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AUTOMATIC CONTROL LABORATORY

Overview on data-driven optimal control via linear programming

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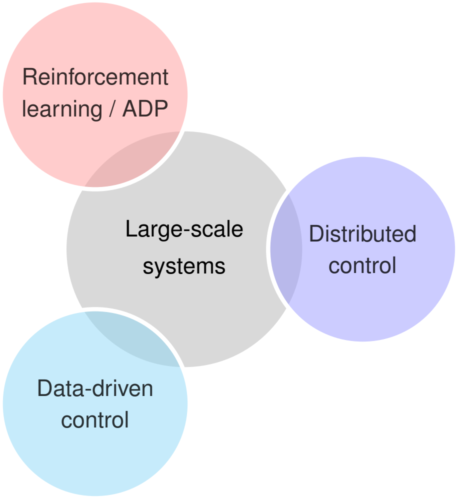
My research map



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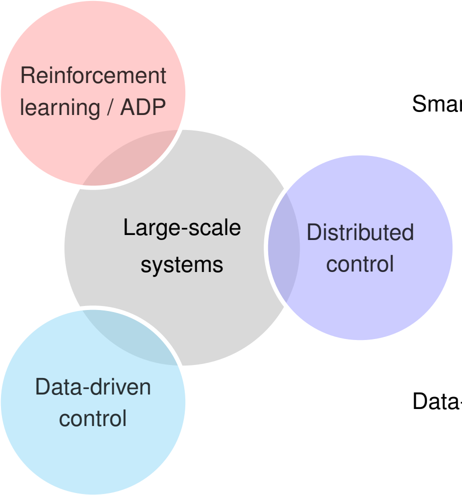

Smart grids

Clustering methods

Stability of dissipative DT systems

Data-driven dissipativity analysis

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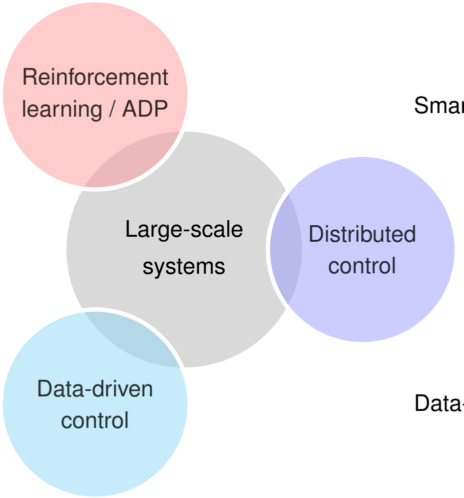

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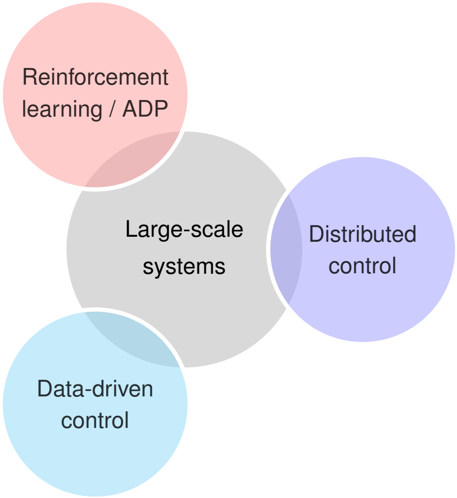

- Martinelli, Gargiani, Draskovic, Lygeros, "Data-driven optimal control of affine systems: A linear programming perspective", *IEEE L-CSS*, 2022
- Martinelli, Gargiani, Lygeros, "Data-driven optimal control with a relaxed linear program", *Automatica*, 2022
- Martinelli, Gargiani, Lygeros, "On the synthesis of Bellman inequalities for data-driven optimal control", *60th CDC*, 2021

Outline

Introduction to data-driven optimal control via linear programming

Estimation of Bellman inequalities from data

Willems' Fundamental Lemma for affine systems

Problem setup: stochastic optimal control

Ingredients:

