Minimality and Observability of Group Systems

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ABSTRACT

Group systems are a generalization of Willems-type linear systems that are useful in error control coding. It is shown that the basic ideas of Willems's treatment of linear systems are easily generalized to linear systems over arbitrary rings and to group systems. The interplay between systems (behaviors) and trellises (evolution laws) is discussed with respect to completeness, minimality, controllability, and observability. It is pointed out that, for trellises of group systems and Willems-type linear systems, minimality is essentially the same as observability. The development is universal-algebraic in nature and holds unconditionally for linear systems over the real numbers.

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I. INTRODUCTION

Willems [1–3] has championed an approach to system theory that is based on defining a system by it behavior, i.e., the set of possible trajectories, without necessarily classifying variables as inputs, outputs, or states. From this point of view, a linear system is simply a subspace of a direct product of vector spaces.

Group systems (in the sense of this paper) are the generalization of Willems-type linear systems to subgroups of direct-product groups. The groups need not be abelian. We will also consider Willems-type linear systems over arbitrary rings, i.e., submodules of direct products of modules. Our development will be "universal-algebraic" in the sense that we will simultaneously discuss group systems and linear systems over fields and rings using only arguments that apply to all these cases. The theory is thus unconditionally valid for Willems-type linear systems over the real numbers.

One motivation for the investigation of group systems comes from error-control coding. It was recognized recently that certain codes for Gaussian communication channels are best described as group systems, and the investigation of the relevant system-theoretic properties of such codes was begun in [4] and [5].

That work may be seen as a generalization of the algebraic structure theory of convolutional codes, which was developed by Massey and Sain, Forney, and others some 20 years ago; cf. [6]. That theory is system-theoretic in nature, but has more in common with the Willems approach than with the traditional input-output framework—convolutional codes are, in fact, Willems-type linear systems over finite fields.

A considerable literature exists on linear systems over rings (cf. [7])—to the knowledge of these authors, all within the input-output framework. Also within the input-output framework are the group homomorphic systems of Brockett and Willsky [8]. Shift-invariant subgroups of the direct product G^Z, for certain groups G (and also more general systems) have been studied by Kitchens and Schmidt [9] and others from a symbolic-dynamics viewpoint.

Our earlier papers [4] and [5] aimed at deriving canonically structured minimal realizations of strongly controllable group systems. The fundamental aspects of the relation between systems (behaviors) and trellises (realizations, evolution laws) were discussed only to the extent that was indispensable for the goals of these papers. The purpose of this paper is a more in-depth discussion of these issues.

The main results of this paper are the following:

(1) It is demonstrated that the basic theory of Willems-type linear systems may be naturally developed in a universal-algebra framework (i.e., simultaneously for systems over groups, rings, and fields).

(2) Minimality, controllability, and observability are studied in detail. It is pointed out that, in the behavioral framework, minimality (of a realization) essentially coincides with observability.

The presentation is self-contained, i.e., no results from [1–5] will be used.

The discussion includes time-variant systems, since finite-time systems (which cannot be time invariant) are important in coding. Also, it is sometimes helpful to analyze an infinite-time system by partitioning the time axis into a finite number of intervals and then treating the system as a finite-time system; cf. [4].

One of the attractive features of system theory over *finite* fields, rings, and groups—which is what is needed in coding—is the possibility of visualizing nontrivial systems by means of finite graphs. Our examples will therefore be of this type. Note, however, that the theory applies without restrictions to linear systems over the real or complex numbers.

The paper is structured as follows. Section 2 is a quick overview of the algebraic concepts that will be needed and establishes our universal-algebra language. Group systems and trellises are introduced in Section 3. The discussion of minimality is begun in Section 4, where the canonical trellis is introduced; most of this section is a straightforward adaptation of ideas developed by Willems. In Section 5, the fundamental minimality conditions are presented. Controllability and observability are discussed in Section 6. The final section provides a very cursory overview of the main results of [4] and [5].

The proofs are collected in Appendix A. A notation index is provided in Appendix B.

2. REVIEW OF ALGEBRAIC CONCEPTS

The algebra that will be used in this paper is essentially a common subset of linear algebra and elementary group theory. We will consider "spaces" that are vector spaces over an arbitrary field, or modules over an arbitrary ring, or just groups. We will treat these three cases simultaneously, using only arguments that apply to all these cases.

It would be annoying, however, to have three separate versions—one for groups, one for modules, and one for vector spaces—for almost every statement of this paper. We will therefore adopt the convention that any statement on group systems implies the corresponding statement for linear systems over rings and over fields; whenever possible, we will give an explicit formulation only for group systems and leave the obvious translation to linear systems to the reader. The purpose of this section is to provide the necessary "translation table" between these algebraic systems.

In order to facilitate this translation, we will use additive notation (i.e., $+$ and $-$) even for noncommutative groups, and the neutral element of a group will be denoted 0 (zero). This implies, of course, that we will not assume commutativity for $+$, which will turn out to be less awkward than it appears at first sight.

We will use the following universal-algebra concepts: subgroups (submodules, subspaces), homomorphisms (linear mappings), direct products, quotient groups (quotient modules, quotient spaces), and the descending-chain condition (finite dimensionality). The first three of these concepts do not need any further comment; a quick look at the remaining two seems appropriate, however.

The *quotient space* G/H of a vector space G with respect to a subspace H is the set of cosets $g + H$, $g \in G$, which is itself a vector space. The same definition (with "space" replaced by "module" or "group," respectively) holds for modules and for commutative groups. For noncommutative groups, the set G/H of cosets forms a group if and only if H is a *normal* subgroup of G: a subgroup H of G is normal in G if $H + g = g + H$ for every $g \in G$. Interestingly, the inclusion of noncommutative groups does not cost us extra effort besides verification of normality for certain subgroups; whenever a universal-algebra argument will require a subgroup to be normal, it will turn out to be so.

If K is a normal subgroup of a group G, then the natural map $G \to G/K: g \mapsto g + K$ is a homomorphism with kernel K; conversely, the kernel K of a surjective (onto) homomorphism $G \to H$ is a normal subgroup of G, and H is isomorphic to G/K. This statement—usually called the "fundamental theorem of homomorphisms"—is also true for vector spaces and modules (where normality is, of course, automatic).

The other algebraic concept to be reviewed here is the descending-chain condition, which is a generalization of finite dimensionality to modules and groups. Let G be a group (module, vector space). A *descending chain* in G is a sequence $H_1, H_2, H_3, \ldots$ of normal subgroups (submodules, subspaces) of G such that $H_1 \supseteq H_2 \supseteq H_3 \supseteq \cdots$. We say that G satisfies the *descending-chain condition* (DCC) if any such chain eventually becomes stationary, i.e., if there exists an integer j such that $H_i = H_j$ for all $i \geqslant j$. Equivalently, G satisfies the DCC if there are at most finitely many steps (strict inequalities) in any descending chain in G.

It is clear that a vector space satisfies the DCC if and only if it is finite-dimensional, and its dimensionality is an upper bound on the number of steps in any descending chain. It is also clear that every finite group (or module, or vector space) satisfies the DCC. Note, however, that a finitely generated module over an arbitrary ring may not satisfy the DCC. For example, the ring of integers Z (considered as a module over itself) does not satisfy the DCC.

3. SYSTEMS AND TRELLISES

Our notion of a dynamical system follows Willems [2], except that we explicitly consider time-varying signal alphabets. The following notions will be used for the formal definition.

A *time axis* is an interval (a gap-free subset) of Z. A *signal alphabet* is an arbitrary nonempty set; usually, it will have some algebraic structure, such as that of a vector space or of a group. The *signal sequence space* determined by a time axis T and a collection $\{W_j : j \in T\}$ of signal alphabets is the Cartesian product $\mathscr{W} = \Pi_{j \in T} W_j$. The time-$j$ component of a signal sequence $c \in \mathscr{W}$ will be denoted by $c(j)$.

DEFINITION 1. A *discrete-time dynamical system* is a triple $\Sigma = (T, \mathscr{W}, \mathscr{B})$ where T is a time axis, $\mathscr{W}$ is a signal sequence space, and the *behavior* $\mathscr{B}$ is a subset of $\mathscr{W}$.

