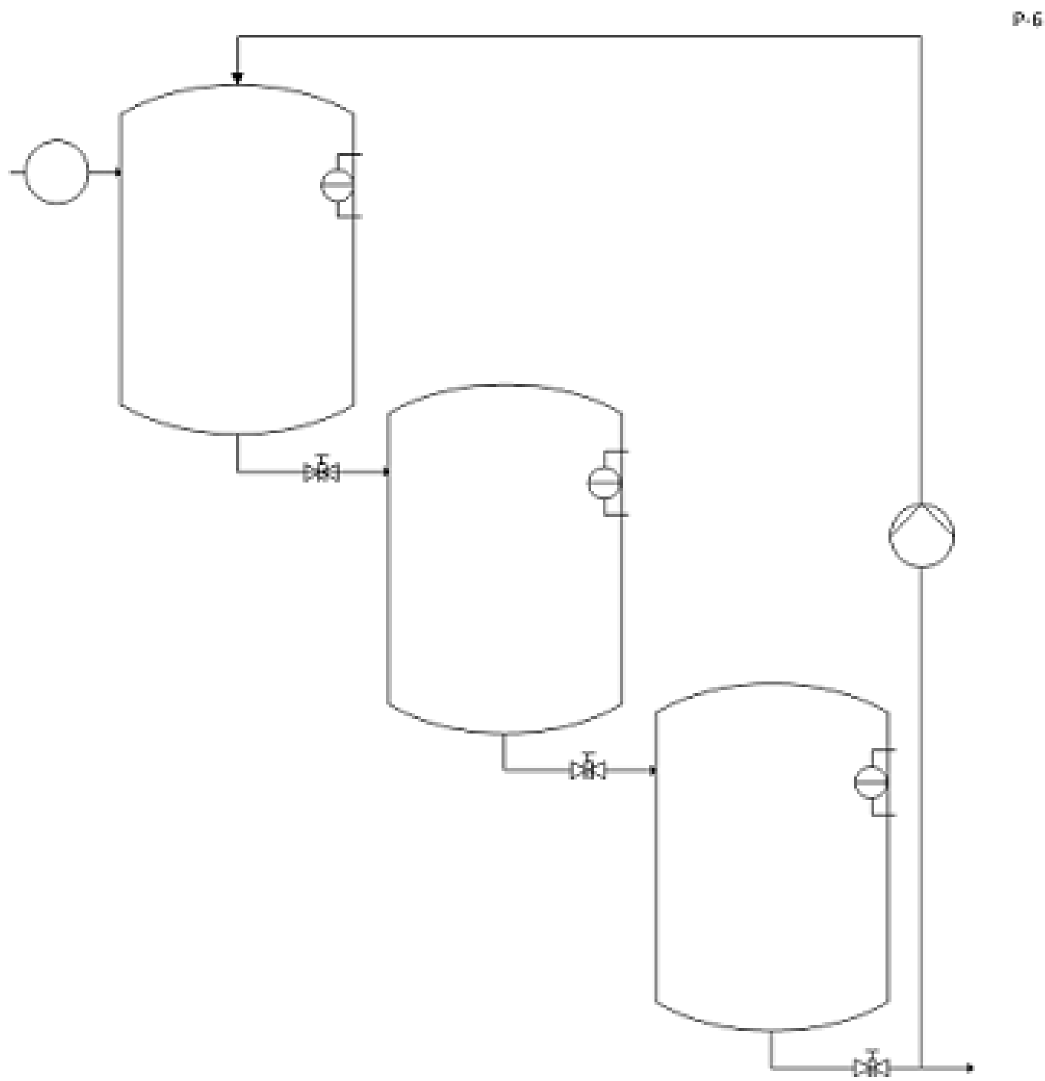


Figure 10.9: Three Interconnected Tanks

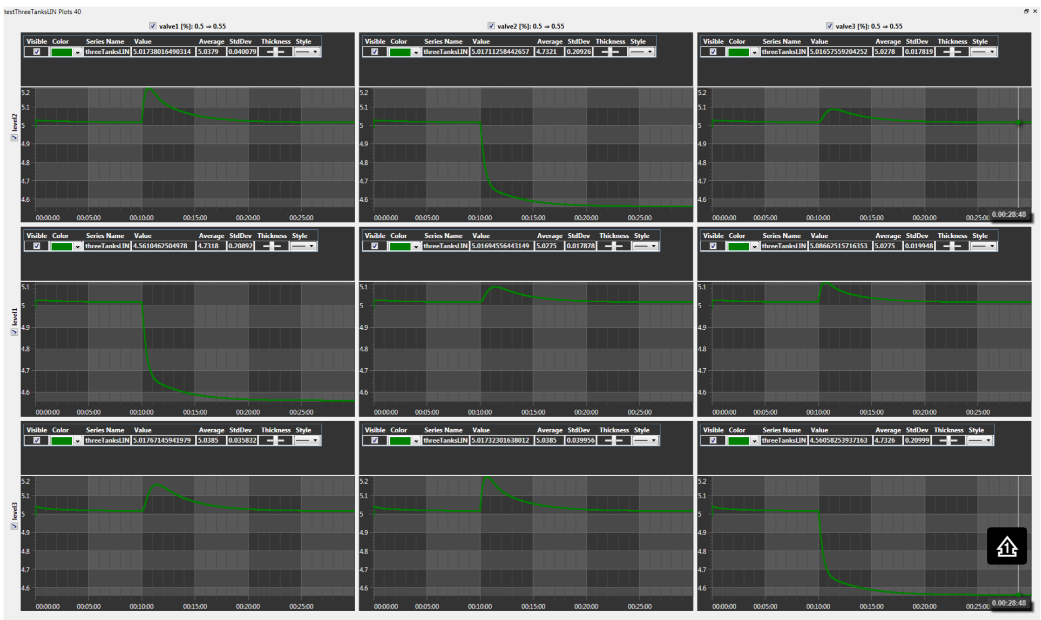


Figure 10.10: Step Response in the Interconnected Tanks Case

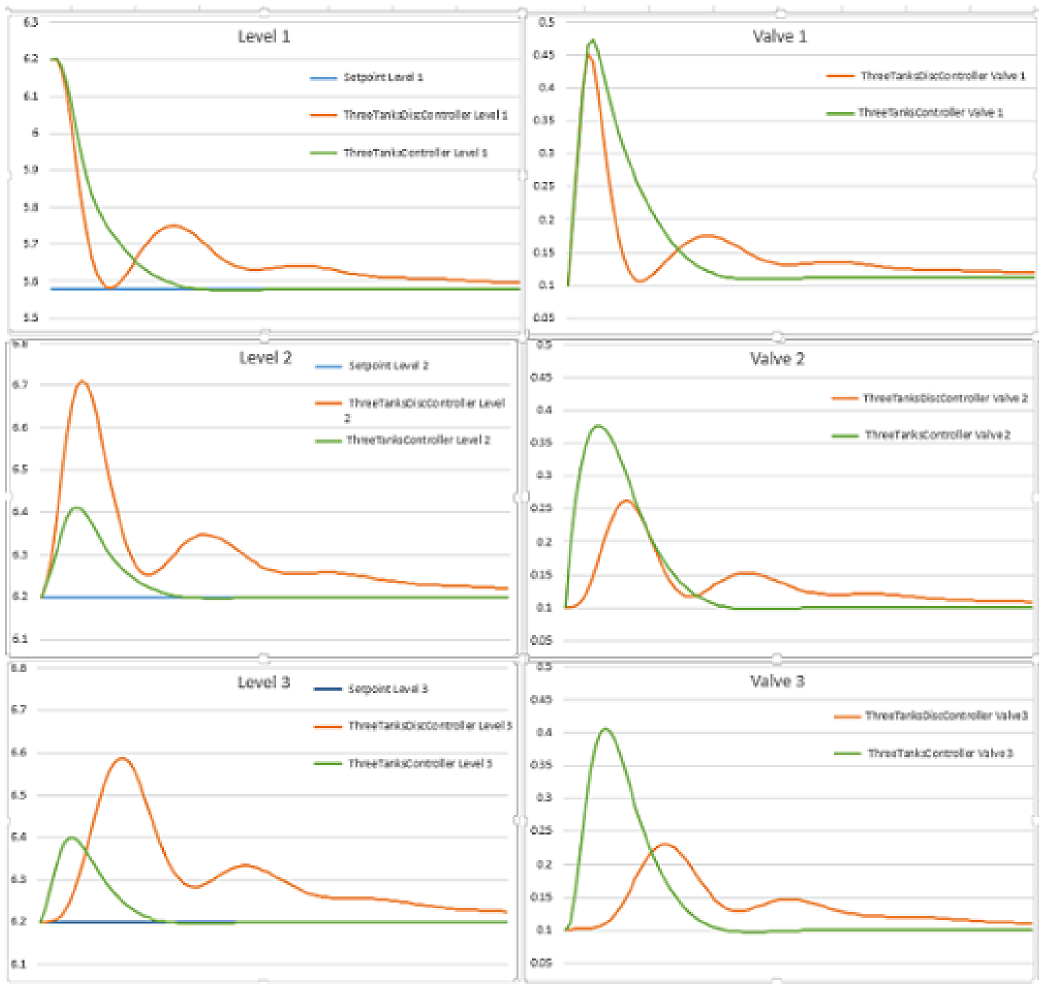


Figure 10.11: MPC and SISO Responses in the Interconnected Tanks Case

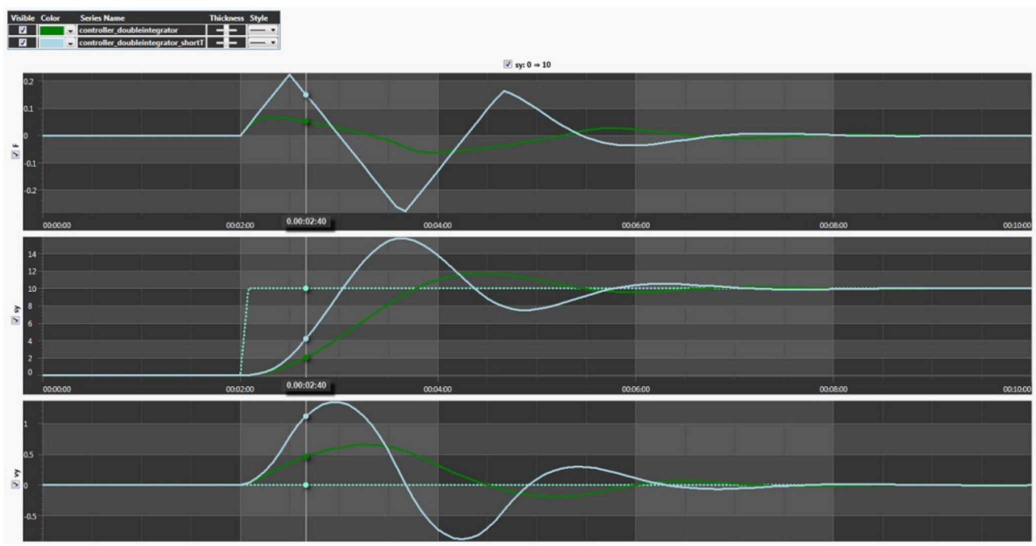


Figure 10.12: Double Integrator looses stability lost for short horizon

Chapter 11

State Estimation and Observers

In practice no perfect observation of the system state is available, either because it is costly, technically unfeasible or because the measurements quality is low. In this case state feedback control laws,

$$u(t) = u(x(t)), \quad t \geq 0$$

as derived in previous lectures is often impractical. There is a need for a systematic approach for the evaluation or estimation of the system state using the information available.

One natural approach is to compute an estimate $\hat{x}$ of the state x and apply the feedback:

$$u(t) = u(\hat{x}(t)), \quad t \geq 0$$

The idea that a stabilizing controller can consist of a state estimator plus (estimated) state feedback is called the *separation principle*. For linear systems this is a valid approach. Indeed, given a linear time invariant system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

Consider the observer

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L\left(y - C\hat{x}\right) \\ &= \left(A - LC\right)\hat{x} + Bu + Ly \end{aligned}$$

Let $e = x - \tilde{x}$, then

$$\begin{aligned} \dot{e} &= (Ax + Bu) - (A - LC)\,\tilde{x} - Bu - LCx \\ &= (A - LC)\,e \end{aligned}$$

Let

$$u = -K\tilde{x} = -K(x - e)$$

Then

$$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} A - BK & BK \\ 0 & A - LC \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}$$

Thus if $A - BK$ and $A - LC$ are stable matrices, the resulting closed loop system is also stable.

Unfortunately, for nonlinear systems this approach does not work in general. The problem is that generally speaking it is not possible to estimate the error dynamics – i.e. the dynamics of the difference between the actual state and the estimated state.

There are several approaches to state estimation that may be applied:

1. Extended Kalman Filter (EKF): Extension of linear Kalman Flter
2. Recursive Prediction Error (RPE): Filter based on the sensitivity equation
3. Unscented Kalman Filter (UKF): Mix of Monte-Carlo with Kalman Filter