- A discrete-time stochastic system $x^+ = f(x, u, \psi)$ with possibly infinite state & action spaces
- A stage-cost function $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_+$

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The γ-discounted ∞-horizon cost associated to a stationary feedback policy $\pi : \mathbb{X} \to \mathbb{U}$ is

$$v_\pi(x) = \mathbb{E}\left[\sum_{k=0}^{\infty} \gamma^k \ell(x_k, \pi(x_k)) \,\middle|\, x_0 = x\right]$$

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Objective: find an optimal policy π^* such that $v_{\pi^*}(x) = \inf_\pi v_\pi(x) = v^*(x)$

Dynamic programming methods

- The value function admits a recursive definition – the **Bellman equation**

$$v_\pi(x) = \underbrace{\ell(x,u) + \gamma \mathbb{E}\left[v_\pi(f(x,u,\psi))\right]}_{(\mathcal{T}_\pi v_\pi)(x)}$$

$$v^*(x) = \underbrace{\inf_{u \in \mathbb{U}} \left\{ \ell(x,u) + \gamma \mathbb{E}\left[v^*(f(x,u,\psi))\right] \right\}}_{(\mathcal{T} v^*)(x)}$$

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- $\mathcal{T}$, $\mathcal{T}_\pi$ are monotone and contractive, hence $v \leq \mathcal{T} v \implies v \leq v^*$, and $\lim_{n\to\infty} \mathcal{T}^n v = v^*$

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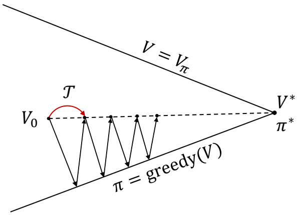
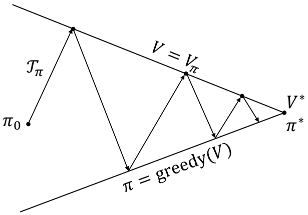
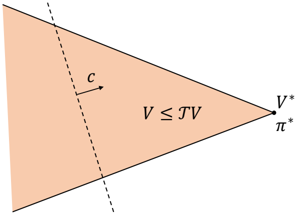
Value Iteration



Policy Iteration



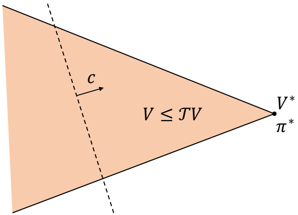
Linear Programming



The linear programming formulation

One can find v^* by solving the ∞-dimensional (nonlinear) program

$$\sup_{v \in \mathbb{V}} \int_{\mathbb{X}} v(x) c(dx)$$
$$\text{s.t.} \quad v(x) \leq (\mathcal{T} v)(x) \quad \forall x,$$



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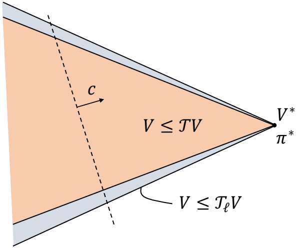

We can relax the constraints by substituting

$$v(x) \leq (\mathcal{T} v)(x) = \inf_{u \in \mathbb{U}} \Big\{ \ell(x,u) + \gamma \mathbb{E}\big[v^*(f(x,u,\psi))\big] \Big\} \quad \forall x$$

with

$$v(x) \leq (\mathcal{T}_\ell v)(x,u) = \ell(x,u) + \gamma \mathbb{E}\big[v(f(x,u,\psi))\big] \quad \forall (x,u)$$

The Q-function

- If one is able to obtain v^*, then

$$\pi^*(x) = \arg\min_{u \in \mathbb{U}} \left\{ \ell(x,u) + \gamma \mathbb{E}\left[v^*(f(x,u,\psi)) \right] \right\}.$$

Problem: policy extraction is in general not possible if f or ℓ are not known

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  Problem: policy extraction is in general not possible if f or ℓ are not known