The elements of $\mathscr{B}$ are the *trajectories* of the system. If the signal alphabet W does not vary with time, we sometimes write $\Sigma = (T, W, \mathscr{B})$ instead of $(T, \mathscr{W}, \mathscr{B})$ in accordance with Willems's notation.

From now on, we will drop the qualifiers "discrete time" and "dynamical" and simply speak about "systems." Also, when the time axis T and the signal sequence space $\mathscr{W}$ are clear from the context, we will sometimes refer to the "system $\mathscr{B}$" as a shorthand for the "system $(T, \mathscr{W}, \mathscr{B})$."

The systems of primary interest to us will be *shift-invariant* systems and *finite-time* systems. In the former case, the time axis T is Z, the signal alphabet is time invariant, and the behavior $\mathscr{B}$ is closed under left and right shifts by one position; in the latter case, T is finite. Finite-time systems are important in coding, where they are called "block codes."

If the signal alphabets W_j, $j \in T$, have an algebraic structure, they induce an algebraic structure on the signal sequence space $\mathscr{W} = \Pi_{j \in T} W_j$ by componentwise application of the operations of W_j. If the alphabets W_j, $j \in T$, are vector spaces over some field F, then $\mathscr{W}$ is also a vector space over F. If, in this case, $\mathscr{B}$ is a sub*space* of $\mathscr{W}$, then the system will be called *linear*. If the alphabets W_j are modules over some fixed ring and if $\mathscr{B}$ is a sub*module* of $\mathscr{W}$, then the system will also be called linear. If W_j are groups and $\mathscr{B}$ is a sub*group* of $\mathscr{W}$, then the system will be called a *group system*.

One of the basic topics in system theory is the interplay between behaviors and realizations. The realizations that will be considered in this paper are what Willems calls "evolution laws." This concept is well established in coding, from where we adopt the name "trellis." (This name is suggested by diagrams as in Figure 1.) Note that slightly more general realizations are considered by Willems and in [4], the difference being irrelevant for well-behaved (complete) systems. (Completeness will be considered further in Section 4.)

We will explicitly include time-varying trellises.

DEFINITION 2. A *trellis section* is a four-tuple $X = (W, S, S', B)$, where W is the *alphabet*; S and S' are the *left state space* and the *right state space*, respectively; and the *branches* B are a subset of $S \times W \times S'$ such that every state is part of at least one branch (i.e., there are no unused states).

If W, S, S' are groups and B is a sub*group* of the direct product $S \times W \times S'$, then we have a *group trellis section*. Similarly, if W, S, and S' are modules (or vector spaces) over some ring (field) and if B is a sub*module* (sub*space*) of the direct sum $S \oplus W \oplus S'$, then we have a *linear trellis section*.

A *trellis* $\mathscr{X} = \{X_j : j \in T\}$, for some time axis T, is a collection (or rather a sequence) of trellis sections $X_j = (W_j, S_j, S'_j, B_j)$ such that, for all j in T, $S_j = S'_{j-1}$. If T has a start k and/or an end l, we further require unique starting and/or ending states, i.e., $|S_k| = |S'_l| = 1$. If all component trellis sections are group trellis sections, then the trellis is a *group trellis*; if all components are linear, then the trellis is *linear*.

We will sometimes refer to the *time-j states* of a trellis $\mathscr{X} = \{X_j : j \in T\}$; by this, we mean the states $S_j = S'_{j-1}$.

In the important special case where $T = Z$ and all trellis sections X_j in $\mathscr{X} = \{X_j : j \in T\}$ are identical, the trellis is called *time invariant*.

If $b = (s, w, s')$ is a branch (i.e., an element of B) from a trellis section $X = (W, S, S', B)$, then we will say that b *starts* in s, *ends* in s', and is *labeled*[1] with w.

A *path* through a trellis $\mathscr{X}$ is a sequence $\dots, b_{-2}, b_{-1}, b_0, b_1, \dots$ of branches $b_j \in B_j$ such that b_{j+1} starts in the state where b_j ends. By a *biinfinite* path, we mean a path that extends both to $-\infty$ (or to the beginning of the time axis T) and to ∞ (or to the end of T). Note that we do not require that such a path "start" in the zero state (unless, of course, the time axis T has a beginning). The set of all biinfinite paths will be denoted by $\Pi(\mathscr{X})$. We will use the term *semiinfinite path* for paths of the form $b_j, b_{j+1}, \dots$ or $\dots, b_{j-1}\, b_j$, i.e., paths that extend from some finite time j to ∞ or $-\infty$ (or to the end or the beginning of the time axis).

By the *branch system* of $\mathscr{X}$, we mean the system with alphabets $\{B_j : j \in T\}$ and behavior $\Pi(\mathscr{X})$.

[1] The terms "branch" and "label" are used differently in [4]; here we follow the terminology of [5].

The set of all label sequences along paths in $\Pi(\mathscr{X})$ will be denoted by $\Lambda(\mathscr{X})$. The *label system* of the trellis $\mathscr{X} = \{X_j: j \in T\}$ is the system $(T, \mathscr{W}, \Lambda(\mathscr{X}))$ with $\mathscr{W} = \Pi_{j \in T} W_j$. Often, we will simply say "the label system $\Lambda(\mathscr{X})$."

The relation between a trellis and its label system is the main subject of this paper. A variety of terms will be used to express that a system Σ is the label system of a trellis $\mathscr{X}$: we may say that $\mathscr{X}$ *generates* Σ, that $\mathscr{X}$ is a *trellis for* Σ, or that $\mathscr{X}$ is a *realization* of Σ. The last of these terms (realization) will be avoided, however, when the difference between a trellis and the more general realizations of Willems and [4] is important, as it is for some subtle issues in Section 4.

For any given trellis $\mathscr{X}$, we define the mapping $\lambda_{\mathscr{X}}: \Pi(\mathscr{X}) \to \Lambda(\mathscr{X})$ that assigns to every path its label sequence. Since we will seldom consider more than one trellis simultaneously, we can usually simplify the notation by dropping the subscript $\mathscr{X}$ and write simply λ.

If $\mathscr{X}$ is a group trellis, then its branch system $\Pi(\mathscr{X})$ is clearly a group system and $\lambda(\cdot)$ is a homomorphism, which implies that the label system $\Lambda(\mathscr{X})$ is also a group system; if $\mathscr{X}$ is linear, both the branch system and the label system are linear, too. (According to the convention of Section 2, the second statement is actually implied by the first one.)

We conclude this section with some examples. Although this paper is about *group* systems, all examples, with one exception, will actually be linear systems over finite fields Z_p (the integers mod p), p prime. Interesting examples of "real" group systems tend to be larger and contribute little to the understanding of this paper. (Such examples are necessary, however, for proper illustration of some concepts discussed in [4] and [5].)

EXAMPLE 1. Consider the trellis $\mathscr{X} = \{X_j: j \in T\}$ that is illustrated by the diagram of Figure 1. The trellis sections $X_j = (W_j, S_j, S_j', B_j), j \in T =$

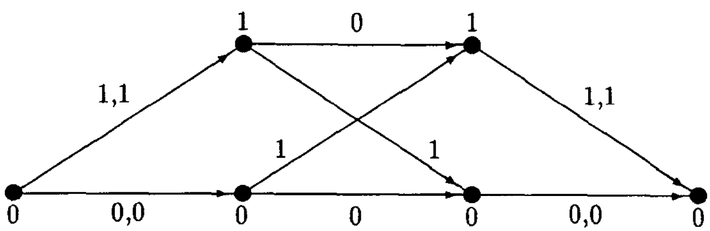

FIG. 1. A finite-time linear trellis (Example 1).

$\{0, 1, 2\}$, are formally defined as follows:

$$W_0 = W_2 = Z_2 \times Z_2, \qquad W_1 = Z_2,$$

$$S_0 = S_2' = \{0\}, \qquad S_0' = S_1 = S_1' = S_2 = Z_2,$$

$$B_0 = \{(0,00,0),(0,11,1)\},$$

$$B_1 = \{(0,0,0),(0,1,1),(1,1,0),(1,0,1)\},$$

$$B_2 = \{(0,00,0),(1,11,0)\}.$$

There are four "biinfinite" paths through the trellis: $\Pi(\mathscr{X}) = \{\pi_1, \pi_2, \pi_3, \pi_4\}$ with

$$\pi_1 = (0,00,0),(0,0,0),(0,00,0),$$

$$\pi_2 = (0,00,0),(0,1,1),(1,11,0),$$

$$\pi_3 = (0,11,1),(1,1,0),(0,00,0),$$

$$\pi_4 = (0,11,1),(1,0,1),(1,11,0).$$

The label system $\Lambda(\mathscr{X})$ consists of the four sequences $\{(00,0,00),(00,1,1),(11,1,00),(11,0,11)\}$. (In fact, this is a single-error-correcting binary linear code.) All three trellis sections are linear over the binary field Z_2. The trellis is therefore also linear over Z_2, and so is the label system.