4. Moving Horizon Estimation (MHE)
5. High Gain Observers

In the sequel we present techniques 1, 4 and 5 for estimating the state of a nonlinear system.

11.1 Least Squares Estimation of Constant Vectors

We consider the process model

$$y = Cx + w, \quad x \in \mathbb{R}^n, \quad y, w \in \mathbb{R}^p, n \geq p.$$

with w denotes white noise.

The goal is to compute the best estimate $\hat{x}$ of x using proces measurements y. Given $\hat{x}$ we can give an estimate $\hat{y}$ of the output by

$$\hat{y} = C\hat{x}$$

Then we define the residual as

$$\epsilon_y = \hat{y} - y$$

We want to obtain the optimal estimate in the sense that the quadratic function

$$J[\hat{x}] = 0.5\epsilon_y^T \epsilon_y$$

It is easily derived that this optimal estimate is given by

$$\hat{x} = (C^T C)^{-1} C^T y$$

Where the matrix inversion is made in the sense of the so called "pseudo inverse". Averaging effect takes place when the dimension of the output is larger than the dimension of the state.

Example 11.1 *Consider a process modelled by the simple linear relationship*

$$y = Cx + w; \quad C = \begin{pmatrix} -5 & 1 \\ 0 & 1 \end{pmatrix}$$

The magnitude y is measured directly and is affected by white noise w. The actual value of x and needs to be reconstructed from the data series

$$Y = [y_1; y_2, \ldots, y_N] = [Cx + w_1; Cx_2 + w_2; \ldots; Cx + w_N]$$

The optimal estimate $\hat{x}$ is then given by

$$\hat{x} = KY, K = (H^T H)^{-1} H^T; \quad H = [C; C; \ldots; C];$$

Figure 11.1 shows how the estimate $\hat{x}$ approaches the true value $x = [5; 5]$ as N, the number of measurements, grows.

Our second example shows how to use the results above in the context of dynamical system parameter estimation.

Example 11.2 *Consider the equations of a simple heat exchanger. Steam at temperature T_{si} enters the system and passes its energy to a gas that enters the system at temperature T_{gi}. The fluids are separated by a metal interface, which has temperature T_m. At the output, the steam and the gas have temperatures*

$$\dot{T}_g = \alpha_g(T_m - T_g) + fg(T_{gi} - T_g) \tag{11.1}$$

$$\dot{T}_m = \alpha_g(T_g - T_m) + \alpha_s(T_s - T_m) \tag{11.2}$$

$$\dot{T}_s = \alpha_s(T_m - T_s) + fs(T_{si} - T_s) \tag{11.3}$$

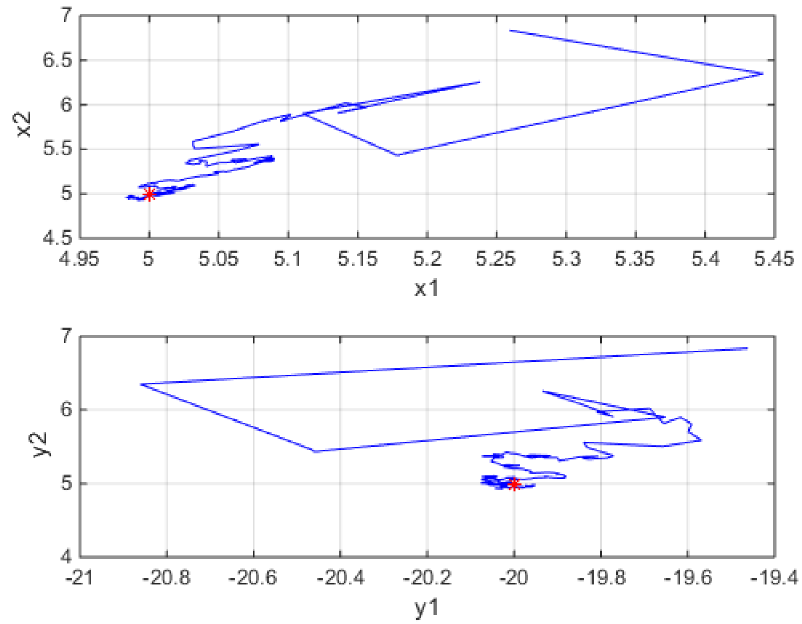


Figure 11.1: Static Estimation. Note how accuracy improves as N grows

Our task is to estimate the values of α_g *and* α_s *from the measurements* T_{gi},T_g,T_{si},T_s *and* T_m.