- Introducing the Bellman operator associated to Q-functions

$$q^*(x,u) = \underbrace{\ell(x,u) + \gamma \mathbb{E}\left[\inf_{w \in \mathbb{U}} q^*(f(x,u,\psi), w) \right]}_{(\mathcal{F}q^*)(x,u)}$$

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- Since $v^*(x) = \min_{u \in \mathbb{U}} q^*(x,u)$, now policy extraction is model-free:

$$\pi^*(x) = \arg\min_{u \in \mathbb{U}} q^*(x,u).$$

LP formulation for Q-functions

- $\mathcal{F}$ is again a monotone contraction mapping, hence

$$\begin{aligned} &\sup_{q \in \mathbb{Q}} \int_{\mathbb{X}\times\mathbb{U}} q(x,u) c(dx,du) \\ &\text{s.t.} \;\; q(x,u) \le (\mathcal{F}q)(x,u) = \ell(x,u) + \gamma \mathbb{E}\left[\inf_{w \in \mathbb{U}} q^*(f(x,\pi^*(x),\psi),w) \right] \quad \forall (x,u) \end{aligned}$$

Problem: We can not relax the constraints due to nesting of $\mathbb{E}$ and $\inf$

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- Introducing the **relaxed** Bellman operator

$$(\hat{\mathcal{F}}q)(x,u) = \ell(x,u) + \gamma \inf_{w\in\mathbb{U}} \mathbb{E}\big[q(f(x,u,\psi),w)\big]$$

The relaxed Bellman operator

Proposition (Properties of the relaxed operator)

(i) *$\hat{\mathcal{F}}$ is a monotone contraction mapping with a unique fixed point $\hat{q}(x, u)$*

(ii) *The fixed point of $\hat{\mathcal{F}}$ is a point-wise upper bound to the fixed point of $\mathcal{F}$, that is,*

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- One can find $\hat{q}(x,u)$ via the following LP:

$$\begin{aligned} &\sup_{q \in \mathbb{Q}} \int_{\mathbb{X}\times\mathbb{U}} q(x,u)c(dx,du) \\ &\text{s.t.} \quad q(x,u) \leq (\hat{\mathcal{F}}_\ell q)(x,u) = \ell(x,u) + \gamma\mathbb{E}\big[q(f(x,u,\psi),w)\big] \quad \forall (x,u,w) \end{aligned}$$

Q: how good is the approximation?

Quantify the approximation introduced by $\hat{\mathcal{F}}$

- In case of linear (affine) systems the relaxed operator is policy-preserving,

$$\arg\min_{u \in \mathbb{U}} \hat{q}(x,u) = \arg\min_{u \in \mathbb{U}} q^*(x,u)$$

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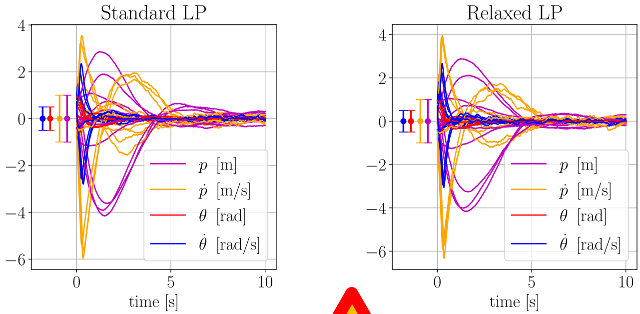
- For nonlinear systems, empirical evidence is encouraging...