EXAMPLE 2. Figure 2 shows a section $X = (W, S, S', B)$ of a time-invariant trellis with $W = Z_3$, $S = S' = Z_3$, and $B = \{(0,0,0),(1,1,2),(2,2,1)\}$. The trellis is linear over the field Z_3 and uncontrollable (cf. Section 6). The

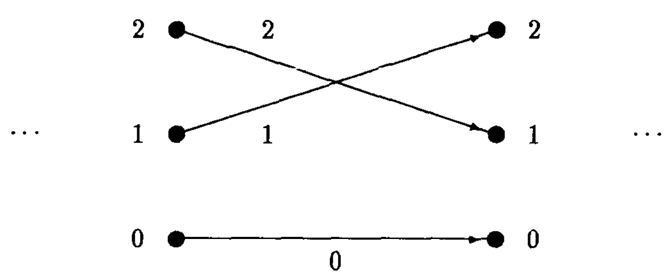

FIG. 2. A section of an uncontrollable, linear, time-invariant trellis (Example 2).

label system has only three trajectories: the all zero sequence and the two phases of $\ldots, 1, 2, 1, 2, \ldots$. It is an example of an *autonomous* system in the sense of [2].

Consider a standard input-state-output system with input $u(j)$, state $s(j)$, and output $y(j)$, e.g., with matrices[2] A, B, C, D such that

$$s(j+1) = As(j) + Bu(j),$$

$$y(j) = Cs(j) + Du(j). \tag{1}$$

Provided that every state s has a predecessor (i.e., the block matrix $[A, B]$ has full rank), such a system gives rise to the trellis with time-j branches

$$B_j = \{(s(j), u(j) | y(j), s(j+1))\}, \tag{2}$$

where the notation $u(j) | y(j)$ indicates that branches are labeled with input-output pairs.

EXAMPLE 3. Consider the binary linear (over Z_2) input-state-output system

$$s(t+1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} s(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} s(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t),$$

where $s(t)$, $u(t)$, and $y(t)$ are binary column vectors of dimension 2, 1, and 2, respectively. The corresponding trellis section has the following branches (for notational convenience, all column vectors are written as row vectors:

$$(00, 0|00, 00), (01, 0|11, 10), (10, 0|01, 00), (11, 0|10, 10),$$
$$(00, 1|11, 11), (01, 1|00, 01), (10, 1|10, 11), (11, 1|01, 01).$$

[2] We ask the reader's pardon for our overuse of the letter B.

A convolutional code, in the traditional sense, is the set of output sequences of a linear "encoder" of the type (1). In this case, the trellis with branches $(s(j), y(j), s(j+1))$, i.e., with output-labeled branches, is more important than the input-output trellis. The distinction between these two trellises is important with respect to minimality (cf. Section 5).

EXAMPLE 3 (Continued). The output trellis section corresponding to Example 3 is obtained from the input-output trellis by dropping the input part of the label, which results in the branches

$$(00,00,00),(01,11,10),(10,01,00),(11,10,10),$$
$$(00,11,11),(01,00,01),(10,10,11),(11,01,01).$$

4. MINIMAL TRELLISES AND THE CANONICAL TRELLIS

The state spaces that have been introduced in the previous section are attributes of a trellis. In this section, we shall define state spaces as attributes of a group system (i.e., of a behavior), construct a canonical trellis of the system using these state spaces, and show that any minimal trellis for the system is equivalent to this canonical trellis. The reader who is familiar with Willems's treatment of canonical realizations will recognize most of this development as a straightforward adaptation of his approach to time-variant group systems.

Let $(T, \mathscr{W}, \mathscr{B})$ be a group system, and let J be a subset of T. We will use the notation $c|_J$ for the restriction of a trajectory c from T to J. We will use the notation

$$\mathscr{B}_J = \{c \in \mathscr{B} : c(j) = 0 \text{ for } j \notin J\}$$

for those elements of $\mathscr{B}$ that are zero outside J, and

$$\mathscr{B}|_J = \{c|_J : c \in \mathscr{B}\}$$

for the restriction of $\mathscr{B}$ to J. In expressions of this type, the interval J will implicitly be understood as $J \cap T$; e.g., for $T = [0, 2]$, $\mathscr{B}_{[1,\infty)}$ means $\mathscr{B}_{[1,2]}$.

It is easily seen that, for any $J \subseteq T$, the sets $\mathscr{B}_J$, $\mathscr{B}_{T \setminus J}$ and $\mathscr{B}_J + \mathscr{B}_{T \setminus J}$ are normal subgroups of $\mathscr{B}$, which is essential for the following definition.[3]

[3]Here is perhaps the only place in the paper where the additive notation for noncommutative groups is a bit disturbing.

DEFINITION 3. The (two-sided) *state space at time j* of a group system $\Sigma = (T, \mathscr{W}, \mathscr{B})$ is the quotient group

$$S_j(\Sigma) = \mathscr{B} / \left(\mathscr{B}_{(-\infty, j)} + \mathscr{B}_{[j, \infty)}\right),$$

and the *time-j state of a trajectory $c \in \mathscr{B}$* is the coset

$$[c]_j = c + \left(\mathscr{B}_{(-\infty, j)} + \mathscr{B}_{[j, \infty)}\right).$$

That "state" is an appropriate name for the elements of $S_j(\Sigma)$ follows from the following theorem.

THEOREM 1. *Let c and c' be trajectories (i.e., elements of $\mathscr{B}$) of a group system $(T, \mathscr{W}, \mathscr{B})$. Then the concatenation of $c|_{(-\infty, j)}$ with $c'|_{[j, \infty)}$ is in $\mathscr{B}$ if and only if $[c]_j = [c']_j$.*

We now address the problem of constructing a trellis $\mathscr{X}$ for a given group system $\Sigma = (T, \mathscr{W}, \mathscr{B})$ such that $\Lambda(\mathscr{X}) = \mathscr{B}$.

Willems has pointed out that a trivial solution exists for this problem, viz., the trellis $\tilde{\mathscr{X}}_\Sigma = \{X_j : j \in T\}$ with $X_j = (W_j, S_j = \mathscr{B}, S_j' = \mathscr{B}, B_j)$ and branches $B_j = \{(c, c(j), c) : c \in \mathscr{B}\}$. This trivial trellis is very uneconomical (except for autonomous [2] systems): its time-j state space is as large as $\mathscr{B}$. A more economical trellis is suggested by Theorem 1:

DEFINITION 4. The *canonical trellis* associated with a group system $\Sigma = (T, \mathscr{W}, \mathscr{B})$ is the trellis $\mathscr{X}_\Sigma = \{X_j : j \in T\}$ whose time-j trellis section is $X_j = (W_j, S_j(\Sigma), S_{j+1}(\Sigma), B_j)$ with branches

$$B_j = \left\{\left([c]_j, c(j), [c]_{j+1}\right) : c \in \mathscr{B}\right\}.$$

It is clear that $\mathscr{X}_\Sigma$ is a group trellis and that $\mathscr{B} \subseteq \Lambda(\mathscr{X}_\Sigma)$. It is not clear at this point, however, whether $\mathscr{X}_\Sigma$ actually generates Σ, i.e., whether $\Lambda(\mathscr{X}_\Sigma) = \mathscr{B}$. Willems [2] has shown—and we will prove below—that $\Lambda(\mathscr{X}_\Sigma) = \mathscr{B}$ indeed holds for complete systems.

DEFINITION 5 (Willems [2]). A system $\Sigma = (T, \mathscr{W}, \mathscr{B})$ is *complete* if any sequence $c \in \mathscr{W}$ such that $c|_J \in \mathscr{B}|_J$ holds for all finite intervals $J \subseteq T$ is actually in $\mathscr{B}$.

