1 Introduction

We shall study a special class of systems called dissipative systems. Intuitively, we can think of systems interacting with the surrounding via some input / output ports, exchanging power with the surrounding, storing some energy and dissipating some. For example, electrical systems exchange power with the surrounding via an inner product between the current and voltage, i.e., $P_e = v^T i$. Energy in these systems may be stored in capacitors and/or inductors in the form of voltage or current, respectively. Another example is mechanical systems; these systems may be supplied with linear and/or rotational power, i.e., $P_m = \omega^T T$ or $P_m = v^T F$, where $v$ is the velocity, $F$ is the applied force, $\omega$ is the rotational speed, and $T$ is the applied torque. This supply may be stored in the form of potential and/or kinetic energy. In what follows, we shall provide a solid foundation for such supply and storage concepts that allows us to describe systems from an input-output perspective.

Example 1 Consider again the mass-spring system from the lecture on Lyapunov Stability Theory, as shown in Figure [4], with an extra input force $u$. Define $x_1 = x$ and $x_2 = \dot{x}$, the position and speed, respectively, let $M = 1$, and assume that we measure the output $y = \dot{x} = x_2$. Then, we can write the dynamical equations of the system as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & x_2 \\ -F(x_1) - \delta x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = x_2 \quad (1)$$

where $F(x_1)x_1 > 0$, $\forall x \neq 0$. How can we make a statement about the stability
of the system? We consider the energy that is stored in the system

\[ S(x) = \int_{0}^{x_1} F(\xi) \, d\xi + \frac{1}{2} x_2^2 \]  

The derivative of \( S(x) \) along the trajectories of (1) is given by

\[ \dot{S}(x) = -\delta x_2^2 + x_2 u = -\delta y^2 + yu \]  

Assume that at \( t = 0, \ x_1 = x_2 = 0 \). Then

\[
S(t) = \int_{0}^{t} \dot{S}(\tau) \, d\tau = \int_{0}^{t} \left( -\rho y^2(\tau) + y(\tau)u(\tau) \right) \, d\tau \\
\leq \int_{0}^{t} y(\tau) \, u(\tau) \, d\tau \leq \int_{0}^{t} |y(\tau)| \, |u(\tau)| \, d\tau
\]

Therefore, we can see that if \( u \) and \( y \) are bounded signals (in some sense), then \( S \) is also bounded. Due to the properties of the function \( S \), we can then limit the state. This reasoning helps us to go from and input-output boundedness property to an internal state boundedness property.

\section{Dissipative Systems}

Consider the nonlinear state-space system, given by

\[
\begin{align*}
\dot{x} &= f(x, u) \\
y &= h(x, u)
\end{align*}
\]  

where \( x \in \mathcal{X} \subseteq \mathbb{R}^n \), \( u \in \mathcal{U} \subseteq \mathbb{R}^m \), and \( y \in \mathcal{Y} \subseteq \mathbb{R}^p \). Associated with (4), we have the following supply rate function

\[ w(u, y) : \mathcal{U} \times \mathcal{Y} \to \mathbb{R} \]
Definition 1 The state-space system (4) is said to be dissipative with respect to the supply rate \( w(u,y) \), if there exists a function \( S : X \rightarrow \mathbb{R}_{\geq 0} \), called the storage function, such that \( \forall x_0 \in X, \forall t_1 > t_0 \), and all input functions \( u \) the following dissipation inequality holds

\[
S(x(t_1)) \leq S(x(t_0)) + \int_{t_0}^{t_1} w(u(t), y(t)) \, dt
\]  

where \( x(t_0) = x_0 \), and \( x(t_1) \) is the state of supply rate at time \( t_1 \) resulting from initial condition \( x_0 \) and the input function \( u(t) \).