...but theoretical bounds are still elusive

Model-free LP formulation

LPs can be solved efficiently in general, but here several sources of intractability arise:

- q is an optimization variable in the ∞-dimensional space $\mathbb{Q}$
- ∞ number of constraints

$$\begin{aligned} &\sup_{q \in \mathbb{Q}} \int_{\mathbb{X}\times\mathbb{U}} q(x,u)c(dx,du) \\ &\text{s.t.} \quad q(x,u) \leq \ell(x,u) + \gamma\mathbb{E}\big[q(f(x,u,\psi),w)\big] \quad \forall (x,u,w) \end{aligned}$$

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We can substitute $\sum_i \varphi_i \phi_i(x,u)$ and sample a finite subset of constraints

$$q(x_i,u_i) \leq \ell(x_i,u_i) + \gamma\mathbb{E}\big[q(x_i^+,w)\big] \quad \forall (x_i,u_i,w_i) \in \mathcal{D}$$

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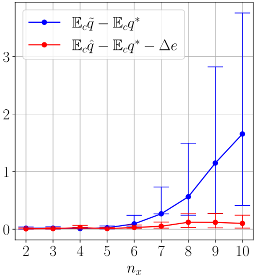
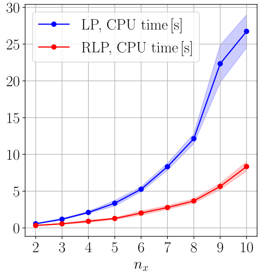
Model-free/RL framework: construct one constraint for each observation

$$\{x_i, u_i, \ell(x_i,u_i), x_i^+\}_{i=1}^d$$

Performance bounds

- Known performance bounds due to function approximation [Beuchat *et al.*, 2020] and constraint sampling [de Farias *et al.*, 2004] tend to be quite loose
- In general, hard to guarantee bounded solutions for large-scale systems





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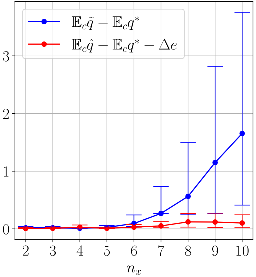
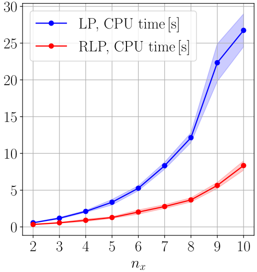




Idea: Select $c(x, u)$ based on sampled data

- Necessary and sufficient conditions on $c(x, u)$ based on duality theory and Farkas' Lemma
- Sufficient conditions on the dataset/sampling logic are under investigation

Outline

Introduction to data-driven optimal control via linear programming

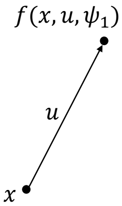
Estimation of Bellman inequalities from data

Willems' Fundamental Lemma for affine systems

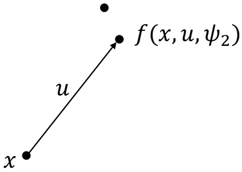
Estimation of Bellman inequalities

x

Estimation of Bellman inequalities



Estimation of Bellman inequalities



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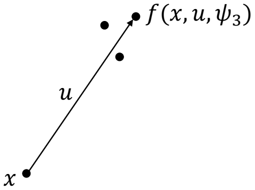

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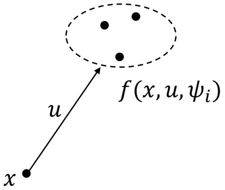

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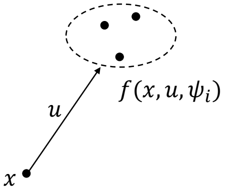
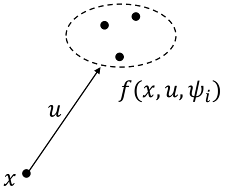


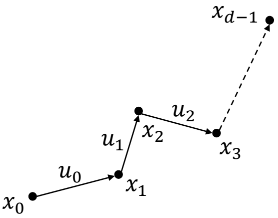
Unbiased estimator

$$\hat{\theta}_{x,u} = \frac{1}{d} \sum_{i=0}^{d-1} v(\underbrace{f(x, u, \psi_i)}_{x_i^+})$$

- $\mathbb{E}[\hat{\theta}_{x,u}] = \mathbb{E}\left[v(f(x, u, \psi))\right]$
- $\textsc{Var}(\hat{\theta}_{x,u}) = \frac{1}{d}\textsc{Var}(v(f(x, u, \psi)))$