For incomplete group systems, the canonical trellis may or may not generate the system. This is illustrated by the following examples.

EXAMPLE 4. Consider the system $\Sigma = \{T = Z, W = Z_2, \mathscr{B}\}$, where $\mathscr{B}$ is the set of all biinfinite sequences over Z_2 with a finite and even number of ones. This system is clearly time invariant, linear, and incomplete. It is easily seen that the canonical trellis $\mathscr{X}_\Sigma$ looks as in Figure 3. It is obvious from Figure 3 that $\Lambda(\mathscr{X}_\Sigma) = W^Z \neq \mathscr{B}$, i.e., the canonical trellis generates all sequences in W^Z, not just those in $\mathscr{B}$.

EXAMPLE 5. Consider the set of all periodic sequences over some field. This system is linear and incomplete. Its canonical trellis coincides with the trivial trellis and generates the system.

It is obvious that, for all engineering purposes, systems should be complete. In fact, Willems [1] has argued that "...the study of non-complete systems does not fall within the competence of system theorists and could better be left to cosmologists or theologians."

However, if a system is defined as the label system of an infinite-time trellis, completeness is not automatic even if the trellis is time-invariant, linear, and strongly controllable. (Controllability will be defined in Section 6.)

EXAMPLE 6. Consider the time-invariant trellis $\mathscr{X}$ with sections $X = (W, S, S', B\}$, where $W = S = S' = Z$ and branches $B = \{(s, s + 2s', s') : s, s' \in Z\}$. This trellis is linear over Z and 1-controllable (every state sequence is possible). The state sequence $\ldots, -8, +4, -2, +1, 0, 0, \ldots$ produces the label sequence $\ldots, 0, 1, 0, \ldots$. The sequence $\ldots, 1, 1, 1, \ldots$, however, is not in $\Lambda(\mathscr{X})$, since the corresponding condition $s(t+1) = (1 - s(t))/2$ on the state sequence $s(t)$ cannot be satisfied for all times t. Thus $\Lambda(\mathscr{X})$ is not complete.

Fortunately, this problem cannot arise for a time-invariant group trellis whose state space satisfies the descending-chain condition: it follows from Theorem 10 in Section 6 that the label system of such a trellis is complete.

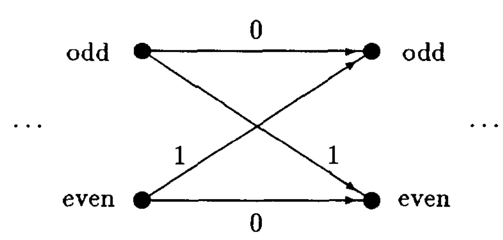

FIG. 3. A section of the canonical trellis of Example 4, which is not a minimal trellis.

We now begin to study minimality. Let $\mathscr{X}$ be a trellis (not necessarily a group trellis) that generates some group system Σ (i.e., the label system of $\mathscr{X}$ is Σ). Let S_j be the set of time-j states of $\mathscr{X}$. One consequence of Theorem 1 is that we can define a mapping

$$\psi_j: S_j \to S_j(\Sigma): S \mapsto [c]_j, \tag{3}$$

where c is the label sequence of an arbitrary biinfinite path through the trellis $\mathscr{X}$ that goes through state S at time j. (If follows from the definition of a trellis that such a path always exists.) For $j \in T$ and $s \in S_j$, let $[s]_j$ be the equivalence class $\{s' \in S_j: \psi_j(s') = \psi_j(s)\}$ of all time-j states that map to the same state in $S_j(\Sigma)$.

PROPOSITION 1. *For any fixed $j \in T$, all equivalence classes $[s]_j$ can be merged into single states without affecting the label system $\Lambda(\mathscr{X})$.*

(The proof is immediate from Theorem 1.) It is therefore natural to define minimality of trellises as follows.

DEFINITION 6. A trellis $\mathscr{X}$ for a group system Σ (i.e., with label system Σ) is *minimal at time j* if $\psi_j(\cdot)$ is one-to-one; $\mathscr{X}$ is *minimal* if $\psi_j(\cdot)$ is one-to-one for all $j \in T$.

EXAMPLE 1 (Continued). As is clear from Figure 1, neither the two time-1 states nor the two time-2 states can be merged without changing the label system. Proposition 1 thus implies that the trellis is minimal.

The following facts are obvious:

(1) Any nonminimal trellis of a group system can be reduced by state merging according to Proposition 1.

(2) If the canonical trellis $\mathscr{X}_\Sigma$ associated with a group system $\Sigma = (T, \mathscr{W}, \mathscr{B})$ satisfies $\Lambda(\mathscr{X}_\Sigma) = \mathscr{B}$ (e.g., if Σ is complete), then $\mathscr{X}_\Sigma$ is minimal.

(3) Any minimal trellis for a group system Σ is equivalent, up to renaming of states, to the canonical trellis $\mathscr{X}_\Sigma$.

Note, however, that the first of these observations does not imply that every trellis for a given group system can be reduced to a *minimal* trellis. In face, a minimal trellis may not exist for certain incomplete systems. It is clear from the observations above that a minimal trellis for a given group system Σ exists if and only if the label system of the canonical trellis $\mathscr{X}_\Sigma$ equals Σ.

The problem whether a minimal trellis exists for some given group system $\Sigma = (T, \mathscr{W}, \mathscr{B})$ is illuminated by the following procedure. We start with the trivial trellis $\tilde{\mathscr{X}}_\Sigma$. Then we apply the state merging according to Proposition 1 for time $j = 0$. The resulting trellis is minimal at time 0. We then continue the state merging for times $1, -1, 2, -2, \ldots$. For any positive integer k, we can in this way obtain a trellis $\mathscr{X}_k$ that is minimal on $[-k, k] \subseteq T$ and satisfies $\Lambda(\mathscr{X}_k) = \mathscr{B}$.

It is interesting to note that, in general, the trellises $\mathscr{X}_k$ are time variant even if the system is time invariant.

In the limit for $k \to \infty$, a trellis $\mathscr{X}_\infty$ is obtained that is equivalent to the canonical trellis. Note, however, that Proposition 1 guarantees $\Lambda(\mathscr{X}_k) = \mathscr{B}$ only for all *finite* k; the label system $\Lambda(\mathscr{X}_\infty) = \Lambda(\mathscr{X}_\Sigma)$ is not necessarily equal to Σ, as we have seen above.

It is clear that this problem disappears if Σ is complete. We have proved:

THEOREM 2. *Any complete group system has a minimal group trellis that is essentially unique and equivalent to the canonical trellis.*

Willems has pointed out that, in general, nonlinear systems do not have a unique minimal realization [2]. Theorem 2 shows, however, that group systems are well behaved in this respect.

5. MINIMALITY CONDITIONS

For deciding whether a given group trellis is minimal or not, Definition 6 is not very helpful. We will now state some more useful minimality conditions. We start with the following modest proposition.

PROPOSITION 2. *If $\mathscr{X}$ is a* group *trellis, then for any $j \in T$, the mapping $\psi_j(\cdot)$ of Equation* (3) *is a homomorphism.*

The condition for minimality at time j thus reduces to the condition that the kernel of $\psi_j(\cdot)$ contains only the zero state, which is the starting point for the proofs of the two theorems of this section.

By a *zero-label path* through a trellis, we mean a path (finite, semiinfinite, or biinfinite) all of whose labels are zero. A *trivial* zero-label path passes through zero states everywhere, i.e., it uses only zero branches.

THEOREM 3. *For any group trellis $\mathscr{X}$, the following conditions are equivalent:*

1. *$\mathscr{X}$ is minimal.*
2. *No state other than the zero state is the starting or ending state of a semiinfinite zero-label path.*
3. *The path-to-label mappings $\lambda|_{(-\infty, j)}$ and $\lambda|_{[j, \infty)}$ are one-to-one for all times j.*

EXAMPLE 1 (Continued). The minimality of the trellis of Figure 1 is easily established by condition 2.

EXAMPLE 4 (Continued). The group trellis of Figure 3 has a nontrivial biinfinite zero-label path and is therefore not minimal.

EXAMPLE 7. The group trellis of Figure 4 has nontrivial semiinfinite zero-label paths (from $-\infty$ to any time j) and is therefore not minimal.

For linear input-state-output systems as described by Equation (1), Theorem 3 can be used to test minimality either with respect to the input-output behavior or with respect to the output behavior.

