Chapter 7 - Sliding Mode Control

Problem 1 - Solution

We define the error as:

\[ e = \begin{bmatrix} x_1 - r \\ x_2 - \dot{r} \end{bmatrix} \]

where the reference derivatives are:

\[ r(t) = \sin t \quad \dot{r}(t) = \cos t \quad \ddot{r}(t) = -\sin t. \]

The error dynamics can then be written as:

\[
\begin{align*}
\dot{e}_1 &= \dot{x}_1 - \dot{r} = e_2 \\
\dot{e}_2 &= \dot{x}_2 - \ddot{r} = x_1^2 + \gamma u - \ddot{r}
\end{align*}
\]

We choose \( k_1 \) such that \( s + k_1 \) is Hurwitz, e.g. \( k_1 = 1 \). Then we define the sliding manifold as:

\[ \sigma = e_1 + e_2. \]

Note that \( \sigma = 0 \Rightarrow e_1 = -e_2 \). By (1), \( \dot{e}_1 = -e_1 \), thus \( e_1 \to 0, e_2 \to 0 \).

Moreover:

\[ \dot{\sigma} = e_2 + x_1^2 + \gamma u + \sin t. \]

By choosing:

\[ u = -\frac{1}{\dot{\gamma}} (e_2 + x_1^2 + \sin t) + v \]

we obtain:

\[ \dot{\sigma} = \gamma v + \Delta(t, x) \]

with

\[ \Delta(t, x) = \left(1 - \frac{\gamma}{\dot{\gamma}}\right) (e_2 + x_1^2 + \sin(t)). \]

We assume that we have the following bound

\[
\left| \frac{\Delta(t, x)}{\gamma} \right| = \left| \left(1 - \frac{1}{\dot{\gamma}}\right) (x_2 - \cos(t) + x_1^2 + \sin(t)) \right|
\leq \left| \left(1 - \frac{1}{\dot{\gamma}}\right) (x_2 - \cos(t) + x_1^2 + \sin(t)) \right|
\leq \left(1 - \frac{1}{\dot{\gamma}}\right) (|x_2| + x_1^2 + \sqrt{2}) =: \rho(x)
\]

1 of 3
where the first inequality is for the lower bound on $\gamma$ in the exercise text, and in the second we use the inequality $\sin(t) - \cos(t) \leq \sqrt{2}$. Note that $\rho(x) \geq 0 \forall x \in \mathbb{R}^2$.

Finally, designing $v = -\left(\left(1 - \frac{1}{\gamma}\right)(|x_2| + x_1^2 + \sqrt{2}) + \beta_0\right)\text{sgn}(\sigma)$, with $\beta_0 > 0$, we achieve asymptotic tracking. Note that the uncertainty on the $\gamma$ affects the amplitude of $v$, hence it requires a more aggressive control.

**Problem 2 - Solution**

1. Take a torque $\tau = H(q)v + C(q, \dot{q})\dot{q} + G(q)$.

Due to the perfect knowledge of the matrices, the equation combines with the model as follows:

$$\ddot{q} = v$$

hence we choose

$$v = \ddot{q}_d - \alpha (q - q_d) - \beta (\dot{q} - \dot{q}_d)$$

and with some diagonal positive definite matrices $\alpha$ and $\beta$ we obtain decoupled control and asymptotic convergence of the state to the desired reference.

2. In the realistic case in which the model matrices are not known exactly, we define the sliding manifold:

$$\sigma(q, \dot{q}) = \dot{q} - \dot{q}_d + K(q - q_d) = \dot{q} - \dot{q}_r = 0$$

for an accordingly defined variable $\dot{q}_r = \dot{q}_r(q, q_d, \dot{q}_d)$.

We consider now a Lyapunov function $V = \frac{1}{2}\sigma^T H(q)\sigma$, which is positive definite since $H(q)$ is positive definite. Then:

$$\dot{V} = \sigma^T H \dot{\sigma} + \frac{1}{2} \sigma^T \dot{H} \sigma$$

$$= \sigma^T H (\ddot{q} - \ddot{q}_r) + \frac{1}{2} \sigma^T \dot{H} \sigma$$

$$= \sigma^T (\tau - C\dot{q} - G - H\ddot{q}_r) + \frac{1}{2} \sigma^T \dot{H} \sigma$$

$$= \sigma^T (\tau - C(\sigma + \dot{q}_r) - G - H\ddot{q}_r) + \frac{1}{2} \sigma^T \dot{H} \sigma$$

$$= \sigma^T (\tau - C\dot{q}_r - G - H\ddot{q}_r) + \frac{1}{2} \sigma^T \left(\dot{H} - 2C\right) \sigma$$

$$= \sigma^T (\tau - C\dot{q}_r - G - H\ddot{q}_r)$$
where the last equality follows from the skew-symmetry of \( \dot{H} - 2C \).

We design the control action \( \tau \) as follows:

\[
\tau = \dot{H}\ddot{q}_r + \dot{C}\dot{q}_r + \dot{G} - \beta \text{sgn}(\sigma).
\]

with \( \beta > 0 \). Hence, the derivative of the Lyapunov function reads as:

\[
\dot{V} = \sigma^\top (\Delta H\ddot{q}_r + \Delta C\dot{q}_r + \Delta G - \beta \text{sgn}(\sigma))
\]

with \( \Delta H = \dot{H} - H \), \( \Delta C = \dot{C} - C \) and \( \Delta G = \dot{G} - G \). The Lyapunov function derivative \( \dot{V} \) is made negative definite by choosing:

\[
\beta > |\Delta H\ddot{q}_r + \Delta C\dot{q}_r + \Delta G|.
\]

Also in this case, the bigger is the uncertainty, the larger is the variable \( \beta \).