The dissipation inequality expresses the concept that the stored energy \( S(x(t_1)) \) of the system (4) at any time \( t_1 \) is at most equal to the sum of the stored energy \( S(x(t_0)) \) present at the time \( t_0 \) and the total energy \( \int_{t_0}^{t_1} w(u(t), y(t)) \, dt \) which is supplied externally during the time interval \([t_0, t_1]\). Hence, as the name suggests, dissipative systems cannot internally create energy, but can rather either store it or dissipate it.

We shall study next a special type of storage functions.

Theorem 1 Consider the system (4) with a supply rate \( w \). Then, it is dissipative with respect to \( w \) if and only if the available storage function

\[
S_a(x) = \sup_{u(\cdot), T \geq 0} \left( -\int_0^T w(u(t), y(t)) \, dt \right), \quad x(0) = x \tag{7}
\]

is well defined, i.e., \( S_a(x) < \infty \), \( \forall x \in X \). Moreover, if \( S_a(x) < \infty \), \( \forall x \in X \), then \( S_a \) is itself a storage function, and it provides a lower bound on all other storage functions, i.e., for any other storage function \( S \)

\[
S_a(x) \leq S(x)
\]

Proof: First note that \( S_a \geq 0 \) (why?). Suppose that \( S_a \) is finite. Compare now \( S_a(x(t_0)) \) with \( S_a(x(t_1)) - \int_{t_0}^{t_1} w(u(t), y(t)) \, dt \), for a given \( u : [t_0, t_1] \rightarrow \mathbb{R}^m \) and resulting state \( x(t_1) \). Since \( S_a \) is given as the supremum over all \( u(.) \) it immediately follows that \( S_a(x(t_0)) \geq S_a(x(t_1)) - \int_{t_0}^{t_1} w(u(t), y(t)) \, dt \) and thus \( S_a \) is a storage function, proving that the system (4) is dissipative with respect to the supply rate \( w \).

In order to show the converse, assume that (4) is dissipative with respect to \( w \). Then there exists a storage function \( S \geq 0 \) such that for all \( u(.) \)

\[
S(x(0)) + \int_0^T w(u(t), y(t)) \, dt \geq S(x(T)) \geq 0
\]
which shows that
\[ S(x(0)) \geq \sup_{u(\cdot), T \geq 0} \left( - \int_0^T s(u(t), y(t)) \, dt \right) = S_a(x(0)) \]
proving finiteness of \( S_a \), as well as \( S_a(x) \leq S(x) \).

\[ \square \]

Remark 1 Note that in linking dissipativity with the existence of the function \( S_a \), we have removed attention from the satisfaction of the dissipation inequality to the existence of the solution to an optimization problem.

Remark 2 The quantity \( S_a \) can be interpreted as the maximal energy which can be extracted from the system \([4]\) starting at an initial condition \( x_0 \).

Consider the dissipation inequality in the limit where \( t_1 \to t_0 \). Then it may be seen that satisfaction of the dissipation inequality is equivalent to fulfilling the partial differential equation (assuming \( S \) is differentiable)
\[ \dot{S}(x) = \frac{\partial S(x)}{\partial x} f(x, u) \leq w(u, h(x, u)), \forall x \in \mathbb{R}^n, u \in \mathbb{R}^m \]
(8)
This version of the dissipation property is called the differential dissipation inequality. Having this differential version, we can establish a connection to what we have already seen in the lecture on Lyapunov stability theory. But before we do that, let us first see what the rate at which the system \([4]\) dissipates energy is, this is given by the following definition.

Definition 2 The function \( d : \mathcal{X} \times \mathcal{U} \to \mathbb{R} \) is the dissipation rate of the dissipative system \([4]\) with supply rate \( w \) and storage function \( S \), if \( \forall t_0, t_1 \in \mathbb{R}^+, x_0 \in \mathcal{X} \), and \( u \in \mathcal{U} \), the following equality holds
\[ S(x(t_0)) + \int_{t_0}^{t_1} (w(t) + d(t)) \, dt = S(x(t_1)) \]
Of course, we would require that \( d \) is non-negative in order to obtain dissipation!
Lemma 1 Let $S$ be a continuously differentiable storage function for the system (4) and assume that the supply rate $w$ satisfies

$$w(0, y) \leq 0, \quad \forall y \in \mathcal{Y}$$

Let the origin $x = 0$ be a strict local minimum of $S(x)$. Then $x = 0$ is a locally stable equilibrium for the unforced system $\dot{x} = f(x, 0)$ and $V(x) = S(x) - S(0) \geq 0$ is a local Lyapunov function.