We proceed by realizing that for any given point in time we can create a linear relationship

$$y = Cx + w$$

where

$$y = \begin{pmatrix} \dot{T}_g - fg(T_{gi} - T_g) \\ \dot{T}_m \\ \dot{T}_s - fs(T_{si} - T_s) \end{pmatrix}$$

and

$$C = \begin{pmatrix} (T_m - T_g) & 0 \\ (T_g - T_m) & (T_s - T_m) \\ 0 & (T_m - T_m) \end{pmatrix}$$

$$x = \begin{pmatrix} \alpha_g \\ \alpha_s \end{pmatrix}$$

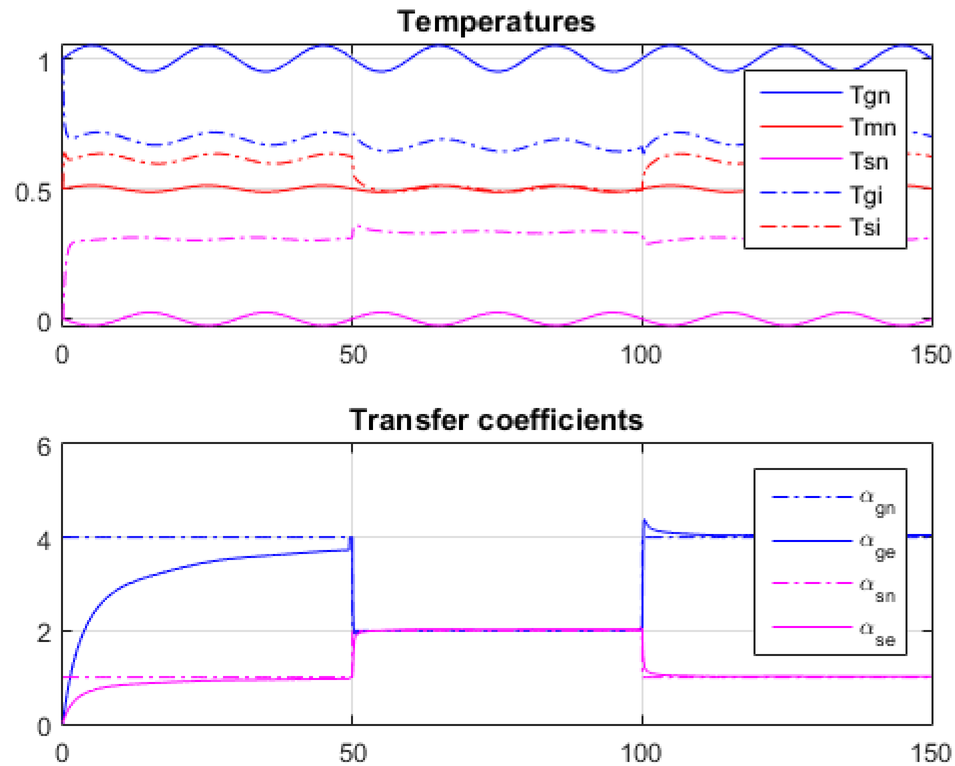


Figure 11.2: Parametric Static Estimation. Note how accuracy improves as N grows

Note that the derivatives $\dot{T}_g, \dot{T}_m, \dot{T}_s$ in y need to be reconstructed from the temperature measurements. Due to measurement noise, one must expect that these "reconstructed" derivatives are noisy measurements of the true derivatives.

Figure 11.2 shows how the estimates α_{ge}, α_{se} approach the true values $[3; 1]$ as N, the number of measurements, grows.

Remark 11.3 *The static method is designed to introduce averaging effects over the whole data set y. Thus, in practice one needs methods to manage which data is presented to the algorithm so that results remain meaningful. For instance, if the process has large steady state phases interrupted by periods* *of transient behavior, then one must make sure that the steady state phases are not presented to the algorithm in the same data set because otherwise the result will be the average of two different steady state modes, which is not what is desired.*

For example, the results in Figure 11.2 were obtained by resetting the time arrays after the change of parameters, i.e. at times $t = 50, 100$. Without this manipulation the observer does not deliver meaningful results.

11.2 Weighted Least Squares Estimator

The measurement error statistical properties can change from one point to another thus it makes sense to give different weight to different points while seeking the optimum. More importantly , as discussed in the previous section, when designing an observer there is always need for methods to fine tune the significance of different values in the method trade off's between model and noise. For instance, the designer may want to make sure that old values are considered less important than recent ones.

If the weights are distributed according to the equation

$$J[\hat{x}] = 0.5\epsilon_y^T\epsilon_y, \quad \epsilon_y = N^{-1}(\hat{y} - y).$$

Making the derivative of this function equal to zero, we derive the following formula for optimal estimate

$$\hat{x} = (C^T S^{-1} C)^{-1} C^T S^{-1} y; \quad S = N^{-T} N^{-1}.$$

This formula is used exactly as in the previous section. Care is of course needed in the choice of the weights N.

11.3 Propagation of the State Estimate and its Uncertainty

The state of a dynamic system changes with time and with its statistical properties. State estimators must take this fact into account. If the initial conditions, the inputs and the dynamics were perfectly known then it would be enough to integrate forward the system equations to have a good estimate of the system state. Unfortunately, this is never the case and uncertainty will always play a role. In this section we must learn to propagate the statistical properties of the state, so that we can use them when calculating the optimal estimator.

Let us assume we have a linear dynamical system

$$\dot{x} = Ax + Bu + Lw$$

Let us replace deterministic values by the "means". Then

$$\mathbb{E}[\dot{x}(t)] = A\mathbb{E}[x(t)] + B\mathbb{E}[u(t)] + L\mathbb{E}[w(t)]$$

Via simple manipulations this implies

$$\begin{aligned} \dot{m}(t) &= Am(t) + Bu(t) \\ \dot{P}(t) &= AP + PA^T + LQL^T. \end{aligned}$$

where

1. $m(t) = \mathbb{E}[x()]$
2. $P(t) = \mathbb{E}[(x(t) - m(t))^T(x(t) - m(t))]$
3. Q is covariance of w.

Note how the uncertainty (covariance) of the mean of the state will always grow due to the driving force LQL^T.

11.4 Kalman Filter

A recursive optimal filter propagates the conditional probability density function from one sampling time to the next, incorporating measurements and statistical of the measurements in the estimate calculation.

The Kalman Filter consists of the following steps

1. State Estimate Propagation
2. State Covariance Propagation
3. Filter Gain Calculation
4. State Estimate Update using the newest measurements

5. State Covariance Update using the newest measurements

Steps 1 and 2 were considered before. Step 3 (calculation of Filter gain) is made using ideas of the first part of this lecture namely weighted least squares. The result is

$$K = P_k C_k^T [C_k P_k C_k^T + R]^{-1}.$$

where R is the covariance of the measurement noise. The formula shows a tradeoff between the measurement noise statistics R and the quality of our estimation as given by P.

The Filter gain K is used in the Steps 4 and 5 as follows.

1. $\hat{x}_{k+1} = x_k + K(y - C\hat{y})$
2. $P_{k+1} = (I - KC_k)P_k$

11.4.1 Extended Kalman Filter

The system is now nonlinear and modifications to the schema above are needed. They are

- Steps 1 is made with the true nonlinear equations
- Steps 2 to 5 are made with linearization of the linear system at the current estimate $\hat{x}$.

A summary of the formulae is given in Figure 11.3.

The same formalism can be used to estimate parameters. Indeed, one can augment the natural system states with states representing the parameters. The dynamic equation for the parameter vector is chosen to be zero, i.e. $\dot{p} = 0 + w_p$, where w_p has covariance Q_{pp}. Substituting these expressions in the standard EKF formulate produces the algorithm represented in Figure 11.4.