EXAMPLE 3 (Continued). The input-output trellis is clearly minimal; the output trellis, however, contains the zero-label loop consisting of the single branch (01, 00, 01) and is therefore not minimal. The given input-state-output system is thus a minimal realization of its input-output behavior but nonminimal as a convolutional encoder.

More convenient formulations are possible for the important special case of time-invariant trellises with a state space that satisfies the descending-chain condition (as is the case for all time-invariant examples of this paper).

THEOREM 4. *Let $\mathscr{X}$ be a time-invariant group trellis whose state group S satisfies the descending chain condition. Then the following conditions are equivalent:*

1. *$\mathscr{X}$ is minimal.*
2. *There is no nontrivial biinfinite zero-label path, and no nontrivial zero-label branch starts or ends in the zero state.*
3. *There exists an integer L such that the mapping $\lambda|_{[0,L)}$ is one-to-one. If N is an upper bound on the number of steps in any descending chain in S (e.g., if $\mathscr{X}$ is linear over some field and S is N-dimensional), then this holds for some $L \leqslant N$.*

The theoretical importance of condition 2 is the separation of two types of nonminimality, viz., the presence of a nontrivial zero-label branch that starts from or ends in the zero state, or the presence of a nontrivial biinfinite zero-label path (a "zero-label loop"). (See, e.g., Figure 4 and Figure 3.) This distinction is important for convolutional codes, where the presence of a zero-label loop makes the trellis "catastrophic."

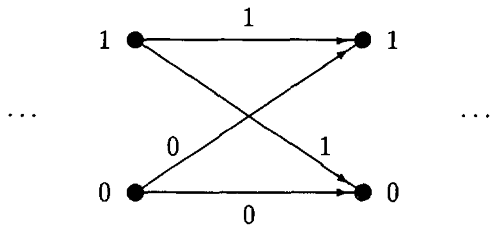

FIG. 4. A section of another nonminimal group trellis (Example 7).

It is also interesting to note that nontrivial biinfinite zero-label paths can always be removed from an arbitrary group trellis (that need not satisfy the preconditions of Theorem 4) without affecting the label system (cf. Lemma 1 in the proof of Theorem 10); this holds even if the label system has no minimal trellis.

The importance of condition 3 of Theorem 4 is that minimality can be tested locally in time, without having to examine the infinite past and future.

It is interesting to apply condition 3 to the standard input-state-output systems with A, B, C, D matrices (1). Assume that the state space of the given A, B, C, D system is N-dimensional (i.e., A is an $N \times N$ matrix). Assume further that every state has a predecessor, i.e., $[A, B]$ has rank N, and the system therefore gives rise to a trellis of the form (2). Then the path-to-label mapping $\lambda|_{[0, N)}$ is one-to-one if and only if the state sequence $s(0), s(1), \dots, s(N)$ is uniquely determined by the sequence $u(0)|y(0)$, $u(1)|y(1), \dots, u(N-1)|y(N-1)$ of input-output pairs; this is easily seen to be equivalent to the condition that the observability matrix $[C', A'C', \dots, (A')^{N-1}C']$ [where $(\cdot)'$ denotes transposition] has rank N. We have thus verified that the following theorem of Willems follows from Theorem 4.

THEOREM 5 (Cf. [2, part 3 of Theorem 4.2]). *An input-state-output system as in Equation* (1) *with an N-dimensional state space is minimal* (*in the sense of Willems*) *if and only if both* $[A, B]$ *and* $[C', A'C', \dots, (A')^{N-1}C']$ *have rank N.*

In other words, for the standard A, B, C, D systems, minimality in the sense of Definition 6 essentially coincides with (classical) observability. In fact, conditions 3 of Theorem 3 and Theorem 4 can be interpreted either as invertibility conditions or as observability conditions. This leads us to the topic of the next section.

6. CONTROLLABILITY AND OBSERVABILITY

In the traditional, input-output-oriented framework of system theory, controllability and observability are properties of *realizations*; they are dual properties, and they are intimately connected to minimality by the formula "minimal $\Leftrightarrow$ controllable and observable" [10].

In Willems's framework [2], on the other hand, controllability is defined as a property of a *behavior*, which is a particularly pleasing feature of his approach. Observability, however, is not considered as an intrinsic property of a system at all. There is thus no duality between controllability and observability, and no connection of these concepts with minimality.

We will combine the advantages of both approaches by defining controllability and observability both for systems (behaviors) and for trellises in such a way that they agree for a minimal group trellis and its label system. We will then formalize the observation from the end of Section 5 that, in the behavioral framework, minimality is essentially the same as observability.

We are primarily interested in *strong* controllability and *strong* observability. The formulation of satisfactory comprehensive notions of (weak-sense) controllability and observability is surprisingly difficult and not attempted here.[4]

DEFINITION 7. A system $\Sigma = (T, \mathscr{W}, \mathscr{B})$ is *$[j, k)$-controllable* if, for any two $c, c' \in \mathscr{B}$, there exists a $c'' \in \mathscr{B}$ such that $c''|_{(-\infty, j)} = c|_{(-\infty, j)}$ and $c''|_{[k, \infty)} = c'|_{[k, \infty)}$. The system is *$l$-controllable* if it is $[j, j + l)$-controllable for all $j \in T$. The system is *strongly controllable* if it is l-controllable for some nonnegative integer l, and the smallest such l is the *controllability index* of the system.

We next define controllability for a *trellis*.

DEFINITION 8. A trellis $\mathscr{X}$ is *$[j, k)$-controllable* (l-controllable, strongly controllable) if the set $\Pi(\mathscr{X})$ of biinfinite paths through $\mathscr{X}$ is so.

Note that Definition 8 is very natural: $[j, k)$-controllability means that every time-k state of the trellis can be reached from every time-j state.

[4] We are grateful to an anonymous reviewer for pointing out that a definition of (weak) controllability proposed by us in a previous version of this paper is not equivalent to Willems's definition for time-invariant systems.

THEOREM 6. *A complete group system is $[j, k)$-controllable (l-controllable, strongly controllable) if and only if its canonical trellis is so.*

We will now repeat this procedure for observability: definition for systems, definition for trellises, proof of equivalence for the canonical trellis.

Our notion of l-observability is identical to what Willems [2] calls "l-finite memory." We feel, however, that it is important to distinguish between controller memory and observer memory, and to preserve duality between them.

DEFINITION 9. A system $\Sigma = (T, \mathscr{W}, \mathscr{B})$ is *$[j, k)$-observable* if, for any c and c' in $\mathscr{B}$ such that $c|_{[j,k)} = c'|_{[j,k)}$, the concatenation of $c|_{(-\infty,j)}$ with $c'|_{[j,\infty)}$ is also in $\mathscr{B}$. The system is *l-observable* if it is $[j, j+l)$-observable for all $j \in T$. The system is *strongly observable* if it is l-observable for some nonnegative integer l, and the smallest such l is the *observability index* of the system.

In other words, in a $[j, k)$-observable system, the portion of the past that lies in the finite interval $[j, k)$ is a "sufficient statistic" for the entire past up to time k, from the point of view of determining what future trajectories are possible.

Note that a system is 0-observable if and only if any past can be connected with any future, which coincides with 0-controllability. Otherwise, however, the observability index of a system is not determined by its controllability index (or vice versa).

A concept closely related to l-observability if l-completeness:

DEFINITION 10 (Willems [2]). A system $\Sigma = (T, \mathscr{W}, \mathscr{B})$ is *l-complete* if every sequence $c \in \mathscr{W}$ such that $c|_{[j,j+l]} \in \mathscr{B}_{[j,j+l]}$ holds for all $j \in T$ is actually in $\mathscr{B}$.

In other words, a system is l-complete if membership in $\mathscr{B}$ is defined by a sliding window of width $l + 1$. This concept is also of basic importance in symbolic dynamics, where l-complete systems are called "subshifts of finite type."

It is clear that l-completeness implies l-observability. Willems has proved that the two notions are equivalent for complete systems. (For groups systems, this follows also from Theorems 7 and 8 below.) For incomplete systems, however, l-observability does not imply l-completeness.

EXAMPLE 8. The set of all binary sequences with only finitely many ones is linear over Z_2, time-invariant, 0-observable, but not complete (and, *a fortiori*, not 0-complete).

