Chapter 1 - Lyapunov Stability Part I

Problem 1 - Solution

Let us use the following Lyapunov function candidate

\[ V(x) = \frac{1}{4}x_1^4 + \frac{1}{2}x_2^2 \]

Once we take the derivative of \( V(x) \) along the trajectories of the system equations we obtain

\[
\dot{V}(x) = x_1^3x_2 + x_2(-x_1^3 - x_2^3) = -x_2^4 \leq 0
\]

And the system is stable in the sense of Lyapunov. In order to show asymptotic stability, we invoke LaSalle’s Invariance Principle as follows. The set for which \( \dot{V}(x) = 0 \) is characterized by the following condition: \( x_2 \equiv 0 \). We don’t know yet about \( x_1 \), but from the condition we see that the system must be confined to the line where \( x_2 = 0 \). However, from the system equations we see that unless \( x_1 = 0 \), this is impossible, the system would diverge from the \( x_2 = 0 \) line. Thus, we find: \( x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow x_1 \equiv 0 \). Therefore, the origin is a globally asymptotically stable equilibrium.

Problem 2 - Solution

Whenever we have \( |2x_1 + x_2| \leq 1 \), the system is described by the following ODE

\[
\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x
\]

where the system matrix is Hurwitz, and the origin is locally asymptotically stable. However, the system is not globally asymptotically stable since there is another equilibrium at the point \((1, 0)\)!

Problem 3 - Solution

The derivative of \( V(x) \) along the trajectories of the system give

\[
\dot{V}(x) = x_1x_2 + x_2(x_1^2 + x_1x_2 + u(x)) = x_2(x_1 + x_1^2 + x_1x_2 + u(x))
\]

Therefore, if we design \( u(x) = -(x_1 + x_1^2 + x_1x_2) - x_2 \), which renders

\[
\dot{V}(x) = -x_2^2
\]
and the closed-loop system is stable. In order to show asymptotic stability, we invoke LaSalle’s Invariance Principle as follows: $\dot{V}(x) \equiv 0 \Rightarrow x_2 \equiv 0 \Rightarrow \dot{x}_2 \equiv 0 \Rightarrow x_1 \equiv 0$, and the result follows.

**Problem 4 - Solution**

$$V(x) = f(x)^T P f(x) \geq \lambda_{\text{min}}(P) \|f(x)\|^2 > 0$$

Thus $V(x)$ is positiv-semidefinite. In order to show that $V(x)$ is pos.-definite, prove that $f(x) = 0$ if and only if $x = 0$. Suppose there exists a $p \neq 0$ such that $f(p) = 0$, then:

$$p^T p \leq -[p^T P f(p) + f^T (p) P p] = 0$$

but from this follows $p = 0$.

To show that $V(x)$ is radially unbounded, we know that

$$x^T P f(x) + f(x)^T P x \leq -x^T x \Leftrightarrow \frac{\|x^T P f(x)\|}{\|x\|^2} \leq -1/2$$

Assume that $f(x) \leq \beta$ as $\|x\| \to \infty$, i.e., $f(x)$ does not grow with the norm of $x$, which is needed for proving that $V(x)$ is radially unbounded. Then

$$\frac{\|x^T P f(x)\|}{\|x\|^2} \leq c \frac{\|x\| \|f(x)\|}{\|x\|^2} = c \frac{\|f(x)\|}{\|x\|^2} \to 0, \quad \text{as} \quad \|x\| \to \infty$$

which is a contradiction. Hence, $V(x)$ is radially unbounded.