Example 11.4 *We consider the same heat exchanger, but now we are prepared to confront the realistic case where all measurements are noisy and thus calculating the derivatives accurately is not realistic. Also, we should drop the assumption that we know when the heat exchanger changes the operational modes. Under these conditions, the static estimation of parameters can not*

Formulae	**Meaning**
At time t_k, given $\hat{x}(t_k)$, $y(t_k)$, $P(t_k)$, Q, R	Data at each step
for $\tau \in [t_k, t_{k+1}^-]$ $$\dot{\hat{x}} = f(\hat{x})$$ $$\dot{P} = \frac{\partial f}{\partial x}(\hat{x}(t_k))P + P\frac{\partial f}{\partial x}^T(\hat{x}(t_k))$$	Step forward in time for estimates and covariances
$$e(t_{k+1}) = y(t_{k+1}) - h(\hat{x}(t_{k+1}^-))$$	Estimation error
$$A(t_{k+1}) = R + \frac{\partial h}{\partial x}(x(t_{k+1}^-))P(t_{k+1}^-)\frac{\partial h}{\partial x}^T(x(t_{k+1}^-))$$	Estimation error covariance
$$K(t_{k+1}) = P(t_{k+1}^-)\frac{\partial h}{\partial x}^T(x(t_{k+1}^-))[A(t_{k+1})]^{-1}$$	Kalman gain
$\dot{\hat{x}}(t_{k+1}) = \hat{x}(t_{k+1}^-) + K(t_{k+1})e(t_{k+1})$ $P(t_{k+1}) = P(t_{k+1}^-) - K(t_{k+1})A(t_{k+1})K^T(t_{k+1})$	Final update

Figure 11.3: Extended Kalman Filter Formulae

be used because both we would not have the elements to construct the matrix C used in Example 2.

In this more realistic case we resort to use the Extended Klaman Filter, whereas we will estimate the states T_g,T_m,T_s and the parameters α_g and α_s. The equations remain the same, i.e. Equations 11.1. The important non trivial step is to create dynamic equations for the parameters. In this case it is enough

$$\begin{aligned}\dot{p}_g &= 0\\ \dot{p}_s &= 0\end{aligned}$$

$$
\begin{aligned}
\dot{x} &= f(x,u,p) + w_1 \\
\dot{p} &= 0 + w_2 \\
y &= h(x,p) + \eta \\
Q &= \mathbb{E}[w_1 w_1^T] \\
Q_{pp} &= \mathbb{E}[w_2 w_2^T] \\
R &= \mathbb{E}[\eta \eta^T]
\end{aligned}
$$

$$
\begin{aligned}
\hat{y}(t_{k+1}) &= h(\hat{x}(t_{k+1}^-), u(t_{k+1}), \hat{p}(t_{k+1}^-)) \\
e(t_{k+1}) &= y(t_{k+1}) - \hat{y}(t_{k+1})
\end{aligned}
$$

$$
\begin{aligned}
\dot{P}_{xx} &= \frac{\partial f}{\partial x^T} P_{xx} + P_{xx} \frac{\partial f}{\partial x}^T + \frac{\partial f}{\partial p^T} P_{xp}^T + P_{xp} \frac{\partial f}{\partial p}^T + Q \\
\dot{P}_{xp} &= \frac{\partial f}{\partial x^T} P_{xp} + \frac{\partial f}{\partial p^T} P_{pp} \\
\dot{P}_{pp} &= Q_{pp}
\end{aligned}
$$

$$
\begin{aligned}
K(t_{k+1}) &= (P_{xx}(t_{k+1}^-) \frac{\partial h}{\partial x}^T + P_{xp}(t_{k+1}^-) \frac{\partial h}{\partial p}^T) A^{-1}(t_{k+1}) \\
L(t_{k+1}) &= (P_{xp}^T(t_{k+1}^-) \frac{\partial h}{\partial x}^T + P_{pp}(t_{k+1}^-) \frac{\partial h}{\partial p}^T) A^{-1}(t_{k+1})
\end{aligned}
$$

$$
\begin{aligned}
\hat{x}(t_{k+1}^+) &= \hat{x}(t_{k+1}^-) + K(t_{k+1}) e(t_{k+1}) \\
\hat{p}(t_{k+1}) &= \hat{p}(t_k) + L(t_{k+1}) e(t_{k+1})
\end{aligned}
$$

Figure 11.4: Augmented State Extended Kalman Filter Formulae

and

$$Q_{pp} = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix}.$$

$$Q_{xp} = 0^{2\times 3}.$$

Note that now there is no need for manipulating the arrays: the algorithm is recursive and reacts correctly to operational mode changes.

11.4.2 Moving Horizon Estimation

In analogy to Model Predictive Control, one can think of the observation problem as one of optimization under a finite horizon. Variables for optimization would be in this case the system states, and parameters if parameter estimation is also required. This approach has the advantage that Model Predictive Control formalism can be reused to deal with this task.

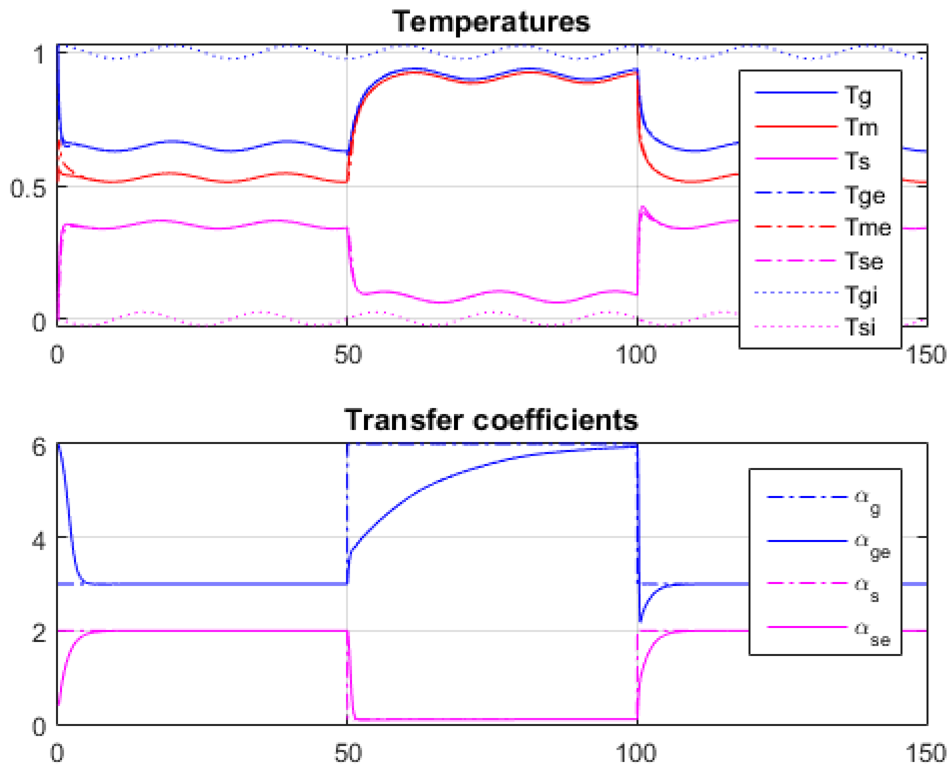


Figure 11.5: Extended Kalman Filter Results for Heat Exchanger

In concrete terms, the "attack plan" is to collect

1. the history of actuators moves and measurements $u_M(\tau), y_M(\tau), \tau \in [t-M:t]$
2. the plant model, including the constraints

with which the observer calculates the trajectory of states $x(\tau), \tau \in [t-M:t]$ that better "explains" the given history and the model.

In a somewhat simplified notation the problem being solved is one of minimization of the function

$$J[x(t-M),\ldots,x(0)] = \sum_{t=-M}^{-1}\{[x(t+1)-f(x(t),u_M(t))]^2_{Qx}+[y_M(t)-h(x(t),u_M(t))]^2_{Qu}\}$$

Tuning parameters are $q_x, q_y > 0$, which denote the degree of trust on the existent measurements (so called measurement noise level) and in the model (process noise level). For large estimation horizon M, this method converges towards a Kalman Filter.

Remark 11.5 *Moving Horizon Estimation allows keeping the problem constraints explicitly, which might be important in application for the sake of eliminating ambiguity.*

Remark 11.6 *(Findeisen, 2003): Stability results of MPC and MHE can be obtained assuming that*

1. *MHE can be made to converge very fast: given error bound is satisfied after a given number of iterations*

2. *MPC has a continuous value function: NMPC robust to small perturbations.*

Neither of these facts are obvious to prove for a given system, but they show in what direction ones needs to work to guarantee stability.

Industrial implementations of this method hide complexity from the user. All is required it is to attach to the model those variables containing the current values of the measurement and actuators. Industrial grade systems automatically:

- collect the data needed, i.e. the values of actuators, measurements and model parameters,

- create the associated mathematical programming problem,

- calculate the estimation of the states as required,

- pass these values to the controller for calculation of the optimal moves.

This capability can also be used in stand-alone fashion for construction of model based soft sensors or even simply for noise filtering purposes. One example of this usage is the implementation of material balancing applications, where the values of certain magnitudes in the process (eg, metal content and/or reagent usage) are reconstructed by looking at magnitudes measured at other points of the circuit and a (simple) mathematical model of the installation.

11.5 High Gain Observers

Another approach to estimate the state is to consider a High Gain Observer.