Proof: Consider the Lyapunov function candidate $V(x) = S(x) - S(0)$, which is positive definite (why?). Under $u = 0$, we have that

$$\dot{V}(x) = \dot{S}(x) = \frac{\partial S(x)}{\partial x} f(x, 0) \leq w(0, y) \leq 0$$

□

We can also show that the feedback interconnection of dissipative systems is stable.

![Feedback interconnection of dissipative systems](image.png)

Figure 2: Feedback interconnection of dissipative systems

Lemma 2 Consider the two systems

$$\begin{align*}
\Sigma_i \quad & \left\{ \begin{array}{l}
\dot{x}_i = f_i(x_i, u_i) \\
y_i = h(x_i, u_i)
\end{array} \right.
\end{align*}$$

(9)

connected in feedback as shown in Figure 2. Assume that both systems are dissipative with respect to supply rates $w_i$ and positive definite storage functions $S_i$. Assume further that

$$w_1(u, y) + w_2(y, -u) = 0, \quad \forall u, y$$

Then, the feedback system is stable.
Proof: Consider the Lyapunov function candidate $V(x) = S_1(x_1) + S_2(x_2)$.

\[ \dot{V}(x_1, x_2) = \dot{S}(x_1) + \dot{S}(x_2) \leq w_1(u_1, y_1) + w_2(u_2, y_2) \\
= w_1(-y_2, y_1) + w_2(y_1, y_2) = 0 \]

and the result follows.

□

Lemma 2 is an extremely powerful one and captures many of the stability results in the frequency domain.

Example 2 Consider the RLC circuit shown in Figure 3. Define the states $x_1 = v_c$ and $x_2 = i$, the input $u = v_{in}$ and the output $y = x_2$. The state-space model is given by

\[ \dot{x} = Ax + Bu = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u, \quad y = Hx = [0 \ 1] x \]

For simplicity, we shall take $L = C = R = 1$. The energy storage in the system is captured by the inductor and the capacitor, i.e., the storage function in the system is given by

\[ S(x) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 \]

which is positive definite, and the supply rate to the system is given by

\[ w(u, y) = uy \quad (10) \]

which is the power injected (extracted) into (from) the system. Now,

\[ \dot{S}(x) = x_1 \dot{x}_1 + x_2 \dot{x}_2 = -x_2^2 + uy \leq w(u, y) \quad (11) \]
Hence, the system is dissipative. Now, if we let $u = 0$, i.e., we short-circuit the terminals, we obtain
\[
\dot{S}(x) = -x_2^2 \leq 0
\]
and the energy that was initially stored in the capacitor and/or inductor is cycled in the system and dissipated in the resistor. We can show in this case, that using the so called strong Lyapunov function with $u = 0$ we get asymptotic stability. Consider the Lyapunov function candidate
\[
V(x) = \frac{1}{2} x_1^2 + \frac{1}{2} x_2^2 + \epsilon x_1 x_2
\]
where $\epsilon \in (0, 1)$ (show that this choice of $\epsilon$ makes $V(x)$ positive definite!). Taking the derivative of $V$ along the trajectories of the closed-loop system, we obtain
\[
\dot{V}(x) = x_1 \dot{x}_1 + x_2 \dot{x}_2 + \epsilon x_1 \dot{x}_2 + \epsilon x_2 \dot{x}_1
\]
\[
= x_1 x_2 + x_2(-x_1 - x_2) + \epsilon x_1(-x_1 - x_2) - \epsilon x_2^2
\]
\[
= -\epsilon x_1^2 - (1 + \epsilon) x_2^2 - \epsilon x_1 x_2
\]
\[
\leq -\epsilon x_1^2 - (1 + \epsilon) x_2^2 + \epsilon^2 x_1^2 + \frac{1}{2} x_2^2
\]
\[
= -\epsilon \left( 1 - \frac{1}{2} \epsilon \right) x_1^2 - \left( \frac{1}{2} + \epsilon \right) x_2^2 < 0
\]
for our choices of $\epsilon$. This way we have avoided using LaSalle’s invariance principle, which we could have used instead to show asymptotic stability of the system.