We now turn to trellises:

DEFINITION 11. A trellis $\mathscr{X}$ is *$[j, k)$-observable* if the path-to-label mapping $\lambda|_{[j,k)}$ is one-to-one; $\mathscr{X}$ is *l-observable* if it is $[j, j + l)$ observable for all $j \in T$.

THEOREM 7. *A complete group system is $[j, k)$-observable if and only if its canonical trellis is so.*

A slightly stronger version holds for a strongly observable trellis with or without a group structure:

THEOREM 8. *The label system of a l-observable trellis is l-complete.*

The connection between strong observability and minimality is obvious from Definition 11. A trivial rewriting of Theorem 4 yields the following theorem.

THEOREM 9. *Let $\mathscr{X}$ be a time-invariant group trellis whose state group S satisfies the descending chain condition. Then $\mathscr{X}$ is minimal if and only if it is strongly observable.*

If N is an upper bound on the number of steps in any descending chain (e.g., if $\mathscr{X}$ is linear over some field and S is N-dimensional), then $\mathscr{X}$ is minimal if and only if it is N-observable.

The difference between the classical minimality concept "minimal $\Leftrightarrow$ controllable and observable" and Willems's (and our) behavioral notion is thus simply that, for the former, the unreachable states of realization are considered to be redundant; as only one-sided (Laurent) sequences are considered in the classical framework, biinfinite paths through unreachable states are not considered to belong to the behavior.

EXAMPLE 2 (Continued). The trellis of Figure 2 is observable, uncontrollable, and minimal.

It is thus not surprising that the application of condition 3 of Theorem 4 to standard A, B, C, D realization yields the rank test on the observability matrix $[C', A'C', \dots, (A')^{N-1}C']$, as was pointed out in Theorem 5. It is notable, however, that this condition was derived without the Cayley-Hamilton theorem, which underlines the power of the abstract, universal-algebra approach of this paper.

It is now clear how weak-sense observability for a trellis should be defined:

DEFINITION 12. A trellis $\mathscr{X}$ is *observable* if every semiinfinite path is uniquely determined by its label sequence.

For this definition, observability of a group trellis always coincides with minimality, as is obvious from Theorem 3.

Further connections exist between strong observability (or l-completeness) and the descending-chain condition. The following one is particularly important.

THEOREM 10. *The label system of a time-invariant group trellis $\mathscr{X}$ whose state group S satisfies the descending-chain condition is complete and strongly observable (i.e., l-complete for some nonnegative integer l).*

Similar results have been obtained by Kitchens and Schmidt [9]. They have shown various generalizations of the fact that, for compact groups, complete shift-invariant group systems that satisfy a descending-chain condition are l-complete.

The difference between Theorem 10 and those results is that Theorem 10 assumes no restrictions for the signal alphabet, and completeness is a result, rather than an assumption, of the theorem.

7. A GLIMPSE AT FURTHER RESULTS

In this final section, some problems with, and results on, group systems are briefly reviewed that are not directly related to minimality and observability. The purpose of this section is to make the paper more useful as a self-contained introduction to group systems.

The development of this paper may have led to the impression that *all* the structure theory of linear systems holds unconditionally for group systems. This is not true, however.

A difficulty arises when it comes to input-state-output realizations. It is well known that, in the field case, every linear behavior can be realized as the output behavior (i.e., the set of possible output sequences) of a linear input-output system; alternatively, every linear behavior can be realized as the kernel behavior (i.e., as the set of input sequences such that the output is always zero) of such a system. In coding, such an input-output system is called an *encoder* (in the former case) or a *syndrome former* (in the latter case).

Interestingly, no encoder with a *homomorphic* input-output mapping exists for certain group systems. (This was first recognized in symbolic dynamics [11].) The problem appears even for linear system over general rings, where there may be no *minimal* linear encoder.

The difficulty can be overcome by introducing a generalized notion of a homomorphic encoder [5]. The output behavior of such generalized homomorphic encoders is always a group system, but such encoders may be nonlinear even in the field case.

The fundamental reason for this problem are the following facts from algebra. If G is a vector space over some field and H is a subspace of G, then G is isomorphic to the direct sum $H \oplus G/H$. This breaks down, however, for general rings and, *a fortiori*, for groups.

The main result of both [4] and [5] is the derivation of a canonically structured minimal encoder for strongly controllable group systems. Due to the mentioned difficulty, the input-output behavior of these encoders is, in general, not homomorphic.

The encoder of [4] is based on the decomposition of an l-controllable system into subsystems of controllability index $0, 1, 2, \ldots, l-1$. A "minimal" set of generators for the system is obtained from suitable generators of the subsystems. It is proved that a minimal such encoder exists for every strongly controllable group system. The output behavior of such an encoder is not necessarily a group system, however.

The encoder structure of [5], on the other hand, does not share this problem, i.e., the output sequences always form a group system. The encoder construction is based on the notion of the *state behavior* of the system, which consists of the set of state trajectories. A minimal encoder is obtained from a minimal encoder for the state behavior, and the encoder structure is defined by a recursive application of this principle. An unsatisfactory feature of this encoder is that the domain of certain homomorphisms is defined somewhat implicitly.

It is interesting that the inclusion of noncommutative groups caused no difficulties in the present paper, nor in [4] and [5]. For the construction of syndrome formers, however, noncommutativity becomes an issue. This problem is the subject of ongoing research.

APPENDIX A. PROOFS

Proof of Theorem 1

Let $\circ$ denote concatenation; let $\mathbf{0}$ denote the all-zero trajectory. Then

$$c|_{(-\infty,j)} \circ c'|_{[j,\infty)} \in \mathscr{B} \Leftrightarrow \mathbf{0}|_{(-\infty,j)} \circ (c'-c)|_{[j,\infty)} \in \mathscr{B}_{[j,\infty)}$$

$$\Leftrightarrow c' - c \in B_{(-\infty,j)} + B_{[j,\infty)},$$

where the second step follows from noting that, for any $\tilde{c} \in \mathscr{B}$,

$$\mathbf{0}|_{(-\infty,j)} \circ \tilde{c}|_{[j,\infty)} \in \mathscr{B}_{[j,\infty)} \Leftrightarrow \tilde{c}|_{(-\infty,j)} \circ \mathbf{0}|_{[j,\infty)} \in \mathscr{B}_{(-\infty,j)}.$$

Proof of Proposition 2

Let $\mathscr{T}$ be a group trellis. Let s and s' be two arbitrary time-j states of $\mathscr{T}$, and let π and π' be paths in $\Pi(\mathscr{T})$ whose time-j branches start in s and s', respectively. Then $\psi_j(s) + \psi_j(s') = [\lambda(\pi)]_j + [\lambda(\pi')]_j = [\lambda(\pi) + \lambda(\pi')]_j = [\lambda(\pi + \pi')]_j = \psi_j(s + s')$.

Proof of Theorem 3

Let $\Pi = \Pi(\mathscr{T})$ be the set of paths through the trellis $\mathscr{T}$, and let $\Lambda = \Lambda(\mathscr{T})$ be the set of label sequences. We will show $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$. For the last of these steps, we will need the following lemma.

LEMMA. *If $\lambda(\Pi_{[j,\infty)}) \neq \Lambda_{[j,\infty)}$ then $\lambda|_{(-\infty,j)}$ is not one-to-one.*

PROOF OF THE LEMMA. It is clear that $\lambda(\Pi_{[j,\infty)}) \subseteq \Lambda_{[j,\infty)}$ always holds. The assumption $\lambda(\Pi_{[j,\infty)}) \neq \Lambda_{[j,\infty)}$ thus implies the existence of a path π in Π such that $\lambda(\pi) \in \Lambda_{[j,\infty)}$ but $\pi \notin \Pi_{[j,\infty)}$, i.e., $\pi|_{(-\infty,j)} \neq 0$. But $\lambda|_{(-\infty,j)}(\pi|_{(-\infty,j)}) = 0$, which implies that $\lambda|_{(-\infty,j)}$ is not one-to-one. ■

$1 \Rightarrow 2$: Assume that $\pi|_{[j,\infty)}$ is a semiinfinite path through $\mathscr{T}$ that starts in a nonzero time-j state s although $\lambda(\pi)|_{[j,\infty)} = 0$, i.e., condition 2 is not satisfied. Since $\lambda(\pi)$ equals the concatenation of $\lambda(\pi)|_{(-\infty,j)}$ with $0|_{[j,\infty)}$, Theorem 1 implies $[\lambda(\pi)]_j = 0$. Thus $\psi_j(s) = 0$, and $\mathscr{T}$ is not minimal. An analogous argument applies if a semiinfinite path with zero labels *ends* in a nonzero state s.