Consider the system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \phi(x,u) \\ y &= x_1 \end{aligned}$$

Suppose that the state feedback $u = \gamma(x)$ is a locally Lipschitz state feedback control that stabilizes the origin $x = 0$ of the closed loop sytem:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \phi(x,\gamma(x)) \\ y &= x_1 \end{aligned}$$

To implement this feedback control using only y, use the following estimation scheme:

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + h_1(y - \hat{x}_1) \\ \dot{\hat{x}}_2 &= \phi_0(\hat{x},\gamma(\hat{x})) + h_2(y - \hat{x}_1) \end{aligned}$$

Where $\phi_0(x,u)$ is a nominal model for $\phi(x,u)$. The estimation error $\tilde{x} = x - \hat{x}$ then satisfies

$$\begin{aligned} \dot{\tilde{x}}_1 &= -h_1\tilde{x}_1 + \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= -h_2\tilde{x}_1 + \delta(x,\hat{x}) \end{aligned}$$

where

$$\delta(x,\hat{x}) = \phi(x,\gamma(x)) - \phi_0(\hat{x},\gamma(\hat{x})).$$

We want *(h_1,h_2)* to be such that $\tilde{x}(t) \to 0$, and if $\delta \equiv 0$, this is easily achieved by making the matrix

$$A_0 = \begin{bmatrix} -h_1 & 1 \\ -h_2 & 0 \end{bmatrix}$$

to be stable. For $\delta \neq 0$, we design (h_1,h_2) in such a way that the transfer function

$$G_0 : \delta \mapsto \tilde{x}$$

is as small as possible, i.e. that the gain of the system is as small as possible. In this case:

$$G_0 = \frac{1}{s^2 + h_1 s + h_2} \begin{bmatrix} 1 \\ s + h_1 \end{bmatrix},$$

which can be made arbitrarily small if $h_2 >> h_1 >> 1$. In particular if

$$h_1 = \frac{\alpha_1}{\varepsilon}, h_2 = \frac{\alpha_2}{\varepsilon^2}, \varepsilon << 1,$$

Then

$$\lim_{\varepsilon \to 0} \|G_0(s)\| = 0$$

Unfortunately there is a problem, the (2,1) term of $\exp(A_0 t)$ grows as h_1, h_2 grow. This gives a *peaking phenomenon*. This means that the estimates of $\tilde{x}$ will become arbitrarily large for some period of time. This will make $u = \gamma(\hat{x})$ large, and may drive the system state out of the region of attraction – the region where the error dynamics are stable.

This situation may be overcome by *saturating* the control action. Since the peaking phenomena is of short duration, we assume that the state estimate will converge to the system state faster than the system diverges, and then the estimated state feedback will be sufficient to stabilize the system.

Example 11.7 *Let us consider the system:*

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_2^3 + u \\ y = x_1 \end{cases}$$

A stabilizing state feedback is given by:

$$u = \gamma(x) = -x_2^3 - x_1 - x_2$$

The output feedback controller is then given by:

$$\begin{cases} \dot{\hat{x}}_1 = \hat{x}_2 + \left(\frac{2}{\varepsilon}\right)(y - \hat{x}_1) \\ \dot{\hat{x}}_2 = -\hat{x}_1 - \hat{x}_2 + \left(\frac{1}{\varepsilon^2}\right)(y - \hat{x}_1) \\ u = sat(-\hat{x}_3^2 - \hat{x}_1 - \hat{x}_2) \end{cases}$$

The following plots compare the behaviour of the system with state feedback, with output feedback and with saturated output feedback for value of ε equal to 0.2 and 0.1. In Figure 11.6 and Figure 11.7 we see how for smaller values of ε, the transient is faster but larger, leading to earlier instabilities of the unsaturated system, but faster convergence of the estimated state to the actual state for the saturated system.

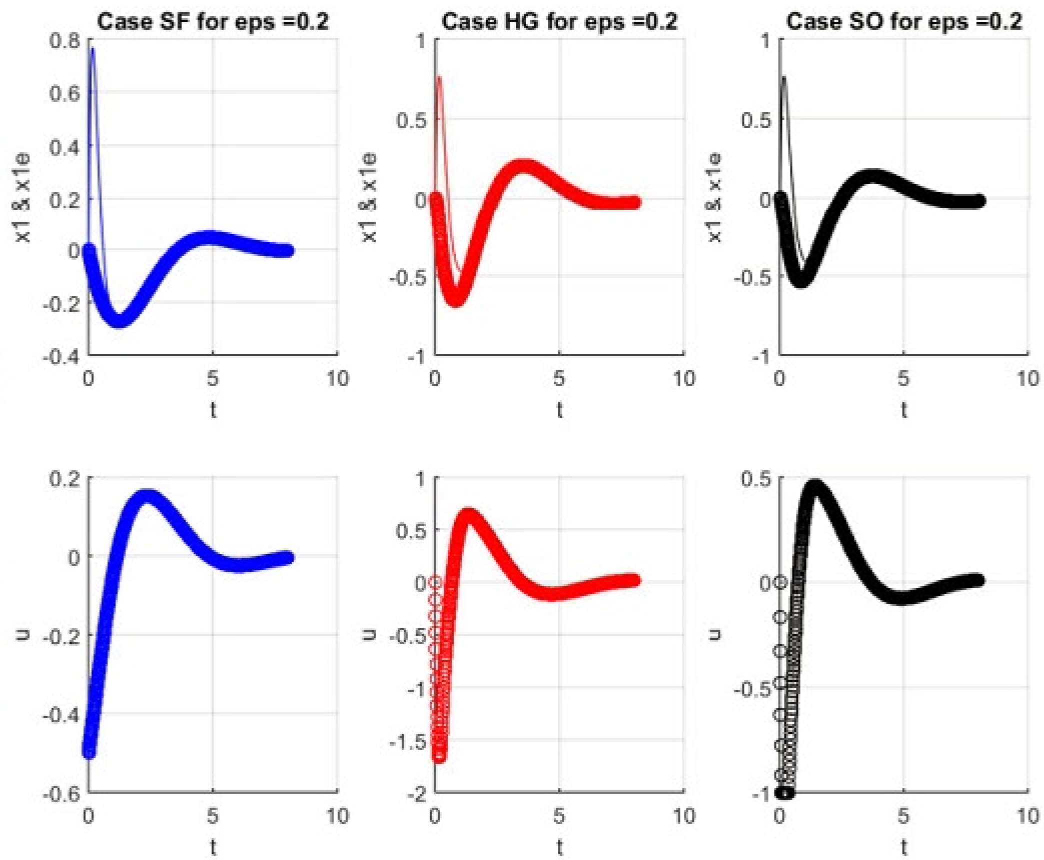


Figure 11.6: High Gain Observer Plots for $\epsilon = 0.2$

In particular, note how for the case $\epsilon = 0.1$ *depicted Figure 11.7, the brief transient is sufficient to drive the controller without saturation into an unstable region (the simulation stops around 0.9s), while in the case with*

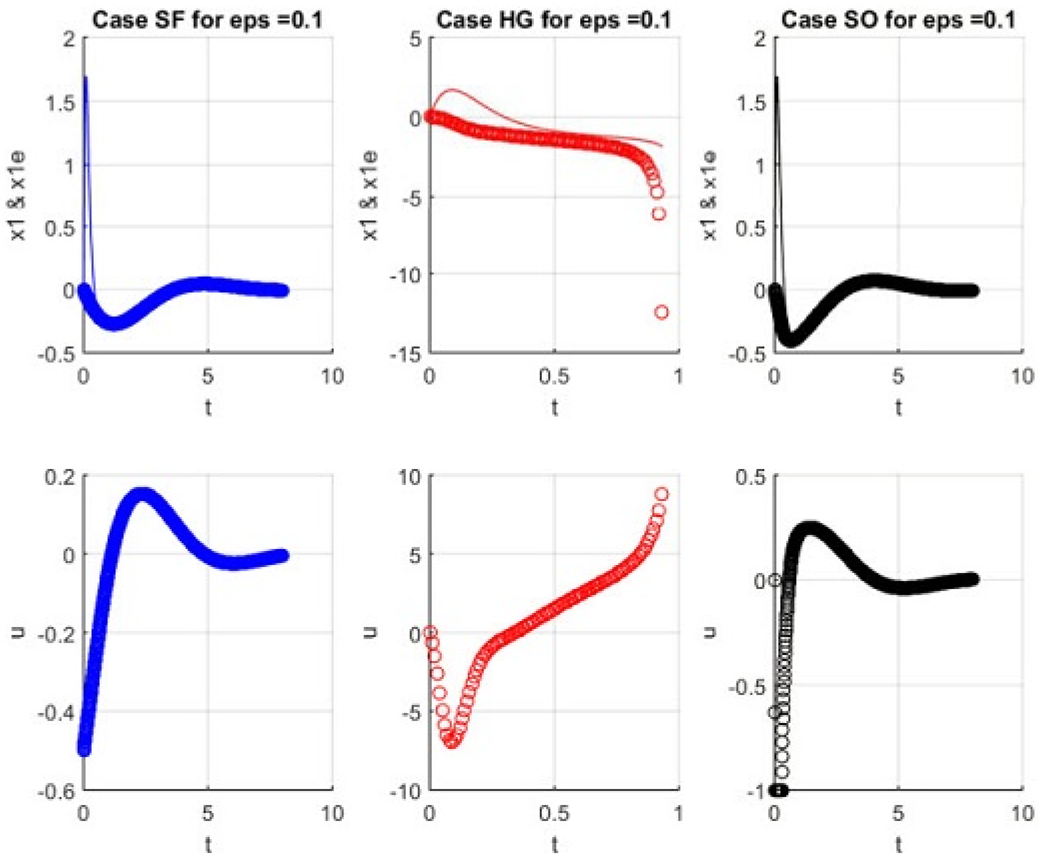


Figure 11.7: High Gain Observer Plots for $\epsilon = 0.1$

saturation, the controller performs almost as well as the state feedback controller.