3 Passive Systems

Passive systems are a special subclass of dissipative systems, and they have a special type of supply rate, given by
\[
w(u, y) = u^T y
\]
with the implicit condition that the number of inputs and outputs is the same, i.e., $u, y \in \mathbb{R}^p$. We can also differentiate among various types of passive dynamical systems according to the following definition:

**Definition 3** A state space system [4] is called
1. passive if it is dissipative with respect to the supply rate \( w(u,y) = u^T y \)

2. lossless if \( \dot{S}(x) = u^T y \)

3. input-feedforward passive if it is dissipative with respect to the supply rate \( w(u,y) = u^T y - u^T \varphi(u) \) for some function \( \varphi \)

4. input strictly passive if it is dissipative with respect to the supply rate \( w(u,y) = u^T y - u^T \varphi(u) \) and \( u^T \varphi(u) > 0, \forall u \neq 0 \)

5. output feedback passive if it is dissipative with respect to the supply rate \( w(u,y) = u^T y - y^T \rho(y) \) for some function \( \rho \)

6. output strictly passive if it is dissipative with respect to the supply rate \( w(u,y) = u^T y - y^T \varphi(y) \) for some function \( y^T \rho(y) > 0, \forall y \neq 0 \)

7. strictly passive if it is dissipative with respect to the supply rate \( w(u,y) = u^T y - \psi(x) \) for some positive definite function \( \psi \)

**Example 3** Consider an integrator model given by

\[
\dot{x} = u, \quad y = x
\]  

with the supply rate \( w(u,y) = uy \). Take the storage function \( S(x) = \frac{1}{2}x^2 \). The derivative \( \dot{S}(x) = x\dot{x} = uy \), and hence the system is lossless.

**Example 4** Now assume that instead of the pure integrator in Example 3 with transfer function \( \frac{1}{s} \), we consider the low pass filter \( \frac{1}{s+1} \), which has the state-space representation

\[
\dot{x} = -x + u, \quad y = x
\]  

Consider the storage function \( S(x) = \frac{1}{2}x^2 \). The derivative \( \dot{S}(x) = x\dot{x} = -x^2 + uy \). Hence the system is strictly dissipative, and globally asymptotically stable for \( u = 0 \) (convince yourself of the latter fact). This is one of the main reasons that we would not implement ‘pure’ integrators in embedded systems, but instead opt for low pass filters due to their inherent stable behavior. Finally, notice from Figure 4 that both systems behave similarly for high frequencies.
3.1 Characterizations of Passivity for Linear Systems

Passive systems are particularly interesting in the linear systems case, because we can get characterizations of passivity both in the frequency domain and in the time domain. In the frequency domain, we think of transfer functions and we can relate passivity to certain conditions being satisfied for the transfer function.

**Definition 4** A $p \times p$ proper transfer function matrix $G(s)$ is called positive real if all the following conditions are satisfied:

1. the poles of all elements of $G(s)$ have none-positive real part

2. for all real frequencies $\omega$ for which $j\omega$ is not a pole of any element of $G(s)$, the matrix $G(j\omega) + G(-j\omega)^T$ is positive semi-definite

3. any pure imaginary pole $j\omega$ of any element of $G(s)$ is a simple pole and the residue matrix $\lim_{s \to j\omega}(s - j\omega)G(s)$ is positive semidefinite Hermitian

$G(s)$ is called strictly positive real if $G(s - \epsilon)$ is positive real for some $\epsilon > 0$.
Remark 3  For \( p = 1 \), the second condition of Definition 4 reduces to \( \text{Re}[G(jw)] \geq 0, \forall w \in \mathbb{R} \). Moreover, this condition is satisfied only if the relative degree of the transfer function \( G(s) \) is at most one.

