Chapter 2
Problem 1 - Solution

Let \( \lambda := [a \ b \ c]^\top \) and consider the vector field \( f(t, x, \lambda) := \begin{bmatrix} \tan^{-1}(ax_1) - x_1x_2 \\ bx_1^2 - cx_2 \end{bmatrix} \).

The Jacobian matrices \( \frac{\partial f(t, x, \lambda)}{\partial x} \) and \( \frac{\partial f(t, x, \lambda)}{\partial \lambda} \) are given by

\[
\frac{\partial f(t, x, \lambda)}{\partial x} = \begin{bmatrix} a + \frac{ax_1^2}{1+(ax_1)^2} - x_2 & -x_1 \\ \frac{ax_1}{2bx_1} & -c \end{bmatrix}, \quad \frac{\partial f(t, x, \lambda)}{\partial \lambda} = \begin{bmatrix} \frac{x_1}{1+(ax_1)^2} & 0 & 0 \\ 0 & x_1^2 & -x_2 \end{bmatrix}.
\]

Let \( \lambda_0 := [a_0 \ b_0 \ c_0] \) be the nominal value of \( \lambda \) and denote by \( x(t, \lambda_0) \) the unique solution of the nominal state equation

\[
\dot{x} = f(t, x, \lambda_0), \quad x(t_0) = x_0.
\]

The sensitivity equation is given by

\[
\dot{S}(t) = A(t, \lambda_0)S(t) + B(t, \lambda_0), \quad S(t_0) = 0,
\]

where

\[
A(t, \lambda_0) = \left[ \frac{\partial f(t, x, \lambda)}{\partial x} \right]_{x(t, \lambda_0), \lambda_0}, \quad B(t, \lambda_0) = \left[ \frac{\partial f(t, x, \lambda)}{\partial \lambda} \right]_{x(t, \lambda_0), \lambda_0}.
\]

One gets that

\[
A(t, \lambda_0) = \begin{bmatrix} 1 + \frac{x_1(t, \lambda_0)^2}{1+x_1(t, \lambda_0)^2} - x_2(t, \lambda_0) \\ 0 & -x_1(t, \lambda_0) \end{bmatrix}, \quad B(t, \lambda_0) = \begin{bmatrix} \frac{x_1(t, \lambda_0)}{1+x_1(t, \lambda_0)^2} & 0 & 0 \\ 0 & \frac{x_1(t, \lambda_0)^2}{1+x_1(t, \lambda_0)^2} & -x_2(t, \lambda_0) \end{bmatrix}.
\]

In this procedure, we need to solve first the nominal state equation and then the linear time-varying sensitivity equation. This is usually done numerically. Another equivalent approach is to solve these two equations simultaneously. To this aim, we set

\[
S = \begin{bmatrix} x_3 & x_5 & x_7 \\ x_4 & x_6 & x_8 \end{bmatrix}.
\]
and we consider the augmented equation

\[
\begin{align*}
\dot{x}_1 &= \tan^{-1}(x_1) - x_1 x_2 \\
\dot{x}_2 &= -x_2 \\
\dot{x}_3 &= \left(\frac{1}{1 + x_1^2} - x_2\right) x_3 - x_1 x_4 + \frac{x_1}{1 + x_1^2} \\
\dot{x}_4 &= -x_4 \\
\dot{x}_5 &= \left(\frac{1}{1 + x_1^2} - x_2\right) x_5 - x_1 x_6 \\
\dot{x}_6 &= -x_6 + x_1^2 \\
\dot{x}_7 &= \left(\frac{1}{1 + x_1^2} - x_2\right) x_7 - x_1 x_8 \\
\dot{x}_8 &= -x_8 - x_2
\end{align*}
\]

with initial conditions \(x_1(t_0) = x_{01}, \ x_2(t_0) = x_{02}, \ x_3(t_0) = x_4(t_0) = x_5(t_0) = x_6(t_0) = x_7(t_0) = x_8(t_0) = 0\).

**Problem 2 - Solution**

1) The system is linear and in particular the vector field is globally Lipschitz in \(x\). By Theorem 2.3, this guarantees existence and uniqueness in \([0, +\infty)\).

2) Expanding in Taylor series the original vector field, one notices that

\[
e^x - 1 - x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots - 1 - x
\]

\[
= \frac{x^2}{2} + \left(\frac{x^3}{6} + \cdots\right).
\]

Consider the initial condition \(x(0) = x_0\), where \(x_0 > 0\). It is well known that the differential equation \(\dot{x} = \frac{x^2}{2}\) has finite escape time. Indeed the solution to the Cauchy problem with initial condition \((0, x_0)\) has the form

\[
x(t) = \frac{2}{\frac{x_0}{2} - t}, \quad t \geq 0.
\]
Therefore, \( \lim_{t \to 2} x_0 - x(t) = +\infty \). Since \( e^x - 1 - x \geq \frac{x^2}{2} \) for \( x \geq 0 \), the solution to the original differential equation \( \dot{x} = e^x - 1 - x \) has a greater slope and so escapes to infinity sooner. This proves that the differential equation 2) does not admit a global solution in \([0, +\infty)\).