Further, as $\varepsilon \to 0$ *the region of attraction under output feedback approaches the region of attraction under state feedback. This holds whenever* $\gamma(x)$ *is a globally bounded stabilizing function. Indeed, as* $\varepsilon \to 0$ *the time in saturation becomes smaller. Eventually this time is so small that the system state does not move during this time.*

In general we have the following result:

Theorem 11.8 *Consider the SISO system*

$$\begin{cases} \dot{x} = A_C x + B_C \phi(x,u) \\ y = C_C x \end{cases}$$

Assume that we have a stabilizing state feedback controller:

$$\begin{cases} \dot{v} = \Gamma(v,x) \\ u = \gamma(v,x) \end{cases}$$

with g and G globally bounded. Consider then the output feedback controller

$$\begin{cases} \dot{v} = \Gamma(v,\hat{x}) \\ \dot{\hat{x}} = A_C \hat{x} + B\phi_0(\hat{x},u) + H\left(y - C_c \hat{x}\right) \\ u = \gamma(v,\hat{x}) \end{cases}$$

where

$$H = \begin{bmatrix} \alpha_1/_{\varepsilon} \\ \alpha_2/_{\varepsilon^2} \\ \vdots \\ \alpha_\rho/_{\varepsilon^\rho} \end{bmatrix} \in \mathbb{R}^\rho, \text{ (r the relative degree)}$$

and $\varepsilon > 0$, and $\alpha_1, \alpha_2, \ldots \alpha_\rho$ are such that the polynomial

$$s^\rho + \alpha_1 s^{\rho-1} + \ldots + \alpha_\rho = 0$$

has all its roots in the complex left half plane. Then $\exists\ \varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, the closed loop system is exponentially stable.

Remark 11.9 *In this formulation we have neglected the unobserved zero dynamics. If they were to exist, then we need to assume that the original system is minimum phase. Then the result will hold.*

11.6 Exercises

1. Observability tells us whether it is possible to design a state estimator for an autonomous system

$$\begin{aligned} x_{k+1} &= (A + BK)\, x_k, \quad x_0 = \text{unknown} \\ y_k &= C x_k. \end{aligned}$$

with neither process nor measurement noise.

   (a) How many observations y_k are necessary to determine the initial state of the system?

   (b) What is the worst-case number of observations necessary to determine the initial state, assuming that the system is observable?

2. Consider the linear stochastic system

$$\begin{aligned} x_{k+1} &= (A + BK)\, x_k, \quad x_0 = \text{unknown} \\ y_k &= C x_k + w_k, \end{aligned}$$

where the measurements y_k are corrupted by independent identically distributed (i.d.d.) measurement noise w_k Laplace distributed. Derive a moving horizon estimator as the maximum likelihood estimator of the state sequence $(x_0, \ldots, x_N)$ given the measurements $(y_0, \ldots, y_N)$.

3. Given the system:

$$\begin{aligned} x_1^+ &= (1 - T_s)x_1 + T_s x_2 + T_s w \\ x_2^+ &= x_2 - T_s x_1^3 + T_s w \\ y &= \sin(x_1) + 2x_2 + v \end{aligned}$$

with $T_s = 1$, unknown initial state x_0, independent identically distributed process noise $w \sim \mathcal{N}(0, 0.2^2)$ and $v \sim \mathcal{N}(0, 0.1^2)$.

Write a MATLAB program to design a discretized Kalman Filter which estimates the state x from output measurements y_k, available at every time step t_k, and simulate its closed loop behavior.

Sample Exam 1

Problem 1: 2D Systems Existence & Uniqueness for ODEs

Consider the following system

$$\begin{cases} \dot{x}_1 = \tan^{-1}(ax_1) - x_1x_2 \\ \dot{x}_2 = bx_1^2 - cx_2 \end{cases}$$

Show that the solution to the system exists and is unique. *Hint: Use the Jacobian and remember that for any function $g(x)$, we have that* $\frac{d}{dx}\tan^{-1}(g(x)) = \frac{\frac{d}{dx}g(x)}{1+g(x)^2}$.

Solution: The Jacobian of the system's vector field $f(x)$ is given by

$$\frac{\partial f(x)}{\partial x} = \begin{bmatrix} \frac{f_1(x)}{x_1} & \frac{f_1(x)}{x_2} \\ \frac{f_2(x)}{x_1} & \frac{f_2(x)}{x_2} \end{bmatrix} = \begin{bmatrix} \frac{a}{1+x_1^2} - x_2 & -x_1 \\ 2bx_1 & -c \end{bmatrix}$$

from which it follows that

$$\left\| \frac{\partial f(x)}{\partial x} \right\|_\infty \leq \max\{a + |x_2| + |x_1|, 2b|x_1|^2 + c\} \tag{11.4}$$

which is bounded on any compact set $\mathbb{R}^2$. Hence we have local existence and uniqueness of the solution.

Problem 2: Lyapunov Stability

Consider the second order system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_1 x_2 \\ \dot{x}_2 &= -x_2\end{aligned}$$

1. Compute the equilibrium point of the system. (4pts)
2. Show using a quadratic Lyapunov function that the obtained equilibrium point is asymptotically stable. Is the obtained result local or global? (6pts)

Solution:

1. Setting the derivative to 0 in both equations, we obtain

$$\begin{aligned}0 &= -x_1 + x_1 x_2 \Rightarrow x_1(x_2 - 1) = 0 \\ 0 &= -x_2 \Rightarrow x_2 = 0\end{aligned}$$

From which it follows that the equilibrium point is the origin $(0, 0)$.

2. Take $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$. Then,

$$\begin{aligned}\dot{V}(x) &= x_1\dot{x}_1 + x_2\dot{x}_2 = -x_1^2 + x_1^2 x_2 - x_2^2 \\ &= -x_1^2(1 - x_2) - x_2^2\end{aligned}$$

Therefore, $\dot{V}(x) \leq 0$, if $1 - x_2 > 0$, i.e., $\forall x_2 < 1$. Hence the system is locally asymptotically stable. To be precise, we need the condition $1 - x_2 \geq \epsilon$, with $\epsilon > 0 \Rightarrow \dot{V}(x) \leq -\epsilon x_1^2 - x_2^2$.

Problem 3: Sliding Mode Control

Consider the system

$$\begin{cases} \dot{x}_1 = x_2 + ax_1 \sin(x_1) \\ \dot{x}_2 = bx_1x_2 + u \end{cases}$$

where the parameters a and b belong to the intervals $[-1\ 2]$ and $[0\ 1]$, respectively. Design using slide mode control a continuous and globally stabilizing state feedback controller.

Hint: Use the sliding surface $s = x_2 + kx_1$, and design the parameter k so that you can achieve stability of the reduced system on this manifold.

Solution: On the manifold $s = x_2 + kx_1$, we have the following system

$$\begin{cases} \dot{x}_1 = -kx_1 + ax_1 \sin(x_1) = -x_1(k - a\sin(x_1)) \\ \dot{x}_2 = -\frac{b}{k}x_2^2 \end{cases}$$

which is stable as long as $k > 2$. This guarantees that the sliding phase is asympotically stable. It remains to show that we can design a control input that renders convergence to the sliding surface in finite time. Consider the following Lyapunov function

$$V = \frac{1}{2}s^2$$

Taking the derivative of V along the trajectory of s, we obtain

$$\begin{aligned} \dot{V} &= s\dot{s} = s(\dot{x}_2 + k\dot{x}_1) \\ &= s(bx_1x_2 + u + kx_2 + kax_1\sin(x_1)) \\ &= s(bx_1x_2 + u + kx_2 + kax_1\sin(x_1)) + su \\ &\le |s||bx_1x_2 + u + kx_2 + kax_1\sin(x_1)| + su \\ &\le |s|(|x_1x_2| + k|x_2| + 2k|x_1|) + su \end{aligned}$$

now designing

$$u = -\mu \cdot sgn(s), \quad \mu > \beta + |x_1x_2| + k|x_2| + 2k|x_1|$$

with $\beta > 0$, yields

$$\dot{V} \le -\beta|s|$$

which implies that the designed controller forces the system towards the sliding surface in finite time.