The positive real property of transfer matrices can be equivalently characterized as follows.

Lemma 3  Let \( G(s) \) be a proper rational \( p \times p \) transfer matrix. Suppose that \( \det(G(s) + G(s)^T) \) is not equivalent to zero for all \( s \). Then \( G(s) \) is strictly positive real if and only if the following three conditions are satisfied

1. \( G(s) \) is Hurwitz,
2. \( G(j\omega) + G(-j\omega)^T \) is positive definite \( \forall \omega \in \mathbb{R} \)
3. either \( G(\infty) + G(\infty)^T \) is positive definite or it is positive semidefinite and \( \lim_{\omega \to \infty} \omega^2 M^T(G(j\omega) + G(-j\omega)^T)M \) is positive definite for any full rank \( p \times (p - q) \) matrix \( M \) such that \( M^T(G(\infty) + G(\infty)^T)M = 0 \), where \( q = \text{rank}(G(\infty) + G(\infty)^T) \).

Example 5  Recall the RLC circuit in Example 2. The transfer function (for \( R = L = C = 1 \)) is given by

\[
G(s) = H(sI - A)^{-1}B = [0 \ 1] \begin{bmatrix} s & -1 \\ 1 & s + 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{s}{s^2 + s + 1} \text{ (16)}
\]

Let us check if it is positive real. Note: We can show that condition 2. of Lemma 3 does not hold, and hence the transfer function is not strictly positive real.

The poles of \( G(s) \) are given by \( s_i = \frac{-1 \pm j\sqrt{2}}{2} \), \( i = 1, 2 \), thus \( G(s) \) is Hurwitz. We also have that

\[
\text{Re}[G(jw)] = \frac{w^2}{(1 - w^2)^2 + w^2} \geq 0, \forall w \in \mathbb{R}
\]

Finally, we have no pure imaginary poles. Therefore, we can conclude that the transfer function (16) is positive real.

One can also look at the state-space system directly and conclude that the transfer function is actually positive real, as shown by the celebrated KYP lemma below.
Lemma 4 (Kalman-Ykubovich-Popov) Consider the \( m \times m \) transfer function matrix \( G(s) = C(sI - A)^{-1}B + D \), where the pair \((A, B)\) is controllable and the pair \((A, C)\) is observable. \( G(s) \) is strictly positive real if and only if there exist matrices \( P = P^T > 0 \), \( L \), and \( W \), and \( \epsilon > 0 \) such that the following equalities hold

\[
PA + A^T P = -L^T L - \epsilon P \tag{17}
\]
\[
PB = C^T - L^T W \tag{18}
\]
\[
W^T W = D + D^T \tag{19}
\]

Remark 4 If \( \epsilon = 0 \) in Lemma 3, then the transfer function \( G(s) \) is simply positive real.

Example 6 Consider again the RLC system in Example 2. Since there is no direct feedthrough in the system, we have \( D = 0 \) and by (19), \( W = 0 \). As such, (18) implies \( P \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow P = \begin{bmatrix} p & 0 \\ 0 & 1 \end{bmatrix}, p > 0 \). Plugging \( P \) into (17)

\[
PA + A^T P = -\begin{bmatrix} 0 & \frac{1-p}{2} \\ \frac{1-p}{2} & 1 \end{bmatrix} \leq 0 \Leftrightarrow p = 1 \Rightarrow L = \begin{bmatrix} 0 \\ \sqrt{2} \end{bmatrix}
\]

Therefore, again we reach the same conclusion that the system is passive. However, we cannot satisfy the condition (17) with \( \epsilon > 0 \) and hence the system is not strictly passive.

Finally, we are ready to state the connection between positive realness and passivity for Linear Time-Invariant (LTI).