Estimation of Bellman inequalities





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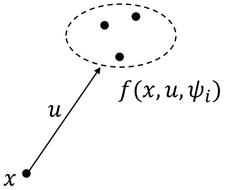
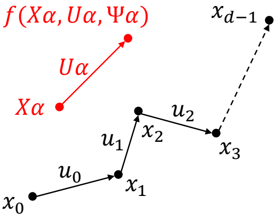
Estimation of Bellman inequalities





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Biased estimator

$$\bar{\theta}_{X\alpha,U\alpha} = v(f(X\alpha, U\alpha, \Psi\alpha))$$

$X = \begin{bmatrix} x_0 & \cdots & x_{d-1} \end{bmatrix}$, $U = \begin{bmatrix} u_0 & \cdots & u_{d-1} \end{bmatrix}$,
$X^+ = \begin{bmatrix} x_0^+ & \cdots & x_{d-1}^+ \end{bmatrix}$ and $\Psi = \begin{bmatrix} \psi_0 & \cdots & \psi_{d-1} \end{bmatrix}$

Problem: no access to f and in general $X^+\alpha \neq f(X\alpha, U\alpha, \Psi\alpha)$

Estimation for linear systems

If $f(x, u, \psi) = Ax + Bu + \psi$ then

$$\begin{aligned} X^+\alpha &= (AX + BU + \Psi)\alpha \\ &= AX\alpha + BU\alpha + \Psi\alpha \\ &= f(X\alpha, U\alpha, \Psi\alpha) \end{aligned}$$

So we can compute $\bar{\theta}_{X\alpha,U\alpha} = v(f(X\alpha, U\alpha, \Psi\alpha))$, but is still biased

- $\mathbb{E}[\bar{\theta}_{X\alpha,U\alpha}] = \mathbb{E}\left[v(f(x, u, \bar{\psi}))\right]$, where $\mathbb{E}[\bar{\psi}] = \mathbb{E}[\psi]$ and $\text{VAR}(\bar{\psi}) = \|\alpha\|_2^2 \text{VAR}(\psi)$
- $\text{VAR}(\bar{\theta}_{X\alpha,U\alpha}) = \|\alpha\|_2^2 \text{VAR}(v(f(x, u, \psi)))$

Estimation for linear systems

Fact

The LP with biased constraints is policy-preserving, i.e.,

$$\bar{q}(x,u) = \hat{q}(x,u) + \frac{\gamma \operatorname{TR}((\|\alpha\|_2^2 - 1) q^* \operatorname{VAR}(\psi))}{1-\gamma}$$

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Fact

If the data matrix $\begin{bmatrix} X \\ U \end{bmatrix}$ *is full row-rank then we can construct (biased) constraints for all* (x,u).
One can meet such condition with a ***persistently exciting*** *input*

Estimation for affine systems

If $f(x, u, \psi) = Ax + Bu + c + \psi$ and $\mathbf{1}^\top \alpha = 1$ then

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Q: Can one can meet such condition with a persistently exciting input?

Outline

Extending the Fundamental Lemma to affine systems

- **Objective:** Guarantee $\textsc{Rank}\begin{bmatrix} X \\ U \\ \mathbf{1}^\top \end{bmatrix} = n + m + 1$ when data are generated by affine dynamics $x^+ = Ax + Bu + c$.