Problem 4: Feedback Linearization

Consider the system

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = a\sin(x_1) - b\cos(x_1)u \end{cases}$$

where a and b are some known positive constants.

1. Is the system feedback linearizable?
2. If so, design using feedback linearization a state feedback controller to stabilize the system at $x_1 = \theta$, for a given $\theta \in [0\ \pi/2)$.

Solution:

1. We can rewrite the equations as

   $$\dot{x} = f(x) + g(x)u$$

   where $f(x) = \begin{bmatrix} x_2 \\ a\sin(x_1) \end{bmatrix}$ and $g(x) = \begin{bmatrix} 0 \\ -b\cos(x_1) \end{bmatrix}$.

   We need to check the two conditions for full state feedback linearization:

   $$\mathcal{G}(x) = \begin{bmatrix} g(x), & ad_f g(x) \end{bmatrix}$$

   Now,

   $$\begin{aligned} ad_f g(x) = [f,g](x) &= \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x) \\ &= \begin{bmatrix} 0 & 0 \\ b\sin(x_1) & 0 \end{bmatrix} \begin{bmatrix} x_2 \\ a\sin(x_1) \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ a\cos(x_1) & 0 \end{bmatrix} \begin{bmatrix} 0 \\ -b\cos(x_1) \end{bmatrix} \\ &= \begin{bmatrix} b\cos(x_1) \\ bx_2\sin(x_1) \end{bmatrix} \end{aligned}$$

   Therefore,

   $$\mathcal{G}(x) = \begin{bmatrix} 0 & b\cos(x_1) \\ -b\cos(x_1) & bx_2\sin(x_1) \end{bmatrix}$$

   And, $det(\mathcal{G}(x)) = b^2\cos(x_1)^2 \neq 0$, $\forall x_1 \in [0\ \pi/2)$.

   We also have that the distribution $\Delta = \{g(x)\}$ being involutive. As such, the system is feedback linearizable.

2. Consider the feedback control law

$$u = \frac{1}{b\cos(x_1)}\left(a\sin(x_1) + k_1(x_1 - \theta) + k_1 x_2\right)$$

Then the closed-loop system is given by

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -k_1(x_1 - \theta) - k_1 x_2 \end{cases}$$

which has the system matrix $A = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}$ which is Hurwitz for all $k_1, k_2 > 0$.

Problem 5: Optimal Control

Consider a boat anchored at a river bank at time t=0. The river current velocity is given by $v = (v_1(x_2), 0)$, i.e. it is parallel to the shore. The boat has a motor $u = (u_1, u_2)$ that can be used to change the boat direction and speed, always acting with total effort equal to 1, i.e.,

$$\|u\|_2^2 = u_1^2 + u_2^2 = 1$$

The dynamics are given by

$$\begin{aligned} \dot{x}_1 &= v_1(x_2) + u_1 \\ \dot{x}_2 &= u_2 \end{aligned}$$

with the initial condition $x(0) = (x_1(0), x_2(0)) = (0, 0)$.

1. Assuming that the river current speed $v = (v_1(x_2), 0) = (x_2, 0)$, find an optimal control $u(t)$ that moves the boat to the maximum distance in the x_1-direction in a fixed time T.

2. How would you solve the problem if the condition is added that the boat must be at the river bank at time $2T$, i.e., $x_2(2T) = 0$?

Solution:

1. The cost to minimize is

$$J(x(0), u) = \phi(x(T)) + \int_0^T L(x, u)dt = -x_1(T)$$

subject to the boat dynamic equations and the motor effort constraint. The Hamiltonian is given by

$$H = 0 + \lambda_1(x_2 + u_1) + \lambda_2 u_2$$

The adjoint equations are given by

$$\begin{aligned} \dot{\lambda}_1 &= -\frac{\partial H}{\partial x_1} = 0 \\ \dot{\lambda}_2 &= -\frac{\partial H}{\partial x_2} = -\lambda_1 \end{aligned}$$

with the boundary conditions

$$\begin{aligned} \lambda_1(T) &= \frac{\partial \phi}{\partial x_1} = -1 \\ \lambda_2(T) &= -\frac{\partial \phi}{\partial x_2} = 0 \end{aligned}$$

The solution to the adjoint equations is given by

$$\begin{aligned} \lambda_1(t) &= -1 \\ \lambda_2(t) &= t - T \end{aligned}$$

for $t \in [0, T]$. The optimal control law $u^*(t)$ is found by mininizing the Hamiltonian function subject to the input constraints, i.e.,

$$\min_{\|u\|_2^2=1} H = \min_{\|u\|_2^2=1} \min_{\|u\|_2^2=1} H$$

which yields

$$u^*(t) = -\frac{\lambda}{\|\lambda\|_2} = \begin{bmatrix} -\frac{\lambda_1(t)}{\|\lambda(t)\|_2} \\ -\frac{\lambda_2(t)}{\|\lambda(t)\|_2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{1+(t-T)^2}} \\ \frac{T-t}{\sqrt{1+(t-T)^2}} \end{bmatrix}$$

2. By the principle of optimality, the optimal trajectory will be one where the Solution a is applied till $t = T$, followed by the symmetric one, i.e.,

$$u^*(t) = -\begin{bmatrix} \frac{1}{\sqrt{1+(t-T)^2}} \\ \frac{T-t}{\sqrt{1+(t-T)^2}} \end{bmatrix}$$

for $t \in (T, 2t]$.

Sample Exam 2

Problem 1: ODEs and Planar Systems

Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= ax_1 - x_1x_2\\ \dot{x}_2 &= bx_1^2 - cx_2\end{aligned}$$

with a, b, and c being positive constants.

1. Determine all equilibrium points of the system.
2. Compute the Jacobian of the system.
3. Determine the type of each equilibrium point.

Solution:

$$\begin{aligned}0 &= ax_1 - x_1x_2 \Rightarrow x_1^* = 0 \text{ or } x_2^* = a\\ 0 &= bx_1^2 - cx_2 \Rightarrow x_1^* = \pm\sqrt{cx_2^*/b}\end{aligned}$$

combining the two equations we obtain the following equilibria

$$x^* = (0,0), \quad x^* = (\pm\sqrt{cx_2^*/b}, a)$$

The Jacobian is computed as

$$J|_{x^*} = \begin{bmatrix} a - x_2 & -x_1 \\ 2bx_1 & -c \end{bmatrix} |_{x^*}$$

Then, we have that

$$J_{x^*=(0,0)} = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$$

which is a saddle point. And for the other two equilibria

$$J_{x^*=\pm(\sqrt{cx_2^*/b},a)} = \begin{bmatrix} 0 & \mp\sqrt{ca/b} \\ \pm 2\sqrt{cab} & -c \end{bmatrix}$$

which has the characteristic polynomial

$$\lambda^2 + c\lambda + 2ac = 0 \Rightarrow \lambda = \frac{-c \pm \sqrt{c^2 - 8ac}}{2}$$

which gives a stable focus for $c < 8a$ and a stable node if $c > 8a$.

Problem 2: Lyapunov Stability

Consider the system

$$\dot{x} = -a\left(\mathbf{I}_n + S(x) + xx^T\right)x$$

where $x \in \mathbb{R}^n$, a is a posivitve constant, $\mathbf{I}_n$ is the $n \times n$ identity matrix, $S(x)$ is a skew symmetric matrix that depends on x, i.e. $S(x)^T = -S(x)$, $\forall x$.

1. Show that the origin is globally asymptotically stable.
2. Show further that the origin is globally exponentially stable.

Solution:

1. Take a quadratic Lyapunov function $V(x) = \frac{1}{2}x^T x$, then
$$\begin{aligned} \dot{V}(x) &= x^T\left(-a\left(\mathbf{I}_n + S(x) + xx^T\right)x\right) \\ &= -a\,\|x\|_2^2 - \underbrace{ax^T S(x) x}_{=0} - ax^T x x^T x \\ &= -a\,\|x\|_2^2 - a\,\|x\|_2^4 \end{aligned}$$
hence GAS.