Theorem 2 The LTI system

\[
\dot{x} = Ax + Bu
\]
\[
y = Cx + Du
\]

with the corresponding transfer matrix \( G(s) = C(sI - A)^{-1}B + D \) is

- passive if \( G(s) \) is positive real
- strictly passive, if \( G(s) \) is strictly positive real.
Proof: Consider the storage function $S(x) = \frac{1}{2}x^TPx$. Then,

$$
\dot{S}(x) = x^TP(Ax + Bu) = \frac{1}{2}x^T(PA + A^TP)x + x^TPBu
$$

$\quad = \frac{1}{2}x^T(PA + A^TP)x + x^TC^Tu - x^TL^Tu
$$

$\quad = \frac{1}{2}x^T(PA + A^TP)x + y^Tu - \frac{1}{2}u^T(D^T + D)u - x^TL^Tu
$$

$\quad = \frac{1}{2}x^T(PA + A^TP)x + y^Tu - \frac{1}{2}u^TW^TWu - x^TL^Tu
$$

$\quad = \frac{1}{2}x^T(PA + A^TP)x + y^Tu - \frac{1}{2}x^TL^Lx
$$

$\quad = \frac{1}{2}x^T(PA + A^TP)x + y^Tu - \frac{1}{2}(Wu + Lx)^T(Wu + Lx) + \frac{1}{2}x^TTL^Lx
$$

$\quad \leq \frac{1}{2}x^T(PA + A^TP)x + y^Tu + \frac{1}{2}x^TTL^Lx
$$

$\quad = \begin{cases} 
-\epsilon \frac{1}{2}x^TPx + y^Tu, & \epsilon > 0, \text{ strictly passive} \\
 y^Tu, & \epsilon = 0, \text{ passive}
\end{cases}$

\[\square\]

### 3.2 Stability of Passive Systems

Let us consider again the nonlinear system (4), where $f : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^n$ is locally Lipschitz, $h : \mathbb{R}^n \times \mathbb{R}^p \to \mathbb{R}^p$ is continuous, with $f(0,0) = 0$ and $h(0,0) = 0$.

**Lemma 5** Assume that the system (4) is passive with a positive definite storage function $S(x)$, then the origin of $\dot{x} = f(x,0)$ is stable.

**Proof:** Take $V(x) = S(x)$ as Lyapunov function candidate. Then $\dot{V}(x) \leq u^Ty = 0$, and stability follows.

\[\square\]

We can further strengthen the previous Lemma, as follows.

**Lemma 6** Assume that the system (4) is strictly passive with some storage function $S(x)$, then the origin of $\dot{x} = f(x,0)$ is asymptotically stable. Furthermore, if $S(x)$ is radially unbounded, then the origin is globally asymptotically stable.
Proof: Let \( V(x) = S(x) \) be a Lyapunov function candidate. Since the system is strictly passive with a storage function \( V(x) \), it follows that (for \( u = 0 \))

\[
\dot{V}(x) \leq -\psi(x) + u^T y = -\psi(x)
\]

Now consider any \( x \in \mathbb{R}^n \) and let \( \phi(t, x) \) be the solution to the differential equation \( \dot{x} = f(x, 0) \), starting at \( x \) and time \( t = 0 \). As such, we have that

\[
V(\phi(t, x)) - V(x) \leq -\int_0^t \psi(\phi(\tau, x)) d\tau \quad \forall t \in [0, \delta] \tag{20}
\]

for some positive constant \( \delta \). Since \( V(\phi(t, x)) \geq 0 \), then

\[
V(x) \geq \int_0^t \psi(\phi(\tau, x)) d\tau
\]