$2 \Rightarrow 3$: Assume that $\lambda|_{[j,\infty)}$ is not one-to-one, i.e., the kernel of $\lambda|_{[j,\infty)}$ contains at least one nonzero element $\pi|_{[j,\infty)}$ in $\Pi_{[j,\infty)}$. Let $s_j, s_{j+1}, \ldots$ be the sequence of states that are used by the path $\pi|_{[j,\infty)}$. At least one of these states must be different from zero; condition 2 is therefore not satisfied. An analogous argument holds for $\lambda|_{(-\infty,j)}$.

$3 \Rightarrow 1$: Assume that both $\lambda|_{(-\infty,j)}$ and $\lambda|_{[j,\infty)}$ are one-to-one for all j. Let s be any time-j state of $\mathscr{T}$ such that $\psi_j(s) = 0$. We have to show that $s = 0$. Let π be a path through $\mathscr{T}$ whose time-j branch starts in s. Since $\psi_j(s) = [\lambda(\pi)]_j = 0$, we have $\lambda(\pi)|_{[j,\infty)} \in (\Lambda_{[j,\infty)})|_{[j,\infty)}$. We know from the lemma that $\lambda(\Pi_{[j,\infty)}) = \Lambda_{[j,\infty)}$; thus $\pi|_{[j,\infty)} \in (\Pi_{[j,\infty)})|_{[j,\infty)}$ by the invertibility of $\lambda|_{[j,\infty)}$, and $s = 0$ follows.

Proof of Theorem 4

The implications 1 ⇒ 2 and 3 ⇒ 1 are immediate consequences of Theorem 3. It remains to show 2 ⇒ 3, for which we need the following lemma.

LEMMA. *Assume that condition 2 of Theorem 4 holds. Then the condition $\lambda(\pi)|_{[0,\infty)} = 0$, for some $\pi \in \Pi(\mathscr{X})$, implies that the branch $\pi(0)$ starts in the zero state of $\mathscr{X}$, i.e., the zero state is the only state of $\mathscr{X}$ in which a semiinfinite zero-label path starts.*

Proof of the lemma.

For $i = 0, 1, 2, \ldots$, let $\tilde{S}_i$ be the set of time-i states in the kernel of $\lambda(\cdot)|_{[0,\infty)}$, i.e., $\tilde{S}_i$ are those states of $\mathscr{X}$ that allow both an infinite zero-label future and a length-i zero-label past. All $\tilde{S}_i$ are clearly normal subgroups of the state group $\tilde{S}$ of $\mathscr{X}$, and $S \supseteq \tilde{S}_0 \supseteq \tilde{S}_1 \supseteq \cdots$ is a descending chain, which implies $\tilde{S}_{l+1} = \tilde{S}_l$ for some large enough l. Since every state in $\tilde{S}_{i+1}$ has a predecessor in $\tilde{S}_i$ to which it is connected by a zero-label branch, the equality $\tilde{S}_{l+1} = \tilde{S}_l$ implies that every state in $\tilde{S}_l$ is part of a biinfinite zero-label path. Condition 2 of the theorem thus implies that $\tilde{S}_l$ contains only the zero state of $\mathscr{X}$. Tracing any zero-label path backwards from time l to time 0, condition 2 prohibits us from leaving the zero state, which implies $\tilde{S}_0 = \{0\}$. ■

Assume that condition 2 of the theorem holds. For $i = 1, 2, \ldots$, let $\tilde{S}_i$ be the set of time-0 states in the kernel of $\lambda(\cdot)|_{[0,i)}$, i.e., $\tilde{S}_i$ are those states of $\mathscr{X}$ that allow a length-i zero-label future. All $\tilde{S}_i$ are clearly normal subgroups of the state group $\tilde{S}$ of $\mathscr{X}$, and $S \supseteq \tilde{S}_1 \supseteq \tilde{S}_2 \supseteq \cdots$ is a descending chain. Thus $\tilde{S}_{L+1} = \tilde{S}_L$ for some integer L; if $\tilde{N}$ is an upper bound on the number of steps in any descending chain, then $\tilde{S}_{L+1} = \tilde{S}_L$ for some $L \leqslant N$.

Every state in $\tilde{S}_{i+1}$ has a successor in $\tilde{S}_i$ to which it is connected by a zero-label branch. The equality $\tilde{S}_{L+1} = \tilde{S}_L$ therefore implies that every state in $\tilde{S}_L$ allows an infinite zero-label future. The lemma thus gives $\tilde{S}_L = \{0\}$, i.e., all paths in the kernel of $\lambda(\cdot)|_{[0,L)}$ start in the zero state of $\mathscr{X}$. Tracing any such path forward, condition 2 prohibits us from leaving the zero state, which implies that $\lambda|_{[0,L)}$ is one-to-one.

Proof of Theorem 6

The "if" part is obvious. For the "only if" part, assume that $\Sigma = (T, \mathscr{W}, \mathscr{B})$ is a complete group system that is $[j,k)$-controllable, and let $\mathscr{X}$ be the canonical (or any minimal) trellis for Σ. Let π and π' be two arbitrary paths in $\Pi(\mathscr{X})$. By the $[j,k)$-controllability of Σ, there exists a $c'' \in \mathscr{B}$ such that $c''|_{(-\infty,j)} = \lambda(\pi)|_{(-\infty,j)}$ and $c''|_{[k,\infty)} = \lambda(\pi')|_{[k,\infty)}$.

Let π'' be a path in $\Pi(\mathscr{X})$ such that $\lambda(\pi'') = c''$. Then Theorem 3 (condition 3) implies that $\pi''|_{(-\infty,j)} = \pi|_{(-\infty,j)}$ and $\pi''|_{[k,\infty)} = \pi'|_{[k,\infty)}$, which shows that the trellis $\mathscr{X}$ is $[j, k)$-controllable.

Proof of Theorem 7

The "if" part is obvious. For the "only if" part, assume that $\mathscr{X}$ is a minimal group trellis such that $\lambda|_{[j,k)}$ is not one-to-one. Then there exist two path π and π', with corresponding label sequences $c = \lambda(\pi)$ and $c' = \lambda(\pi')$, such that $c|_{[j,k)} = c'|_{[j,k)}$ but $\pi|_{[j,k)} \neq \pi'|_{[j,k)}$. By condition 3 of Theorem 3, the paths π and π' have different time-j states.

Again because of condition 3 of Theorem 3, $\pi|_{(-\infty,j)}$ is the unique path for $c|_{(-\infty,j)}$, and $\pi'|_{[j,\infty)}$ is the unique path for $c'|_{[j,\infty)}$. As π and π' have different time-j states, the concatenation of $c|_{(-\infty,j)}$ with $c'|_{[j,\infty)}$ is not in $\Lambda(\mathscr{X})$, which shows that the label system of $\mathscr{X}$ is not $[j, k)$ observable.

Proof of Theorem 8

Let $\mathscr{X}$ be an l-observable trellis, and let $\Sigma = (T = Z, \mathscr{W}, \mathscr{B} = \Lambda(\mathscr{X}))$ be its label system. Let c be a sequence in $\mathscr{W}$ such that, for all $j \in T$, $c|_{[j,j+l]} \in \mathscr{B}|_{[j,j+l]}$. We have to show that $c \in \mathscr{B}$.

For all $j \in T$, let π_j be a path in $\Pi(\mathscr{X})$ such that $\lambda(\pi_j)|_{[j,j+l]} = c|_{[j,j+l]}$. Then $\lambda(\pi_{j-1})|_{[j,j+l)} = c|_{[j,j+l)} = \lambda(\pi_j)|_{[j,j+l)}$, which, by the invertibility of $\lambda|_{[j,j+l)}$, implies $\pi_{j-1}|_{[j,j+l)} = \pi_j|_{[j,j+l)}$. The path segments $\pi_j|_{[j,j+l]}$, $j \in T$, thus agree on their overlapping parts and can be glued together to form a path $\pi \in \Pi(\mathscr{X})$ such that $\lambda(\pi) = c$.