Extending the Fundamental Lemma to affine systems

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- Nontrivial extension since
  - Trajectories do not form a linear subspace anymore
  - We must guarantee $\mathbf{1}^\top \notin \textsc{Rowsp}\begin{bmatrix} X \\ U \end{bmatrix}$
- Some tricks that don't work:
  - $x^+ = Ax + \begin{bmatrix} B & I \end{bmatrix}\begin{bmatrix} u \\ c \end{bmatrix} \longrightarrow (A, \begin{bmatrix} B & I \end{bmatrix})$ is controllable but $\begin{bmatrix} u \\ c \end{bmatrix}$ is not p.e.
  - $\begin{bmatrix} x^+ \\ y^+ \end{bmatrix} = \begin{bmatrix} A & c \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} u \longrightarrow u$ is p.e. but $(\tilde{A}, \tilde{B}) = (\begin{bmatrix} A & c \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} B \\ 0 \end{bmatrix})$ is not controllable

A key result

A sequence $S = \begin{bmatrix} S_1 & \cdots & S_d \end{bmatrix} \in \mathbb{R}^{m \times d}$ is persistently exciting of order K if the associated Hankel matrix of depth K,

$$\mathcal{H}_K(S) = \begin{bmatrix} S_1 & S_2 & \cdots & S_{d-K+1} \\ S_2 & S_3 & \cdots & S_{d-K+2} \\ \vdots & \vdots & & \vdots \\ S_K & S_{K+1} & \cdots & S_d \end{bmatrix} \in \mathbb{R}^{mK \times (d-K+1)},$$

is full row-rank, *i.e.* $\textsc{Rank}(\mathcal{H}_K(S)) = mK$.

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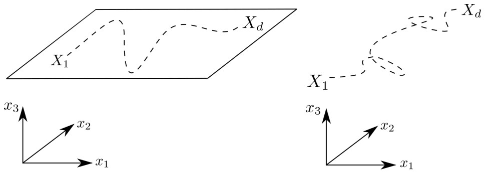
Proposition

If S is persistently exciting of order K, then for all $K' < K$ it holds that

$$\mathbf{1}^\top \notin \textsc{Rowsp}\, \mathcal{H}_{K'}(S)$$



Willems' Fundamental Lemma for affine systems

Theorem

Consider a dataset (X, U, X^+, Y) of length d with $X^+ = AX + BU + c\mathbf{1}^\top$ and $Y = CX + DU + r\mathbf{1}^\top$.
If U is a persistently exciting input of order $n + L + 1$ and (A, B) is a controllable pair, then

1. $\textsc{Rank}\begin{bmatrix} \mathcal{H}_1(X_{1:d-L+1}) \\ \mathcal{H}_L(U) \\ \mathbf{1}^\top \end{bmatrix} = n + mL + 1$,
2. $(\tilde{X}, \tilde{U}, \tilde{X}^+, \tilde{Y})$ *is a dataset of length L if and only if there exists $g \in \mathbb{R}^{d-L+1}$ such that*

$$\begin{bmatrix} \textsc{Vec}\,\tilde{U} \\ \textsc{Vec}\,\tilde{Y} \\ 1 \end{bmatrix} = \begin{bmatrix} \mathcal{H}_L(U) \\ \mathcal{H}_L(Y) \\ \mathbf{1}^\top \end{bmatrix} g.$$

- 1. $\implies$ 2. was proven in [1]
- Inspired by state-space proof in [2], we show that higher order p.e. + controllability $\implies$ 1.

[1] Berberich *et al.*, "Linear tracking MPC for nonlinear systems Part II: The data-driven case", *IEEE TAC*, 2022
[2] van Waarde *et al.*, "Willems' FL for state-space systems and its extension to multiple datasets", *IEEE L-CSS*, 2020

Some consideration on the Fundamental Lemma

- One can “compensate” a constant disturbance with higher p.e. order... can we do the same with richer disturbances, *e.g.*, periodic signals?
- More generally, for multi-input systems $x^+ = Ax + \sum_{i=1}^{k} B_i u_i$
  - Current conditions tend to break down easily: by re-writing $x^+ = Ax + \begin{bmatrix} B_1 & \cdots & B_k \end{bmatrix} \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix}$, one should require that $(A, \begin{bmatrix} B_1 & \cdots & B_k \end{bmatrix})$ is controllable and $\begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix}$ is p.e.
  - The signals u_i can represent non-manipulable inputs or even coupling with other systems in a network

Thank you!

References:

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