Suppose now that there exists some \( \bar{x} \neq 0 \) such that \( V(\bar{x}) = 0 \). This implies that

\[
\int_0^t \psi(\phi(\tau, \bar{x})) d\tau = 0, \forall t \in [0, \delta] \Rightarrow \psi(\phi(\tau, \bar{x})) \equiv 0 \Rightarrow \bar{x} = 0
\]

which gives a contradiction. Hence, \( V(x) > 0 \) for all \( x \neq 0 \), i.e., positive definite. Combining this with the \( \dot{V}(x) \leq -\psi(x) \), yields asymptotic stability of the origin. Finally, if \( V(x) \) is radially unbounded, we obtain asymptotic stability.

\[\square\]

We shall look at some of the stability properties of passive systems, when connected in a feedback structure as shown in Figure 5.

\[\text{Figure 5: Feedback interconnection of passive systems}\]

**Theorem 3** The feedback connection of two passive systems is passive.
Proof: Let $S_i(x_i)$ be the storage function of system $\Sigma_i$, $i = 1, 2$. Since both systems are passive, we have that

$$\dot{V}_i(x_i) \leq e_i^T y_i$$

Using the feedback structure in Figure 5, we have that

$$\dot{S}(x) = \dot{S}_1(x_1) + \dot{S}_2(x_2) \leq e_1^T y_1 + e_2^T y_2 \leq (u_1 - y_2)^T y_1 + (u_2 + y_2)^T y_2 = u_1^T y_1 + u_2^T y_2 = u^T y$$

and the result follows. Note that if any of the two systems are memoryless, i.e., $y_i = h_i(u_i)$, then the corresponding storage function can be taken to be 0.

\[ \square \]

3.3 Passivity-Based Control

Having done some analysis on passive systems, we can now show a glimpse of how this theory can be used to design the so-called passivity-based controllers. Consider the dynamical system

$$\dot{x} = f(x, u)$$
$$y = h(x)$$

with the usual Lipschitz assumption on $f$ and continuity of $h$. Moreover, assume that $f(0, 0) = 0$ and $h(0) = 0$.

Definition 5 The system (23) is called zero-state observable, if no solution of the unforced system $\dot{x} = f(x, 0)$ can stay identically in the set $\{h(x) = 0\}$ other than the trivial solution $x(t) \equiv 0$.

Theorem 4 Assume that the system (23) is

1. passive with a radially unbounded positive definite storage function, and
2. is zero-state observable,

then the origin can be globally stabilized with a control law $u = -\phi(y)$, where $\phi$ is any locally Lipschitz function such that $\phi(0) = 0$, and $y^T \phi(y) > 0$, $\forall y \neq 0$. 

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Proof: Let $V(x)$ be a storage function of the system and use it as a candidate Lyapunov function for the closed-loop system with $u = -\phi(y)$. We have that 

$$
\dot{V}(x) \leq u^T y = -\phi(y)^T y \leq 0
$$

Hence, $\dot{V}(x) \leq 0$, and $\dot{V}(x) = 0$ if and only if $y = 0$. By zero-state observability, the only solution that can stay in the set $\{y = h(x) = 0\}$ is the trivial solution $x(t) \equiv 0$, and we can conclude using LaSalle’s invariance principle that the origin is globally asymptotically stable.

This last theorem is very useful when designing control laws for a large number of electrical and mechanical systems. Moreover, instead of starting with systems for which the origin is open-loop stable, we can design control laws that convert a nonpassive system into a passive one, a technique known as feedback passivation.

Example 7 Consider the system

\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_1^2 + u \\
y &= x_2
\end{align*}

The open-loop system ($u = 0$) is unstable, and hence the system is not passive. However, we can design the control law 

$$u = -x_2^2 - x_1^3 + v$$

that yields the system passive with respect to the supply rate $w = vy$. Let $S(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2$, then 

$$\dot{S}(x) = -x_2^2 + vy$$

and the system is passive. Noting that $v = 0$ and $y(t) \equiv 0$ imply $x(t) \equiv 0$. Therefore, all the conditions of Theorem 4 are satisfied, and accordingly we can design a globally stabilizing control law, for example $v = -kx_2$ or $v = -\tan^{-1}(x_2)$ and any $k > 0$.

For further reading on the subject, please see [1, 2, 3].
References