3) The vector field \( f(x) = e^{-|x|} - 1 \) is continuous. Moreover, for \( x \neq 0 \) it is easy to show that
\[
\left| \frac{d}{dx} f(x) \right| < 1 \quad \forall x \neq 0.
\]
The only thing that one needs to check is whether or not the derivative of \( f \) is bounded for \( x = 0 \). One notices that
\[
\left| \left( \frac{d}{dx} f(0) \right)^+ \right| = 1, \quad \left| \left( \frac{d}{dx} f(0) \right)^- \right| = 1.
\]
Therefore, we deduce that \( f \) is globally Lipschitz. Therefore, by Theorem 2.3 the solution exists and is unique for \( t \in [0, +\infty) \).

4) We will first show that the o.d.e.
\[
\begin{align*}
\dot{x}_1(t) &= -x_1(t)^3, \quad t \geq 0 \\
x_1(0) &= x_{01},
\end{align*}
\]
admits a global unique solution. Note that the vector field \( f(x) := -x^3 \) is not globally Lipschitz and so we cannot use Theorem 2.3. However, it is locally Lipschitz at \( x \), for every \( x \in \mathbb{R} \). Moreover, when \( x_1(t) > 0 \), the derivative \( \dot{x}_1(t) < 0 \) and when \( x_1(t) < 0 \), the derivative \( \dot{x}_1(t) > 0 \). Thus for any solution \( x_1(\cdot) \) of the o.d.e. we have \( x_1(t) \in K \), for all \( t \geq 0 \), where \( K := \{ x \in \mathbb{R} : |x| \leq |x_{01}| \} \) is compact. \(^1\) Therefore, by Theorem 2.6 we conclude that the o.d.e. admits a unique global solution in \([0, +\infty)\). Finally, note that the o.d.e.
\[
\begin{align*}
\dot{x}_2(t) &= x_1(t) - x_2(t) - x_1(t)^3, \quad t \geq 0 \\
x_2(0) &= x_{02},
\end{align*}
\]
\(^1\)W.l.o.g., assume that \( x_{01} > 0 \). If \( x_1(0) = x_{01} \), we claim that \( |x_1(t)| \leq x_{01} \), for all \( t \geq 0 \). Assume for the sake of contradiction that this is not the case. Let \( \mu > 0 \) be the first time for which \( |x_1(\mu)| > x_{01} \). W.l.o.g., \( x_1(\mu) > x_{01} > 0 \). By continuity of \( x_1(\cdot) \), there exists \( \delta > 0 \), such that \( x_1(t) > 0 \), for all \( t \in (\mu - \delta, \mu + \delta) \). But then, \( x_1(\cdot) \) is decreasing in \((\mu - \delta, \mu + \delta)\) and thus \( x_1(t) > x(\mu) > x_{01} \), for \( t \in (\mu - \delta, \mu) \). This contradicts the fact that \( \mu \) is the first time that \( x_1(\cdot) \) is larger that \( x_{01} \).
admits a unique global solution in $[0, \infty)$ as well, because the vector field $f(t, x) := x_1(t) - x - x_1(t)^3$ is continuous and globally Lipschitz in $x$ uniformly in $t$. Indeed $f$ is continuously differentiable and $|\frac{\partial f(t, x)}{\partial x}| = 1$.

**Problem 3 - Solution**

Consider the o.d.e.

$$\begin{align*}
\dot{x}(t) &= \sqrt{x(t)}, \quad t \geq 0 \\
x(0) &= 1.
\end{align*}$$

Let $x(\cdot)$ be a solution of the system. Then, since $\dot{x}(t) \geq 0$, for all $t \geq 0$, the trajectory $x(\cdot)$ is monotonically increasing. In particular, $x(t) \geq x(0) = 1$, for all $t \geq 0$. We will show that $f(x) := \sqrt{x}$, $x \geq 1$ is globally Lipschitz. Indeed, $f$ is continuously differentiable and $|f'(x)| \leq \frac{1}{2}$, for all $x \geq 1$. Therefore by Theorem 2.3 we conclude that the system admits a unique solution. The answer is different if we consider the system

$$\begin{align*}
\dot{x}(t) &= \sqrt{x(t)}, \quad t \geq 0 \\
x(0) &= 0,
\end{align*}$$

since $f$ is not even locally Lipschitz for $x = 0$. This is because $\lim_{x \to 0^+} f'(x) = +\infty$. In this case, the assumptions of Theorem 2.3 are not satisfied and thus we cannot conclude that the system admits a unique solution. In fact the system has infinitely many solutions. Two different solutions are for example

$$x(t) = 0 \quad \forall t \geq 0 \quad \text{and} \quad x(t) = \frac{t^2}{4} \quad \forall t \geq 0.$$