Proof of Theorem 10

Before we can start with the actual proof, we have to make some preparations.

Let $\mathscr{X} = \{X_j : j \in T\}$ be a group trellis with sections $X_j = (W_j, S_j, S_j', B_j)$. (Time invariance and the descending-chain condition are not necessary at this point.) The set $\tilde{B}_j$ of *neutral branches* of B_j consists of those branches in B_j through which a biinfinite zero-label path exists. The following lemma shows that one can get rid of the nontrivial neutral branches $\tilde{B}_j$ simultaneously for all times j without affecting the label system.

LEMMA 1. *For any time $j \in T$, the neutral branches $\tilde{B}_j$ are a normal subgroup of B_j, and the time-j states $\tilde{S}_j$ and $\tilde{B}_j$ are a normal subgroup of S_j. Moreover, if $\mathscr{X}' = \{X_j' : j \in T\}$ is the group trellis that is obtained from $\mathscr{X}$ by merging the states in every coset $s\tilde{S}_j$, for all $j \in T$, i.e., with $X_j' = (W_j, S_j/\tilde{S}_j, S_j'/\tilde{S}_j', B_j')$ and $B_j' = \{(s\tilde{S}_j, w, s\tilde{S}_j') : (s, w, s') \in B_j\}$, then $\Lambda(\mathscr{X}') = \Lambda(\mathscr{X})$.*

PROOF OF LEMMA 1. That $\tilde{B}_j$ is a normal subgroup of B_j, and $\tilde{S}_j$ a normal subgroup of S_j, follows from the observation that the set of all biinfinite zero-label paths is a normal subgroup of $\Pi(\mathscr{X})$. The relation $\Lambda(\mathscr{X}) \subseteq \Lambda(\mathscr{X}')$ is clear. It remains to show $\Lambda(\mathscr{X}') \subseteq \Lambda(\mathscr{X})$.

Let π' be a path in $\Pi(\mathscr{X}')$. A path $\pi \in \Pi(\mathscr{X})$ such that $\lambda(\pi) = \lambda(\pi')$ can be constructed as follows. By the definition of B_j', there exists, for every time j, a branch $b_j = (s_j, w_j, s_j')$ in B_j such that $\pi'(j) = (s_j\tilde{S}_j, w_j, s_j'\tilde{S}_j')$. Let $\pi(0) = b_0$. Since the starting state s_1 of b_1 and the ending state s_0' of b_0 are in the same coset $s_1\tilde{S}_1 = s_0'\tilde{S}_0'$, there exists a neutral branch $\tilde{b}_1$ in $\tilde{B}_1$ such that the branch $b_1\tilde{b}_1$ starts in the ending state s_0' of $\pi(0)$. Let $\pi(1) = b_1\tilde{b}_1$. Note that the branch $\pi(1)$ has the same label as $\pi'(1)$ and ends in one of the states in the coset $s_1'\tilde{S}_1'$. Continuing in this way, we can construct $\pi(2), \pi(3), \ldots$ and $\pi(-1), \pi(-2), \ldots$. ∎

We also need the following lemma.

LEMMA 2. *If, for all π in $\Pi(\mathscr{X})$ and all $j \in T$, the branch $\pi(j)$ is uniquely determined by $\lambda(\pi)|_{(j-l/2,j+l/2]}$, then the label system $\Lambda(\mathscr{X})$ is l-complete.*

PROOF OF LEMMA 2. (Note that the interval $(-l/2, l/2]$ always contains precisely l integers.) Let c be a sequence in $\mathscr{W}$ such that, for all $j \in T$,

$$c|_{(j-l/2-1,j+l/2]} \in \Lambda(\mathscr{X})|_{(j-l/2-1,j+l/2]}.$$

We have to show that $c \in \Lambda(\mathscr{X})$.

For all $j \in T$, there exists a path π_j in $\Pi(\mathscr{X})$ such that

$$c|_{(j-l/2-1,j+l/2]} = \lambda(\pi_j)|_{(j-l/2-1,j+l/2]}.$$

Since $\pi_j|_{(j-l/2,j+l/2]} = \pi_{j+1}|_{(j-l/2,j+l/2]}$, we have $\pi_j(j) = \pi_{j+1}(j)$ by the assumption of the lemma. In particular, the starting of the branch $\pi_{j+1}(j+1)$ equals the ending state of $\pi_j(j)$. The branch sequence π defined by $\pi(j) = \pi_j(j)$ is therefore a valid path through the trellis, and it is clear that $\lambda(\pi) = c$. ∎

We are now ready for the proof of the theorem. We thus assume that $\mathscr{X}$ is a time-invariant group trellis whose state group S satisfies the descending-chain condition. Because of Lemma 1, we can further assume that $\mathscr{X}$ has no nontrivial biinfinite zero-label path; for, if there were such paths, we could merge the corresponding states without changing the label system, and the new state group would still satisfy the descending chain condition.

For $i = 0, 1, 2, \ldots,$ consider the paths $P_i = \{\pi \in \Pi(\mathscr{X}) : \pi|_{[-i,i]} = \mathbf{0}\}$. Let $\tilde{B}_i = \{\pi(0) : \pi \in P_i\}$ be the corresponding set of time-0 branches, which is clearly a normal subgroup of the branches B of $\mathscr{X}$. Let $\tilde{S}_i$ and $\tilde{S}'_i$ be the set of starting and ending states, respectively, of $\tilde{B}_i$.

We have the relations $\tilde{S}_{i+1} \subseteq \tilde{S}_i$, $\tilde{S}'_{i+1} \subseteq \tilde{S}'_i$, $\tilde{S}_{i+1} \subseteq \tilde{S}'_i$, and $\tilde{S}'_{i+1} \subseteq \tilde{S}_i$. The descending-chain condition implies $\tilde{S}_{i+1} = \tilde{S}_i$ and $\tilde{S}'_{i+1} = \tilde{S}'_i$ for all sufficiently large i. But then $\tilde{S}_{i+1} \subseteq \tilde{S}'_i = \tilde{S}'_{i+1} \subseteq \tilde{S}_i = \tilde{S}_{i+1}$, from which we conclude $\tilde{S}_i = \tilde{S}'_i$, since both inclusions must be equalities. But every state in $\tilde{S}_i = \tilde{S}'_i$ has both a predecessor and a successor in $\tilde{S}_i = \tilde{S}'_i$ such that the corresponding branch label is zero. But this implies $\tilde{B}_i = 0$, since the trellis has no nontrivial biinfinite zero-label paths.

We have thus shown that, for any path $\pi \in \Pi(\mathscr{X})$, the time-zero branch $\pi(0)$ is uniquely determined by $\lambda(\pi)|_{[-i,i]}$, which implies $(2i + 1)$-completeness by Lemma 2.

APPENDIX B. NOTATION

T	a time axis
W_j	a signal alphabet at time j
$\mathscr{W}$	the signal sequence space $\Pi_{j \in T} W_j$
$\mathscr{B}$	the set of trajectories (the behavior) of a system, $\mathscr{B} \subseteq \mathscr{W}$
Σ	a discrete-time dynamical system $(T, \mathscr{W}, \mathscr{B})$
S_j	left state space of a trellis section at time j
S'_j	right state space of a trellis section at time j
B_j	branch space of a time-j trellis section, $B_j \subseteq (S_j, W_j, S'_j)$
X_j	a time-j trellis section (W_j, S_j, S'_j, B_j)
$\mathscr{X}$	the trellis $\{X_j : \mathrm{j} \in T\}$
$\Pi(\mathscr{X})$	the branch system of a trellis $\mathscr{X}$
$\Lambda(\mathscr{X})$	the label system of a trellis $\mathscr{X}$
λ	the mapping from branch sequences to label sequences
$\mathscr{B}_J$	the set of trajectories that are zero outside J, $J \subseteq T$
$\mathscr{B}\|_J$	behavior restricted to J, $J \subseteq T$
$S_j(\Sigma)$	the canonical state space of a group system Σ at time j
$[c]_j$	the canonical state of a trajectory $c \in \mathscr{B}$ at time j
$\mathscr{X}_\Sigma$	the canonical trellis of a group system Σ
ψ_j	the mapping (3) from the time-j trellis states to the time-j states of the label system

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