2. We also have that $\frac{1}{2}\|x\|_2^2 \leq V(x) \leq \frac{1}{2}\|x\|_2^2$, and $\dot{V}(x) \leq -a\,\|x\|_2^2$ from which it follows that
$$\dot{V}(x) \leq -a\,\|x\|_2^2 \leq -2aV(x) \Rightarrow V(x(t)) \leq e^{-2at}V(x(0))$$
where the last inequality follows from the comparison lemma. Finally, we have that
$$\frac{1}{2}\|x(t)\|_2^2 \leq V(x(t)) \leq e^{-2at}V(x(0)) \leq e^{-2at}\frac{1}{2}\|x(0)\|_2^2 \Rightarrow \|x(t)\|_2 \leq e^{-at}\|x(0)\|_2$$
and exponential stability follows.

Problem 3: Radial Unboundedness of Lyapunov Functions

Consider the following positive definite function

$$V(x) = \frac{(x_1 + x_2)^2}{(x_1 + x_2)^2 + 1} + (x_1 - x_2)^2$$

When checking radial unboundedness of the function $V(x)$, we may think that it is sufficient to check what happens to $V(x)$ as $\|x\| \to \infty$ along the principal axes. However, this is not true!

1. Show that $V(x)$ is positive definite.
2. Show that $V(x) \to \infty$ as $\|x\| \to \infty$ along the principal axes/lines $x_1 \equiv 0$ or $x_2 \equiv 0$.
3. Show that $V(x)$ is *not* radially unbounded?
4. Why do we need radial unboundedness in Lyapunov's theory?

Solution:

1. We can show that $V(x)$ is positive definite via the following argument:

$$\begin{aligned} V(x) &= \frac{(x_1 + x_2)^2 + (x_1 - x_2)^2 + (x_1 - x_2)^2 (x_1 + x_2)^2}{(x_1 + x_2)^2 + 1} \\ &= \frac{x_1^2 + x_2^2 + (x_1 - x_2)^2 (x_1 + x_2)^2}{(x_1 + x_2)^2 + 1} \geq \frac{x_1^2 + x_2^2}{(x_1 + x_2)^2 + 1} > 0 \end{aligned}$$

2. Along $x_1 = 0$, we have that

$$V(0, x_2) = \frac{x_2^2}{x_2^2 + 1} + x_2^2 \geq x_2^2 \to \infty, \text{ as } |x_2| \to \infty$$

   The same holds for the case of $x_2 = 0$.

3. No, because for $x_1 = x_2 = \xi$, we have that

$$V(x) = \frac{4\xi^2}{4\xi^2 + 1} \to 1, \text{ as } |\xi| \to \infty$$

4. For showing global asymptotic stability

Problem 4: Sliding Mode Control

Consider the nonlinear system

$$\dot{x}_1 = x_2 + x_1 e^{-|x_2|}$$
$$\dot{x}_2 = \psi(x_1, x_2) + u$$

where the function ψ is unknown and satisfies $|\psi(x_1, x_2)| \leq \|x\|^4, \forall x \in \mathbb{R}^2$.

1. Design as state feedback sliding mode controller to globally stabilize the system to the origin. You need to prove that you controller is actually stabilizing using a proper Lyapunov function.

2. Is the resulting controller continuous? If not, can you make it continuous? Hint: remember the discussion in the lecture regarding alternative choices to the *sign* function.

Solution:

1. Consider the manifold $s = x_2 - kx_1$. On the manifold, we have that

   $$\dot{x}_1 = -kx_1 + x_1 e^{-|x_2|} = -(k - e^{-|x_2|})x_1$$

   which is exponentially asymptotically stable for any choice of $k > 1$. It remains to show that we can drive the system towards the mainfold in finite time. First, we have that

   $$\dot{s} = \dot{x}_2 - k\dot{x}_1 = \psi(x_1, x_2) + u - kx_2 - kx_1 e^{-|x_2|}$$

   Consider the Lyapunov function candidate $V = \frac{1}{2}s^2$, then

   $$\dot{V} = s\dot{s} = s(\psi(x_1, x_2) + u - kx_2 - kx_1 e^{-|x_2|}) \leq |s| \|x\|^4 + s(u - kx_2 - kx_1 e^{-|x_2|})$$

   taking $u = kx_2 + kx_1 e^{-|x_2|} - (\|x\|^4 + \beta)\text{sgn}(s)$ we obtain

   $$\dot{V} \leq -\beta |s|$$

   which implies that the system using the sliding mode control above converges to the manifold in finite time.

2. In order to make the sliding mode controller continuous, we can use the following alternative input

   $$u = kx_2 + kx_1 e^{-|x_2|} - (\|x\|^2 + \beta) sat\left(\frac{s}{\epsilon}\right), \quad \epsilon > 0 \tag{11.5}$$

Problem 5: Feedback Linearization

Consider the nonlinear system

$$\begin{aligned}
\dot{x}_1 &= -x_1 - x_2 + \psi(x_2)\\
\dot{x}_2 &= x_1 + \psi(x_2) + u\\
y &= x_2
\end{aligned}$$

where the nonlinear function ψ is continuously differentiable and satisfies:

$$\psi(0) = 0, \quad \frac{d\psi(0)}{dx_2} = 1, \quad 0 < \frac{d\psi(x_2)}{dx_2} < 1, \forall x_2 \neq 0$$

1. Is the system input-output linearizable? What is the relative degree?
2. Is it full state feedback linearizable? (Hint: you need to check some conditions involving Lie brackets)

Solution:

1.
$$\dot{y} = \dot{x}_2 = x_1 + \psi(x_2) + u$$
hence the system has relative degree $\rho = 1$ and is input-output linearizable.

2. We can rewrite the equations as
$$\dot{x} = f(x) + g(x)u$$
where $f(x) = \begin{bmatrix} -x_1 - x_2 + \psi(x_2) \\ x_1 + \psi(x_2) \end{bmatrix}$ and $g(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
We need to check the two conditions for full state feedback linearization:
$$\mathcal{G}(x) = \begin{bmatrix} g(x), & ad_f g(x) \end{bmatrix}$$
Now,
$$\begin{aligned}
ad_f g(x) = [f,g](x) &= \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x)\\
&= -\begin{bmatrix} -1 & -1 + \frac{d\psi(x_2)}{dx_2} \\ 1 & \frac{d\psi(x_2)}{dx_2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - \frac{d\psi(x_2)}{dx_2} \\ -\frac{d\psi(x_2)}{dx_2} \end{bmatrix}
\end{aligned}$$

Therefore,

$$\mathcal{G}(x) = \begin{bmatrix} 0 & 1 - \frac{d\psi(x_2)}{dx_2} \\ 1 & -\frac{d\psi(x_2)}{dx_2} \end{bmatrix}$$

And, $det(\mathcal{G}(x)) = \frac{d\psi(x_2)}{dx_2} - 1 \neq 0$, $\forall x_2 \neq 0$.

We also have that the distribution $\Delta = \{g(x)\}$ being involutive. As such, the system is feedback linearizable.

Problem 6: Stability of Model Predictive Control Designs

Consider

1. Discrete time, time invariant system

$$x(k+1) = f(x(k), u(k)), \quad f(0,0) = 0, \quad x(0) = x_0$$

2. Objective function

$$J[x_0, u(\cdot)][= \sum_{l=k}^{k+N} L(x(l), u(l))$$

Consider an MPC algorithm for this system, where

- The problem contains a terminal constraint $x(k+N) = 0$
- The function L in the cost function is positive definite in both arguments.

Prove that if the optimization problem is feasible at time k, then the coordinate origin is a stable equilibrium point.

Solution: Clearly, the origin is an equilibrium point of this discrete time system because $f(0,0) = 0$. In order to prove stability of the origin, we use the Lyapunov result on stability of discrete time systems introduced in the Lyapunov stability lecture. Indeed, consider the function

$$V(x) = J^\star(x),$$

where $J^\star$ denotes the performance index evaluated at the optimal trajectory.

We note that:

- $V(0) = 0$
- $V(x)$ is positive definite.
- $V(x(k+1)) - V(x(k)) < 0$. The later is seen by noting the following argument. Let

$$u_k^\star(l), \quad l = k : k+N$$

be the optimal control sequence at time k. Then, at time $k+1$, it is clear that the control sequence $u(l)$, $l = k+1 : N+1$, given by

$$\begin{aligned} u(l) &= u_k^\star(l), \quad l = k+1 : N \\ u(N+1) &= 0 \end{aligned}$$

generates a feasible albeit suboptimal trajectory for the plant. Then, we observe

$$V(x(k{+}1)) - V(x(k)) < J(x(k{+}1), u(\cdot)) - V(x(k)) = -L(x(k), u^\star(k)) < 0$$

which proves the theorem.

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