Moreover, it is easy to check that for any fixed $c > 0$, the trajectory

$$x(t) = \begin{cases} 
\frac{(t-c)^2}{4}, & t \in [c, +\infty) \\
0, & t \in [0, c)
\end{cases},$$

is a solution as well.
Problem 4 - Solution

Since \( f_1 \) is locally Lipschitz at \( x_0 \), there exist \( r_1 > 0 \) and \( L_1 > 0 \) such that
\[
|f_1(x) - f_1(y)| \leq L_1|x - y|,
\] (1)
for all \( x, y \in \{ z \in \mathbb{R} : |z - x_0| \leq r_1 \} \). Similarly, since \( f_2 \) is locally Lipschitz at \( x_0 \), there exist \( r_2 > 0 \) and \( L_2 > 0 \) such that
\[
|f_2(x) - f_2(y)| \leq L_2|x - y|,
\] (2)
for all \( x, y \in \{ z \in \mathbb{R} : |z - x_0| \leq r_2 \} \). Set \( r := \min\{r_1, r_2\} \). Then (1) and (2) hold for all \( x, y \in B(x_0, r) := \{ z \in \mathbb{R} : |z - x_0| \leq r \} \).

1) It follows by the triangle inequality that
\[
|(f_1 + f_2)(x) - (f_1 + f_2)(y)| = |f_1(x) - f_1(y) + f_2(x) - f_2(y)|
\leq |f_1(x) - f_1(y)| + |f_2(x) - f_2(y)| \leq (L_1 + L_2)|x - y|,
\]
for all \( x, y \in B(x_0, r) \). This proves that the sum is locally Lipschitz at \( x_0 \).

2) It follows using the triangle inequality that
\[
|(f_1f_2)(x) - (f_1f_2)(y)| = |f_1(x)f_2(x) - f_1(y)f_2(y)| = |f_1(x)f_2(x) - f_1(x)f_2(y) + f_1(x)f_2(y) - f_1(y)f_2(y)|
\leq |f_1(x)f_2(x) - f_1(x)f_2(y)| + |f_1(x)f_2(y) - f_1(y)f_2(y)|
\leq |f_1(x)||L_2||x - y| + |f_2(y)||L_1||x - y| \leq (|f_1(x)||L_2 + |f_2(y)||L_1||x - y|,
\]
for all \( x, y \in B(x_0, r) \). Let \( M_1 := \sup_{x \in B(x_0, r)} |f_1(x)| \) and \( M_2 := \sup_{x \in B(x_0, r)} |f_2(x)| \). Then
\[
|(f_1f_2)(x) - (f_1f_2)(y)| \leq (M_1L_2 + M_2L_1)|x - y|,
\]
for all \( x, y \in B(x_0, r) \). This proves that the product is locally Lipschitz at \( x_0 \).

3) Assume that \( f_1, f_2 \) are globally Lipschitz functions with Lipschitz constants \( L_1 \) and \( L_2 \) respectively. Then, using the definition of Lipschitz continuity, one gets that
\[
|(f_2(f_1(x))) - f_2(f_1(y))| \leq L_2|f_1(x) - f_1(y)| \leq L_2L_1|x - y|,
\]
for all \( x, y \in \mathbb{R} \). This proves that the composition is globally Lipschitz continuous as well.
Problem 5 - Solution

\( P \) is contractive \iff \exists \rho \in [0, 1) : \|Ax - Ay\| \leq \rho \|x - y\|, \forall x, y \in \mathbb{R}^n
\iff \exists \rho \in [0, 1) : \|Av\| \leq \rho \|v\|, \forall v \in \mathbb{R}^n.

Taking the square on both sides and rearranging one gets

\[ \|Av\|^2 \leq \rho^2 \|v\|^2 \iff v^\top A^\top Av \leq \rho^2 v^\top v \iff v^\top (A^\top A - \rho^2 I)v \leq 0. \]

So,

\( P \) is contractive \iff \exists \rho \in [0, 1) : v^\top (A^\top A - \rho^2 I)v \leq 0, \forall v \in \mathbb{R}^n.

This holds if and only if all the eigenvalues of \( A^\top A - \rho^2 I \) are nonpositive. In particular one knows that if \( \{\lambda_1, \ldots, \lambda_n\} \subset [0, +\infty) \) are the eigenvalues of \( A^\top A \), then \( \{\lambda_1 - \rho^2, \ldots, \lambda_n - \rho^2\} \) are the eigenvalues of \( A^\top A - \rho^2 I \). Thus,

\( P \) is contractive \iff \exists \rho \in [0, 1) : \lambda_i \leq \rho^2, \forall i = 1, \ldots, n
\iff \lambda_i < 1, \forall i = 1, \ldots, n
\iff \sqrt{\lambda_i} < 1, \forall i = 1, \ldots, n.

Thus \( P \) is contractive if and only if all the singular values of \( A \) are smaller than 1. Moreover, when \( A^\top = A \), \( P \) is contractive if and only if all the eigenvalues of \( A \) have absolute value smaller than 1.